D-modules in birational geometry

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Rio de Janeiro, August 4, 2018

Some questions/results

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2) Let (A, Θ) be an irreducible principally polarized abelian variety. Give a sharp bound for $\mathrm{mult}_x \Theta$, for isolated singular points $x \in \Theta$.

3) Let X be a smooth projective variety of general type. Then every 1-form on X vanishes at some point. (Suitable statement in arbitrary Kodaira dimension.)

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5) Base spaces of families with maximal variation – e.g. subvarieties of moduli – of (not necessarily canonically polarized) smooth projective varieties are hyperbolic in various senses: log general type, Brody hyperbolic, ...

6) If $f \in \mathcal{O}_X$ and $b_f(s)$ is its Bernstein-Sato polynomial, let $\tilde{b}_f(s) = \frac{b_f(s)}{(s+1)}$ be the *reduced* Bernstein-Sato polynomial. Well known fact:

- $-\big(\text{largest root of } b_f(s)\big) = \text{log canonical threshold } \mathrm{lct}(f)$.

Similarly, give a bound in terms of invariants of a resolution of singularities for largest root of $\tilde{b}_f(s)$ (the "microlocal log canonical threshold").

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What is it? Rough answers:

(i) Vast generalization of the theory of **variations of Hodge structure** in the presence of singularities (of maps, of connections, of spaces...)

(ii) Special more manageable case of the theory of **filtered** D**-modules**, where tools from Hodge theory and birational geometry can be brought into play.

Filtered D-modules

- X smooth complex variety; $D_X =$ sheaf of differential operators on X, with standard filtration given by:

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- D-module $\mathcal{M}$ is an $\mathcal{O}_X$-module carrying an action of D_X; equivalently, have operation

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_X$$

satisfying the properties of an integrable connection (but $\mathcal{M}$ not necessarily a vector bundle).

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- (Good) filtration on $\mathcal{M}$ is a $\mathbf{Z}$-indexed increasing filtration $F_\bullet \mathcal{M}$ such that

$$F_k D_X \cdot F_\ell \mathcal{M} \subseteq F_{k+\ell} \mathcal{M}$$

plus other (well, good...) properties, e.g. all $F_\ell \mathcal{M}$ coherent.

Localization

Typical example: Let D be a hypersurface in X, and

$$\mathcal{O}_X(*D) := \bigcup_{k\geq 0} \mathcal{O}_X(kD)$$

be the quasi-coherent sheaf of functions with poles along D. Locally, if $D = (f = 0)$, this is the localization $\mathcal{O}_X[f^{-1}]$; natural action of D_X by quotient rule.

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- Pole order filtration: $P_k\mathcal{O}_X(*D) = \mathcal{O}_X\big((k+1)D\big)$, for $k \geq 0$.

Not the right filtration (unless D is smooth)! We want something with Hodge theoretic meaning, which also captures the singularities of D.

De Rham complex

De Rham complex. $(\mathcal{M}, F)$ = filtered D-module.

$$\mathrm{DR}(\mathcal{M}) := \big[0 \to \mathcal{M} \to \mathcal{M} \otimes \Omega_X^1 \to \cdots \to \mathcal{M} \otimes \Omega_X^n \to 0\big]$$

Placed in degrees $-n, \ldots, 0$; if

$$\mathrm{gr}_k^F \mathcal{M} := F_k\mathcal{M}/F_{k-1}\mathcal{M},$$

then this induces associated graded complexes

$$\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}) = \big[0 \to \mathrm{gr}_k^F \mathcal{M} \to \cdots \to \mathrm{gr}_{k+n}^F \mathcal{M} \otimes \Omega_X^n \to 0\big]$$

which are objects in $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ (i.e. maps become $\mathcal{O}_X$-linear).

Hodge modules

Here we consider a Hodge module as being a (regular, holonomic) D_X-module $\mathcal{M}$ with good filtration $F_\bullet \mathcal{M}$, satisfying a number of special properties; e.g.:

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- When $X = \mathrm{pt}$, it is the complexification of a **Hodge structure**.

- An **inductive condition** on the dimension of the support of $\mathcal{M}$ is satisfied: for each $f \in \mathcal{O}_X$ (locally), the graded quotients of the monodromy weight filtration on the **nearby and vanishing cycles** with respect to f (defined at the level of D-modules using the Kashiwara-Malgrange V-filtration) are again Hodge modules.

