Arithmetic models for Shimura varieties

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Modular curves

Complex upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, $\mathrm{SL}_2(\mathbb{R})$ acts: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$.

$p =$ a prime number. $\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, p|c \right\}$.

Modular curve $X_0(p) := \Gamma_0(p)\backslash\mathfrak{H}$.

$X_0(p) \xrightarrow{\sim}$ Isomorphism classes of elliptic curve isogenies $E_1 \xrightarrow{\phi} E_2$ of degree p over $\mathbb{C}$.

The curve $X_0(p)$ has a planar model over $\mathbb{Z}$ given by

$$\Phi_p(X, Y) = 0$$

where $\Phi_p(X, Y) \in \mathbb{Z}[X, Y]$ is the "modular polynomial".

Kronecker congruence: $\Phi_p(X, Y) \equiv (X - Y^p)(Y - X^p) \bmod p$

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Theorem (Shimura, Igusa, Deligne, Deligne-Rapoport):

Assume $p \geq 5$.

There is a model $\mathcal{X}_0(p)$ of $X_0(p)$ over $\mathbb{Z}$, (i.e. $\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \mathbb{C} = X_0(p)$), such that:

- If $\ell \neq p$ is another prime, then $\mathcal{X}_0(p) \bmod \ell$ is a *smooth* curve over the finite field $\mathbb{F}_\ell$.
- If $\ell = p$, then $\mathcal{X}_0(p) \bmod p$ is a *singular* curve over the finite field $\mathbb{F}_p$.

  There are two smooth irreducible components intersecting transversely.

  The curve $\mathcal{X}_0(p) \bmod p$ is (étale) locally isomorphic to $\mathrm{Spec}(\mathbb{F}_p[x, y]/(xy))$.

Proof: $\mathcal{X}_0(p)$ is defined as the moduli scheme over $\mathbb{Z}$ of elliptic isogenies of degree p. Deformation theory of such elliptic isogenies determines the local structure. □

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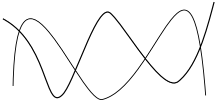

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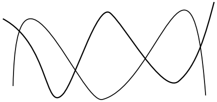

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  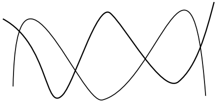

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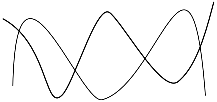

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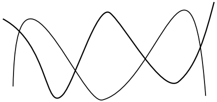

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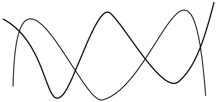

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Shimura varieties

Shimura data (after Deligne): Pairs (G, X) with

- $G =$ connective reductive group over $\mathbb{Q}$, and
- $X = \{h\}$ the $G(\mathbb{R})$-conjugacy class of a homomorphism of algebraic groups over $\mathbb{R}$

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}, \quad \text{(i.e. } h(\mathbb{R}) : \mathbb{C}^* \to G(\mathbb{R})\text{)}$$

satisfying certain conditions. These conditions, in particular, imply that

X is a disjoint union of symmetric Hermitian domains, and
$G(\mathbb{R})$ acts on X (transitively) via holomorphic automorphisms.

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Let $X^\vee := G/P_\mu$ be the homogeneous space of parabolic subgroups of G of type μ; it is a smooth projective variety defined over E.

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Theorem (Shimura, Deligne, Milne, Borovoi) The Shimura variety $\mathrm{Sh}_K(G, X)$ has a canonical model defined over $E(G, X)$.

Types of Shimura varieties (data): (PEL) $\subset$ (Hodge) $\subset$ (Abelian) $\subset$ (General).

Shimura varieties of PEL type $=$ moduli spaces of abelian varieties with Polarization, Endomorphisms and Level structures. (Examples: Modular curves, Siegel modular varieties.)

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Definition: The *reflex field* $E = E(G, X) \subset \mathbb{C}$ is the field of definition of the $G(\mathbb{C})$-conjugacy class $\{\mu\}$ of μ. It is a number field.

Let $X^\vee := G/P_\mu$ be the homogeneous space of parabolic subgroups of G of type μ; it is a smooth projective variety defined over E.

