Kähler-Einstein metrics on Fano manifolds: variational and algebro-geometric aspects

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- Kähler forms in $[\omega] \in H^2(X)$ are parametrized by the space of **Kähler potentials**
$$\mathcal{H} = \left\{ \varphi \in C^\infty(X) \mid \omega + i\partial\bar{\partial}\varphi > 0 \right\}.$$

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  - X **canonically polarized** ($K_X > 0$),
  - X **Calabi-Yau** ($K_X \equiv 0$), or
  - X **Fano** ($K_X < 0$).

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Theorem (Chen-Donaldson-Sun '15, Tian '15, Berman '16, Datar-Szekelyhidi '16)

A Fano manifold X is KE iff it is K-(poly)stable.

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where μ is a volume form determined by ω and $c > 0$ normalizes the mass.

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- *If $\mathrm{Aut}^0(X) = \{1\}$, a KE potential exists iff D is 'proper'.*

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- minimizers of D in $\mathcal{E}^1$ $=$ KE potentials;
- the growth of D in $\mathcal{E}^1$ can be tested along geodesic rays.

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- *Solutions in* $\mathcal{E}^1$ *of* $\mathrm{MA}(\varphi) = c\, e^{-\varphi}\mu$ *are smooth, and hence KE potentials in* $\mathcal{H}$*.*

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Corollary (Tian '97, Phong-Song-Sturm-Weinkove '08, Darvas-Rubinstein '15)

If X *is Fano with* $\mathrm{Aut}^0(X) = \{1\}$, *then* X *KE* $\iff$ D *coercive.*

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Theorem (Chen-Tang '08, Phong-Sturm '10)

Every test configuration $(\mathcal{X}, \mathcal{L})$ determines a unique algebraic geodesic ray (φ_t) emanating from 0.

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- $\lim_{t\to\infty} t^{-1} L(\varphi_t) = L^{\mathrm{alg}}(\mathcal{X}, \mathcal{L})$;

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(v) *X is uniformly K-stable (Dervan, BHJ).*

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Muito obrigado pela vossa atenção!