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As with VHS, there are pure, polarizable, mixed Hodge modules, etc.

Vanishing and positivity package

Key feature: Hodge modules satisfy vanishing and positivity properties extending those of (relative) canonical bundles in birational geometry. Starting point:

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Theorem (Saito Vanishing)

If X is a smooth projective variety, $(\mathcal{M}, F) =$ Hodge module and $L =$ ample line bundle on X, then

$$\mathbf{H}^i\big(X, \mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}) \otimes L\big) = 0 \quad \text{for all} \quad i > 0.$$

Examples

(1) Variations of Hodge structure: Any VHS $\big((\mathcal{V}, \nabla), F^\bullet \mathcal{V}\big)$ defines a (pure) Hodge module.

- In particular, the trivial VHS on X of dimension n defines the trivial Hodge module $\mathbf{Q}_X^H[n]$. The D-module is $\mathcal{O}_X$ with usual differentiation, and with filtration

$$F_k\mathcal{O}_X = 0, \quad k < 0 \quad \text{and} \quad F_k\mathcal{O}_X = \mathcal{O}_X, \quad k \geq 0.$$

Have

$$\mathrm{gr}^F_{-k}\mathrm{DR}(\mathcal{O}_X) = \Omega_X^k[n-k], \quad \text{for all} \quad k,$$

so Saito's theorem specializes to Kodaira-Nakano vanishing.

Examples

(2) Localization: $\mathcal{O}_X(*D)$, for $D \subset X$ a hypersurface, underlies the (mixed) Hodge module $j_*\mathbf{Q}_U^H[n]$, where $U = X \smallsetminus D$ and $j\colon U \hookrightarrow X$. Filtration hard to describe, but know

$$F_0\mathcal{O}_X(*D) = F_{\mathrm{lowest}}\mathcal{O}_X(*D) \simeq \mathcal{O}_X(D) \otimes \mathcal{I}\big((1-\epsilon)D\big),$$

the **multiplier ideal** of the **Q**-divisor $(1-\epsilon)D$. More later.

- In particular

$$\mathrm{gr}_0^F \mathrm{DR}(\mathcal{O}_X(*D)) \simeq \omega_X(D) \otimes \mathcal{I}\big((1-\epsilon)D\big),$$

so we recover (a special case of) Nadel vanishing.

Examples

(3) Direct images: $f : X \to Y$ projective morphism; Saito **decomposition theorem** (generalizing BBD):

$$f_+(\mathcal{O}_X, F) \simeq \bigoplus (\mathcal{M}_i, F)[-i]$$

in the derived category of filtered D-modules, and each $(\mathcal{M}_i, F)$ is a (pure) Hodge module on Y. Again, filtration hard to describe, but can at least show

$$\mathrm{gr}^F_{\mathrm{lowest}} \mathrm{DR}(\mathcal{M}_i) \simeq R^i f_* \omega_X,$$

so we recover Kollár vanishing.

Vanishing and positivity package

- Also have a natural extension of semi (and weak) positivity results of Griffiths, Fujita, Kawamata, Viehweg, Zuo,...

Theorem (P.–Wu)

$(\mathcal{M}, F) =$ polarizable pure Hodge module with strict support X. For each p the dual $K_p^\vee$ of the kernel of the generalized Kodaira-Spencer map

$$\theta_p \colon \mathrm{gr}_p^F \mathcal{M} \to \mathrm{gr}_{p+1}^F \mathcal{M} \otimes \Omega_X^1$$

is weakly positive.

Main constructions

Two main constructions based on the framework above are used towards the applications described at the beginning:

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Two main constructions based on the framework above are used towards the applications described at the beginning:

(1) Hodge ideals associated to $\mathbf{Q}$-divisors

(2) Hodge modules and graded systems arising from families of varieties

Hodge ideals (work with Mustață)

- Back to $\mathcal{O}_X(*D)$, for $D \subset X$ a reduced hypersurface. Hodge filtration coming from mixed Hodge module structure satisfies

$$F_k\mathcal{O}_X(*D) \subseteq P_k\mathcal{O}_X(*D) = \mathcal{O}_X\big((k+1)D\big), \quad \text{for all} \quad k \geq 0.$$

- Defines

$$F_k\mathcal{O}_X(*D) = \mathcal{O}_X\big((k+1)D\big) \otimes I_k(D).$$

with $I_k(D)$ = the k-th Hodge ideal of D.

