D-modules in birational geometry

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Some questions/results

- **Basic geometric problems** we were recently able to solve (many in joint works with M. Mustață or C. Schnell); in most cases no proofs are known via more established methods. Some were familiar conjectures – will return to them at the end.

1. “Bound” the geometry of $S = \text{set of isolated singular points of multiplicity } m \geq 2$ of a hypersurface $D \subset P^n$ of fixed degree $d$ (e.g. number of conditions imposed on hypersurfaces in $P^n$).

2. Let $(A, \Theta)$ be an irreducible principally polarized abelian variety. Give a sharp bound for $\text{mult}_x \Theta$, for isolated singular points $x \in \Theta$.

3. Let $X$ be a smooth projective variety of general type. Then every 1-form on $X$ vanishes at some point. (Suitable statement in arbitrary Kodaira dimension.)
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5) Base spaces of families with maximal variation – e.g. subvarieties of moduli – of (not necessarily canonically polarized) smooth projective varieties are hyperbolic in various senses: log general type, Brody hyperbolic, ...

6) If $f \in \mathcal{O}_X$ and $b_f(s)$ is its Bernstein-Sato polynomial, let $\tilde{b}_f(s) = \frac{b_f(s)}{(s+1)}$ be the reduced Bernstein-Sato polynomial. Well known fact:

- (largest root of $b_f(s)$) = log canonical threshold $\text{lct}(f)$.

Similarly, give a bound in terms of invariants of a resolution of singularities for largest root of $\tilde{b}_f(s)$ (the “microlocal log canonical threshold”).
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(i) Vast generalization of the theory of variations of Hodge structure in the presence of singularities (of maps, of connections, of spaces...)

(ii) Special more manageable case of the theory of filtered $D$-modules, where tools from Hodge theory and birational geometry can be brought into play.
Filtered $D$-modules

- $X$ smooth complex variety; $D_X =$ sheaf of differential operators on $X$, with standard filtration given by:

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  \[ F_k D_X = \text{differential operators of order at most } k, \text{ for } k \geq 0. \]
- $D$-module $\mathcal{M}$ is an $\mathcal{O}_X$-module carrying an action of $D_X$; equivalently, have operation
  \[ \nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega^1_X \]
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- (Good) filtration on $\mathcal{M}$ is a $\mathbb{Z}$-indexed increasing filtration $F_\bullet \mathcal{M}$ such that
  \[
  F_k D_X \cdot F_\ell \mathcal{M} \subseteq F_{k+\ell} \mathcal{M}
  \]
  plus other (well, good...) properties, e.g. all $F_\ell \mathcal{M}$ coherent.
**Typical example:** Let $D$ be a hypersurface in $X$, and

$$\mathcal{O}_X(*D) := \bigcup_{k \geq 0} \mathcal{O}_X(kD)$$

be the quasi-coherent sheaf of functions with poles along $D$. Locally, if $D = (f = 0)$, this is the localization $\mathcal{O}_X[f^{-1}]$; natural action of $D_X$ by quotient rule.
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- Pole order filtration: $P_k \mathcal{O}_X(*D) = \mathcal{O}_X((k + 1)D)$, for $k \geq 0$.

Not the right filtration (unless $D$ is smooth)! We want something with Hodge theoretic meaning, which also captures the singularities of $D$. 
De Rham complex. \((\mathcal{M}, F) = \text{filtered } D\text{-module.}\)

\[
\text{DR}(\mathcal{M}) := [0 \to \mathcal{M} \to \mathcal{M} \otimes \Omega^1_X \to \cdots \to \mathcal{M} \otimes \Omega^n_X \to 0]
\]

Placed in degrees \(-n, \ldots, 0\); if

\[
\text{gr}^F_k \mathcal{M} := F_k \mathcal{M} / F_{k-1} \mathcal{M},
\]

then this induces associated graded complexes

\[
\text{gr}^F_k \text{DR}(\mathcal{M}) = [0 \to \text{gr}^F_k \mathcal{M} \to \cdots \to \text{gr}^F_{k+n} \mathcal{M} \otimes \Omega^n_X \to 0]
\]

which are objects in \(D_{\text{coh}}^b(X)\) (i.e. maps become \(O_X\)-linear).
Hodge modules

Here we consider a Hodge module as being a (regular, holonomic) $D_X$-module $\mathcal{M}$ with good filtration $F_\bullet \mathcal{M}$, satisfying a number of special properties; e.g.:

• When $X = \text{pt}$, it is the complexification of a Hodge structure.

• An inductive condition on the dimension of the support of $\mathcal{M}$ is satisfied: for each $f \in \mathcal{O}_X$ (locally), the graded quotients of the monodromy weight filtration on the nearby and vanishing cycles with respect to $f$ (defined at the level of $D$-modules using the Kashiwara-Malgrange $V$-filtration) are again Hodge modules.