The complex variety $X^\vee_{\mathbb{C}} = X^\vee \otimes_E \mathbb{C}$ is the compact dual of the symmetric Hermitian domain X.

Theorem (Shimura, Deligne, Milne, Borovoi) The Shimura variety $\mathrm{Sh}_K(G, X)$ has a canonical model defined over $E(G, X)$.

Types of Shimura varieties (data): (PEL) $\subset$ (Hodge) $\subset$ (Abelian) $\subset$ (General).

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Goal: Describe "good" models of the varieties $\mathrm{Sh}_K(G, X)$ over the integers $\mathcal{O}_E$.

Work prime-by-prime:

Notations: Fix $(v) \subset \mathcal{O}_E$, with $(v)|(p)$, $E_v =$ completion of E at v, $\mathcal{O}_v =$ integers of E_v, $\pi_v =$ uniformizer of $\mathcal{O}_v$, $k_v =$ finite residue field of $\mathcal{O}_v$.

Assume $K = \prod_\ell K_\ell$ with $K_\ell \subset G(\mathbb{Q}_\ell)$ and $K^p = \prod_{\ell \neq p} K_\ell$ is "sufficiently small".

We want models $\mathcal{S}_K(G, X)$ over $\mathcal{O}_v$, i.e. schemes over $\mathcal{O}_v$ with

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- Description of $\overline{k}_v$–points,
- Description of local structure (singularities),
- Some sort of characterization among all possible models.

A motivating application is to Langlands' program relating automorphic representations and Galois representations. This correspondence is expressed by writing the Hasse-Weil zeta functions of Shimura varieties as a product of automorphic L-functions.

Several other applications, for example to: 1) Integral and p-adic theory of automorphic forms, 2) formulae ("of Gross-Zagier type") for special values of derivatives of L-functions.

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Conjecture (Langlands, Langlands-Rapoport): Suppose that G extends to a reductive group over $\mathbb{Z}_p$ and that $K_p = G(\mathbb{Z}_p)$.

a) There is a "canonical" *smooth* model $\mathcal{S}_K(G, X)$ over $\mathcal{O}_v$,

b) The set $\mathcal{S}_K(G, X)(\bar{k}_v)$ with its Hecke and $\mathrm{Gal}(\bar{k}_v/k_v)$-action has a "motivic" - "group-theoretic" description: There is a $\langle \mathrm{Frob}_v \rangle \times G(\mathbb{A}_f^p)$-equivariant bijection

$$\varprojlim_{K^p} \mathcal{S}_{K_pK^p}(G, X)(\bar{k}_v) \xrightarrow{\sim} \bigsqcup_{[\phi]} \varprojlim_{K^p} I_\phi(\mathbb{Q}) \backslash (X_p(\phi) \times (X^p(\phi)/K^p))$$

where: $[\phi]$ runs over isomorphism classes of "G-pseudo-motives" ϕ over $\bar{k}_v$,

$I_\phi = \mathrm{Aut}(\phi)$, an algebraic group over $\mathbb{Q}$,

$X^p(\phi) \simeq G(\mathbb{A}_f^p)$,

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In general, $\mathcal{G} = \mathcal{G}_x$ depends on the choice of a point x in the Bruhat-Tits building of the p-adic group $G(\mathbb{Q}_p)$. Recall, the group $G(\mathbb{Q}_p)$ acts on the building.

The group scheme $\mathcal{G}_x$ is constructed as a smooth connected affine group scheme over $\mathbb{Z}_p$ which

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Local model conjecture (Rapoport, P.): Fix (G, X) and a parahoric subgroup $K_p \subset G(\mathbb{Q}_p)$, with $K_p = \mathcal{G}_x(\mathbb{Z}_p)$.

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2) There should exist a "relatively proper system" of models $\mathcal{S}_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ over $\mathcal{O}_v$, each with the following property:

For each closed point $a \in \mathcal{S}_K(G,X)$, there is a closed point $b \in \mathrm{M}^{\mathrm{loc}}$, such that an étale local nbd of a in $\mathcal{S}_K(G,X)$ is isomorphic to an étale local nbd of b in $\mathrm{M}^{\mathrm{loc}}$.

