The Subconvexity Problem for L-functions

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L-functions

Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

- Analytic continuation (simple pole at $s = 1$)

$$\zeta : \mathbb{C} - \{1\} \to \mathbb{C}$$

- Functional equation

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \xi(1 - s)$$

Γ-factor $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$

Dirichlet L-function $L(s,\chi)$

$$\chi \in \mathrm{Hom}\left((\mathbb{Z}/M\mathbb{Z})^\star, \mathbb{C}^\star\right)$$

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Note $\chi(p) = 0$ for $p|M$

- Conv. abs. for $\sigma > 1$, and extends to an entire function
- For $a = \chi(-1)$, $\omega_\chi = i^{-a} G(\chi)/\sqrt{M}$

$$\Lambda(s,\chi) := M^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+a) L(s,\chi) = \omega_\chi \Lambda(1-s,\bar{\chi})$$

- $L_p(s,\chi)^{-1} = 1 - \chi(p)p^{-s}$ degree one in p^{-s}

Discriminant modular form

Set $q = e(z) = e^{2\pi i z}$, $z \in \mathbb{H}$

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

One has

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12} \Delta(z), \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Δ is a modular form (cuspform) of weight 12, level 1

Ramanujan Conjectures:

- $\tau(mn) = \tau(m)\tau(n)$ for $(m, n) = 1$
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$
- $|\tau(p)| < 2p^{\frac{11}{2}}$

Hecke L-function $L(s,\Delta)$

Set $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau_0(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\tau_0(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}$$

- Conv. abs. for $\sigma > 1$, and extends to an entire function
- Functional equation

$$\Lambda(s,\Delta) := (1/2\pi)^s \Gamma(s + \tfrac{11}{2}) L(s,\Delta) = \Lambda(1-s,\Delta)$$

- $L_p(s,\Delta)^{-1} = 1 - \tau_0(p)p^{-s} + p^{-2s}$ degree 2 in p^{-s}

Convolution

$$L(s,f_i)=\prod_p\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1}\left(1-\frac{\beta_i(p)}{p^s}\right)^{-1},\ i=1,2$$

- (**Twists**) χ mod M primitive $L(s,f_1\otimes\chi)=\prod_p L_p(s,f_1\otimes\chi)$

$$L_p(s,f_1\otimes\chi)^{-1}=\left(1-\frac{\alpha_1(p)\chi(p)}{p^s}\right)\left(1-\frac{\beta_1(p)\chi(p)}{p^s}\right)$$

- (**Rankin-Selberg**) $L(s,f_1\otimes f_2)$

$$\left(1-\frac{\alpha_1(p)\alpha_2(p)}{p^s}\right)\left(1-\frac{\beta_1(p)\alpha_2(p)}{p^s}\right)\left(1-\frac{\alpha_1(p)\beta_2(p)}{p^s}\right)\left(1-\frac{\beta_1(p)\beta_2(p)}{p^s}\right)$$

- (**Symmetric Square**) $L(s,\mathrm{Sym}^2 f_1)$

$$\left(1-\frac{\alpha_1(p)^2}{p^s}\right)\left(1-\frac{\alpha_1(p)\beta_1(p)}{p^s}\right)\left(1-\frac{\beta_1(p)^2}{p^s}\right)$$

Automorphic L-function $L(s,\pi)$

Let π be an automorphic form for $GL_d(\mathbb{A}_{\mathbb{Q}})$. Then $\pi = \otimes_p \pi_p$, and we define

$$L(s,\pi) = \prod_{p \text{ prime}} L_p(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

with

$$L_p(s,\pi) = L(s,\pi_p) = \prod_{i=1}^{d} (1 - \alpha_i(p)p^{-s})^{-1}.$$

The local factor at infinity is defined as

$$L(s,\pi_\infty) = \prod_{i=1}^{d} \Gamma_{\mathbb{R}}(s-\mu_i)$$

- Conv. abs. for $\sigma > \sigma_0$ and extends to an entire function
- Functional equation

$$\Lambda(s,\pi) := q_\pi^{s/2} L(s,\pi_\infty) L(s,\pi) = \omega_\pi \Lambda(1-s,\tilde{\pi})$$

for some $q_\pi \in \mathbb{N}$, $|\omega_\pi| = 1$ (Godement-Jacquet)