• Have decomposition by strict support.

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Vanishing and positivity package

Key feature: Hodge modules satisfy vanishing and positivity properties extending those of (relative) canonical bundles in birational geometry. Starting point:
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Theorem (Saito Vanishing)

If $X$ is a smooth projective variety, $(\mathcal{M}, F) = \text{Hodge module and } L = \text{ample line bundle on } X$, then

$$H^i(X, \text{gr}_k^F \text{DR}(\mathcal{M}) \otimes L) = 0 \quad \text{for all } i > 0.$$
(1) Variations of Hodge structure: Any VHS \((\mathcal{V}, \nabla), F^\bullet \mathcal{V}\) defines a (pure) Hodge module.

- In particular, the trivial VHS on \(X\) of dimension \(n\) defines the trivial Hodge module \(Q^H_X[n]\). The \(D\)-module is \(\mathcal{O}_X\) with usual differentiation, and with filtration

\[ F_k \mathcal{O}_X = 0, \quad k < 0 \quad \text{and} \quad F_k \mathcal{O}_X = \mathcal{O}_X, \quad k \geq 0. \]

Have

\[ \text{gr}^F_{-k} \text{DR}(\mathcal{O}_X) = \Omega^k_X[n - k], \quad \text{for all} \quad k, \]

so Saito’s theorem specializes to Kodaira-Nakano vanishing.
(2) Localization: $\mathcal{O}_X(\ast D)$, for $D \subset X$ a hypersurface, underlies the (mixed) Hodge module $j_* \mathcal{Q}^H_U[n]$, where $U = X \setminus D$ and $j: U \hookrightarrow X$. Filtration hard to describe, but know

$$F_0 \mathcal{O}_X(\ast D) = F_{\text{lowest}} \mathcal{O}_X(\ast D) \simeq \mathcal{O}_X(D) \otimes \mathcal{I}((1 - \epsilon)D),$$

the multiplier ideal of the $\mathbb{Q}$-divisor $(1 - \epsilon)D$. More later.

• In particular

$$\text{gr}^F_0 \text{DR}(\mathcal{O}_X(\ast D)) \simeq \omega_X(D) \otimes \mathcal{I}((1 - \epsilon)D),$$

so we recover (a special case of) Nadel vanishing.
(3) **Direct images:** $f : X \rightarrow Y$ projective morphism; Saito *decomposition theorem* (generalizing BBD):

$$f_+(\mathcal{O}_X, F) \simeq \bigoplus (\mathcal{M}_i, F)[-i]$$

in the derived category of filtered $D$-modules, and each $(\mathcal{M}_i, F)$ is a (pure) Hodge module on $Y$. Again, filtration hard to describe, but can at least show

$$\text{gr}^F_{\text{lowest}} \text{DR}(\mathcal{M}_i) \simeq R^i f_*\omega_X,$$

so we recover **Kollár vanishing**.
Vanishing and positivity package

• Also have a natural extension of semi (and weak) positivity results of Griffiths, Fujita, Kawamata, Viehweg, Zuo,...

**Theorem (P.–Wu)**

\((\mathcal{M}, F) = \text{polarizable pure Hodge module with strict support } X\). For each \(p\) the dual \(K_p^\vee\) of the kernel of the generalized Kodaira-Spencer map

\[
\theta_p : \text{gr}_p^F \mathcal{M} \to \text{gr}_{p+1}^F \mathcal{M} \otimes \Omega^1_X
\]

is weakly positive.
Two main constructions based on the framework above are used towards the applications described at the beginning:
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1. Hodge ideals associated to $\mathbb{Q}$-divisors

2. Hodge modules and graded systems arising from families of varieties
Hodge ideals (work with Mustață)

- Back to $\mathcal{O}_X(*D)$, for $D \subset X$ a reduced hypersurface. Hodge filtration coming from mixed Hodge module structure satisfies

$$F_k\mathcal{O}_X(*D) \subseteq P_k\mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D), \quad \text{for all } k \geq 0.$$ 

- Defines

$$F_k\mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k(D).$$

with $I_k(D)$ = the $k$-th Hodge ideal of $D$. 

Theorem

(i) (also Saito) $I_0(D)$ is the multiplier ideal $\mathcal{I}((1-\epsilon)D)$; so

\[ I_0(D) = \mathcal{O}_X \iff (X, D) \text{ is log canonical}. \]

(ii) $I_k(D)$ determine Deligne’s Hodge filtration on the singular cohomology $H^\bullet(U, \mathbb{C})$, where $U = X \setminus D$.

(iii) $D$ is smooth if and only if $I_k(D) = \mathcal{O}_X$ (i.e. $F_k = P_k$) for all $k$.