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where q is smooth and $\mathcal{G}_x$-equivariant, π a torsor for a smooth quotient of $\mathcal{G}_x$.

Morevover, the generic fibers of these diagrams should agree with the diagrams obtained by the principal automorphic bundle.

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Theorem 1 (P.- Zhu, '13): Assume G splits over a tamely ramified extension of $\mathbb{Q}_p$. Then a local model $\mathrm{M}^{\mathrm{loc}}$ with the properties (a), (b), (c) exists.

(When p divides the order of $\pi_1(G_{\mathrm{der}})$, the reduced special fiber property (b) holds, after slightly modifying the definition of P.-Zhu; He-P.-Rapoport '18.)

Proof: For simplicity, assume the derived group G_{der} is simply connected.

1) Carefully extend the parahoric group scheme $\mathcal{G} = \mathcal{G}_x$ to an affine smooth group scheme $\underline{\mathcal{G}}$ over $Y = \mathrm{Spec}(\mathbb{Z}_p[u])$ using Bruhat-Tits theory. The extension is in the sense that

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2) Consider the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}} \to Y = \mathrm{Spec}(\mathbb{Z}_p[u])$:

By definition, $\mathrm{Gr}_{\underline{\mathcal{G}}}$ parametrizes isomorphism classes of triples $(\mathcal{E}, y, \beta)$:

$\mathcal{E}$ a $\underline{\mathcal{G}}$-torsor over Y, y a point of Y, β a section of $\mathcal{E}$ over $Y - y$.

The homogeneous space $X^{\vee}_{E_v}$ embeds in $\mathrm{Gr}_{\underline{\mathcal{G}}} \times_Y \mathrm{Spec}(E_v)$. (The base change is by $u \mapsto \pi_v$.)

Definition: The local model $\mathrm{M}^{\mathrm{loc}}$ is the Zariski closure of $X^{\vee}_{E_v}$ in $\mathrm{Gr}_{\underline{\mathcal{G}}} \times_Y \mathrm{Spec}(\mathcal{O}_v)$.

Property (a) follows from the definition but showing (b) and (c) is hard:

The proof of (b) and (c) can be reduced to showing the equality of two Hilbert polynomials. This equality is the "coherence conjecture" of P.-Rapoport. It is enough to prove the corresponding equality in the equal characteristic case. This was done by Zhu ('11), using methods from the theory of geometric Langlands correspondence. □

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The proof of (b) and (c) can be reduced to showing the equality of two Hilbert polynomials. This equality is the "coherence conjecture" of P.-Rapoport. It is enough to prove the corresponding equality in the equal characteristic case. This was done by Zhu ('11), using methods from the theory of geometric Langlands correspondence. □

2) Consider the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}} \to Y = \mathrm{Spec}(\mathbb{Z}_p[u])$:

By definition, $\mathrm{Gr}_{\underline{\mathcal{G}}}$ parametrizes isomorphism classes of triples $(\mathcal{E}, y, \beta)$:

$\mathcal{E}$ a $\underline{\mathcal{G}}$-torsor over Y, y a point of Y, β a section of $\mathcal{E}$ over $Y - y$.

The homogeneous space $X^{\vee}_{E_v}$ embeds in $\mathrm{Gr}_{\underline{\mathcal{G}}} \times_Y \mathrm{Spec}(E_v)$. (The base change is by $u \mapsto \pi_v$.)

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Remarks: 1) Levin has an extension of the construction to most wildly ramified groups.

2) Scholze gives a characterization of local models as the only flat projective $\mathcal{G}$-schemes satisfying (a), (b), and an extra property, expressed in terms of their associated diamonds: These should embed in a diamond Beilinson-Drinfeld affine Grassmannian in a way that resembles the embedding in the definition above. This property, together with the resulting unique characterization, was recently proven for $\mathrm{M}^{\mathrm{loc}}$, in many cases (He-P.-Rapoport and Lourenço).

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Theorem 2 (Kisin-P. '15): Assume that (G, X) is of abelian type, G splits over a tamely ramified extension of $\mathbb{Q}_p$ and p is odd. Then the local model conjecture is true, when we take the local model $\mathrm{M}^{\mathrm{loc}}$ of P.-Zhu in (1).