Hodge ideals

Theorem

(i) (also Saito) $I_0(D)$ *is the multiplier ideal* $\mathcal{I}\big((1-\epsilon)D\big)$*; so*

$$I_0(D) = \mathcal{O}_X \iff (X, D) \text{ is log canonical.}$$

(ii) $I_k(D)$ *determine Deligne's Hodge filtration on the singular cohomology* $H^\bullet(U, \mathbf{C})$*, where* $U = X \smallsetminus D$.

(iii) D *is smooth if and only if* $I_k(D) = \mathcal{O}_X$ *(i.e.* $F_k = P_k$*) for all* k.

(iv) If $I_k(D) = \mathcal{O}_X$ *for some* $k \geq 1$*, then* D *is normal with rational singularities.*

Theorem (Cont'd)

(v) There are non-triviality criteria for $I_k(D)$ at a point $x \in D$ in terms of the multiplicity of D at x.

(vi) On smooth projective varieties, $I_k(D)$ satisfy an analogue of Nadel Vanishing for multiplier ideals.

(vii) $I_k(D)$ satisfy restriction (inversion of adjunction), subadditivity and semicontinuity theorems.

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Recent development: Similar (but substantially more complicated) picture for Hodge ideals associated to $\mathbf{Q}$-divisors; survey in the proceedings paper. Important for applications, e.g. the bound for the largest root of the reduced Bernstein-Sato polynomial mentioned at the beginning.

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Applications based on a combination of local criteria and vanishing. Quick example (special case of a conjecture on theta divisors):

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Theorem

Let (A, Θ) *be an irreducible principally polarized abelian variety,* $\dim A = g$. *If* Θ *has isolated singularities, then*

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Idea: $\mathrm{mult}_x \Theta \ge \frac{g+2}{2} \implies I_1(\Theta) \subseteq \mathfrak{m}_x^2$ by the local non-triviality criterion; vanishing theorem for $I_1(\Theta)$ then implies that the linear system $|2\Theta|$ separates tangent vectors, **contradiction** (Kummer map is ramified at the 2-torsion points of A).

Hodge modules from maps (work with Schnell)

- $f : X \to Y$ be a morphism of smooth projective varieties, L **ample** line bundle on Y.

$$B := \omega_{X/Y} \otimes f^* L^{-1}.$$

Assume: $(*)$ $\exists$ generically finite map $\varphi \colon Z \to X$ such that

$$H^0(Z, \varphi^* B) \neq 0,$$

Let $h \colon Z \to Y$ as in

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & X \\ & {\scriptstyle h}\searrow & \downarrow {\scriptstyle f} \\ & & Y \end{array}$$

Hodge modules from maps

- Hodge module construction for map h:

$$h_+(\mathcal{O}_Z, F) \simeq \bigoplus (\mathcal{M}_i, F)[-i],$$

and we focus on $(\mathcal{M}_0, F)$ (extension of VHS given by middle cohomology of smooth fibers of h).

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- Inspired by beautiful construction of Viehweg-Zuo: there is a graded submodule

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which remembers all the data:

- G is "nice" away from $D =$ singular locus of f.
- $G_0 \simeq L$.
- G has the weak positivity properties of Hodge modules.

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- $\mathrm{Supp}(G) \subseteq$ image in $T^*Y = Y \times H^0(X, \Omega_X^1)$ of

$$Z_f = \{(x, \omega) \mid \omega(x) = 0\}\big) \subset X \times H^0(X, \Omega_X^1).$$

- $\mathrm{Supp}(G) \twoheadrightarrow H^0(X, \Omega_X^1)$, using **vanishing theorems** for Hodge modules.

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- $\mathrm{Supp}(G) \twoheadrightarrow H^0(X, \Omega_X^1)$, using **vanishing theorems** for Hodge modules.

- We deduce that every 1-form on X of general type has zeros, a conjecture of Hacon-Kovács and Luo-Zhang.

Hodge modules from maps

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Use $G_0 = L$ and the **weak positivity** of $K_p^\vee$, with

$$K_p = \mathrm{Ker}\big(\theta_p \colon G_p \to G_{p+1} \otimes \Omega_Y^1\big),$$

plus criterion of Campana-Păun, to get that (Y, D) is of log general type (i.e. $\omega_Y(D)$ is big).

This is an extension of a hyperbolicity conjecture due to Viehweg.

THANK YOU!