(iv) If $I_k(D) = \mathcal{O}_X$ for some $k \geq 1$, then $D$ is normal with rational singularities.
Theorem (Cont’d)

(v) There are non-triviality criteria for $I_k(D)$ at a point $x \in D$ in terms of the multiplicity of $D$ at $x$.

(vi) On smooth projective varieties, $I_k(D)$ satisfy an analogue of Nadel Vanishing for multiplier ideals.

(vii) $I_k(D)$ satisfy restriction (inversion of adjunction), subadditivity and semicontinuity theorems.
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Recent development: Similar (but substantially more complicated) picture for Hodge ideals associated to $\mathbb{Q}$-divisors; survey in the proceedings paper. Important for applications, e.g. the bound for the largest root of the reduced Bernstein-Sato polynomial mentioned at the beginning.
Example: application to theta divisors

Applications based on a combination of local criteria and vanishing. Quick example (special case of a conjecture on theta divisors):

\[ \text{Theorem} \]

Let \((A, \Theta)\) be an irreducible principally polarized abelian variety, \(\dim A = g\). If \(\Theta\) has isolated singularities, then

\[ \text{mult}_x \Theta \leq g + \frac{1}{2} \]

for all \(x \in \Theta\).

- In general, earlier result of Kollár gives \(\text{mult}_x \Theta \leq g\) for all \(x \in \Theta\). Equality in the Theorem is possible.

Idea: \(\text{mult}_x \Theta \geq g + 2\)

\[ \Rightarrow \text{I}_1(\Theta) \subseteq \text{m}_2 \] by the local non-triviality criterion; vanishing theorem for \(\text{I}_1(\Theta)\) then implies that the linear system \(\mid 2\Theta \mid\) separates tangent vectors, contradiction (Kummer map is ramified at the 2-torsion points of \(A\)).
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**Idea:** \(\text{mult}_x \Theta \geq \frac{g+2}{2} \implies I_1(\Theta) \subseteq m_x^2\) by the local non-triviality criterion; vanishing theorem for \(I_1(\Theta)\) then implies that the linear system \(|2\Theta|\) separates tangent vectors, **contradiction** (Kummer map is ramified at the 2-torsion points of \(A\)).
• $f : X \to Y$ be a morphism of smooth projective varieties, $L$ ample line bundle on $Y$.

$$B := \omega_{X/Y} \otimes f^* L^{-1}.$$ 

Assume: $(\ast)$ \exists generically finite map $\varphi : Z \to X$ such that

$$H^0(Z, \varphi^* B) \neq 0,$$

Let $h : Z \to Y$ as in

$$Z \xrightarrow{\varphi} X \xrightarrow{f} Y$$
Hodge modules from maps

- Hodge module construction for map $h$:

$$h_+(O_Z, F) \simeq \bigoplus (\mathcal{M}_i, F)[-i],$$

and we focus on $(\mathcal{M}_0, F)$ (extension of VHS given by middle cohomology of smooth fibers of $h$).
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- Inspired by beautiful construction of Viehweg-Zuo: there is a graded submodule $G = \bigoplus G_k \subseteq \bigoplus \text{gr}_k^F \mathcal{M}_0$

which remembers all the data:

- $G$ is “nice” away from $D = \text{singular locus of } f$.
- $G_0 \simeq L$.
- $G$ has the weak positivity properties of Hodge modules.
Condition (\(\ast\)) is satisfied, for example, in the following cases:

- \(X\) is of general type, and \(f: X \to Y\) is its Albanese map. We have:
  - \(\text{Supp}(G) \subseteq \text{image in } T^*Y = Y \times H^0(X, \Omega^1_X)\) of \(Z_f = \{ (x, \omega) | \omega(x) = 0 \}\) ⊂ \(X \times H^0(X, \Omega^1_X)\).
  - \(\text{Supp}(G) \twoheadrightarrow H^0(X, \Omega^1_X)\), using vanishing theorems for Hodge modules.
  - We deduce that every 1-form on \(X\) of general type has zeros, a conjecture of Hacon-Kovács and Luo-Zhang.
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- \( \text{Supp}(G) \to H^0(X, \Omega^1_X) \), using **vanishing theorems** for Hodge modules.
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(2) Fibers of $f$ are of general type, and vary maximally in “birational moduli” (using results of Viehweg, Kollár, Kawamata).
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Use $G_0 = L$ and the weak positivity of $K_p^\vee$, with

$$K_p = \text{Ker}(\theta_p : G_p \to G_{p+1} \otimes \Omega^1_Y),$$

plus criterion of Campana-Păun, to get that $(Y, D)$ is of log general type (i.e. $\omega_Y(D)$ is big).

This is an extension of a hyperbolicity conjecture due to Viehweg.
THANK YOU!