Remarks: 1) In fact, the more precise conjecture with (2) replaced by the diagram property (2+) also holds, under some additional technical assumptions.

2) Assume the Shimura variety is of PEL type (i.e. it is a moduli space of abelian varieties with polarization, endomorphisms and level structures). Then integral models, local models and a diagram as in (2+) were given by Rapoport and Zink ('94). This uses the moduli interpretation and generalizes work of de Jong, and Deligne-P. ('93). The Rapoport-Zink local models were given explicitly via certain linked Grassmannians for lattice chains. It was not clear that the difficult properties (b) and (c) were true for them, or for their flat closures in general. When G is tamely ramified, their flat closures agree with the local models of P.-Zhu.

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Proof: First reduce to the case (G, X) is of Hodge type. This reduction uses an integral version of Deligne's theory of the connected component of Shimura varieties. For simplicity, assume $E = \mathbb{Q}$. For (G, X) of Hodge type, there is

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given by a symplectic representation of G. Then $\mathrm{Sh}_K(G, X)$ is a moduli space of abelian varieties of dimension g equipped with Hodge cycles.

The Siegel Shimura variety S_g has a (canonical) integral model $\mathcal{S}_g$ given as moduli of p.p. abelian varieties of dimension g.

We show that the symplectic representation giving ι can be chosen so that:

a) The parahoric group scheme $\mathcal{G} = \mathcal{G}_x$ embeds "nicely" in a reductive group scheme which extends the symplectic group.

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Definition: The model $\mathcal{S}_K(G, X)$ is given as the normalization of $\mathrm{Sh}_K(G, X) \hookrightarrow S_g$ in $\mathcal{S}_g$.

It remains to show that, given (a) and (b) above, $\mathcal{S}_K(G, X)$ and $\mathrm{M}^{\mathrm{loc}}$ have the same étale local structure.

This is done by studying the crystalline avatars of the universal Hodge cycles on the universal abelian scheme. We determine how these control the étale local structure of $\mathcal{S}_K(G, X)$ via deformation theory.

We first use:

- Integral p-adic Hodge theory (Breuil-Kisin modules).
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Definition: The model $\mathcal{S}_K(G,X)$ is given as the normalization of $\mathrm{Sh}_K(G,X) \hookrightarrow S_g$ in $\mathcal{S}_g$.

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We deduce that the crystalline realizations of the universal Hodge cycles can be put "in standard position" over each dvr-valued point a of $\mathcal{S}_K(G, X)$.

Then, the Hodge filtration of the abelian scheme given by a defines a corresponding point $b = s(a)$ of $\mathrm{M}^{\mathrm{loc}}$. It is enough to show how to construct deformations of a in $\mathcal{S}_K(G, X)$ from deformations of b in $\mathrm{M}^{\mathrm{loc}}$.

By Grothendieck-Messing theory, the deRham cohomology $\mathrm{H}^1_{\mathrm{DR}}(A)$ of an abelian scheme A is a crystal and deformations of A are given, roughly speaking, by deformations of the Hodge filtration $F^0(A) \subset \mathrm{H}^1_{\mathrm{DR}}(A)$. However, here we need to be careful and show that if the Hodge filtration deforms inside $\mathrm{M}^{\mathrm{loc}}$, then the corresponding abelian scheme stays in $\mathcal{S}_K(G, X)$.

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We use that points of $\mathrm{M}^{\mathrm{loc}}$ give values of a functor that resembles a Witt vector affine Grassmannian for $\mathcal{G}$. These give Zink displays "with $\mathcal{G}$-structure" and, so, corresponding suitable p-divisible groups with corresponding points that "stay" in $\mathcal{S}_K(G, X)$. □

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Examples:

1) $G = \mathrm{GL}_2, \quad X = \mathbb{C} - \mathbb{R} \simeq \mathfrak{H} \sqcup \mathfrak{H}, \quad X^\vee = \mathbb{P}^1,$

$$K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, p|c \right\} \subset \mathrm{GL}_2(\mathbb{Z}_p),$$

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$$\mathrm{Spec}(\mathbb{Z}_p[x, y, z]/(x(x + yz) - p)) \subset \mathrm{M}^{\mathrm{loc}}$$

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