- $L(s, \mathrm{Sym}^2 f) \rightsquigarrow GL(3)$ (Gelbart-Jacquet)
- $L(s, f \otimes g) \rightsquigarrow GL(4)$ (Ramakrishnan)

Main Conjectures

Grand Riemann Hypothesis

All non-trivial zeros of $L(s, \pi)$ are on the central line

$$\frac{1}{2} + it$$

Generalised Lindelöf Hypothesis

Analytic conductor:

$$C(\pi, t) = q_\pi \prod_{i=1}^{d} (1 + |\mu_i + it|) \to \begin{cases} q_\pi & \text{level} \\ t^d & t \\ \Lambda_\pi = \prod \mu_i & \text{spectral} \end{cases}$$

GRH $\implies$ For any $\varepsilon > 0$ one has

$$|L(\tfrac{1}{2} + it, \pi)| \leq c(\varepsilon) C(\pi, t)^\varepsilon$$

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GRH $\implies$ For any $\varepsilon > 0$ one has

$$L(\tfrac{1}{2} + it, \pi) \ll C(\pi, t)^\varepsilon.$$

The Subconvexity Problem

Convexity bound

Functional equation + Phragmen-Lindelöf principle $\implies$

$$L(\tfrac{1}{2}+it,\pi) \ll C(\pi,t)^{\frac{1}{4}+\varepsilon}.$$

The Subconvexity Problem

To establish a bound of the form (with $\delta > 0$)

$$L(\tfrac{1}{2}+it,\pi) \ll C(\pi,t)^{\frac{1}{4}-\delta}.$$

We can also formulate the problem w.r.t. each parameters separately (level aspect q_π, t-aspect, depth aspect μ_i)

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The Subconvexity Problem

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$$L(\tfrac{1}{2}+it,\pi) \ll q_\pi^{\frac{1}{4}-\delta}.$$

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The Subconvexity Problem

To establish a bound of the form (with $\delta > 0$)

$$L(\tfrac{1}{2} + it, \pi) \ll t^{\frac{d}{4}-\delta}.$$

We can also formulate the problem w.r.t. each parameters separately (t-aspect, level aspect q_π, depth aspect μ_i)

The Subconvexity Problem

Convexity bound

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The Subconvexity Problem

To establish a bound of the form (with $\delta > 0$)

$$L(\tfrac{1}{2}+it,\pi) \ll \Lambda_\pi^{\frac{1}{4}-\delta}.$$

We can also formulate the problem w.r.t. each parameters separately (t-aspect, level aspect q_π, depth aspect μ_i)

Applications

Application - 1 : Hilbert's 11th Problem

Asks which integers are represented by a given n-ary quadratic form Q over a number field F. Siegel's Mass formula is the key in most cases, except when Q is $+$ve definite ternary form. The answer follows from subconvexity of $L(\frac{1}{2}, \pi \otimes \chi)$ where π is fixed $GL_2(\mathbb{A}_F)$ form. (Duke–Schulze-Pillot (1990) for $F = \mathbb{Q}$, Cogdell–Piatetski-Shapiro–Sarnak for F totally real.)

Application - 2 : Linnik's Problem

Equidistribution of points on the sphere/ellipsoid

Application - 3 : Equidistribution of Heegner Points

Follows from subconvexity of the Rankin-Selberg $L(\frac{1}{2} + it, f \otimes g)$. (Michel et al (2002-2006))

Application - 4 : Quantum Unique Ergodicity

Projection map $P_q : X_0(q) \to X_0(1)$. Let $f \in S_2(q)$, then

$$\mu_f(z) = \frac{|f(z)|^2}{\langle f, f \rangle} \mathrm{d}x\mathrm{d}y$$

is a prob measure on $X_0(q)$. Let $\mu_f^\star$ be its pushforward to $X_0(1)$.
QUE conjecture:

$$\mu_f^\star \xrightarrow{W} \frac{1}{\mathrm{vol}(X_0(1))} \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Follows from subconvexity of $L(\frac{1}{2}, \mathrm{Sym}^2 f)$ & $L(\frac{1}{2}, \mathrm{Sym}^2 f \otimes g)$.

Degree one L-functions

- How does one estimate the size on the central line?
- Functional equation $\implies$

$$\zeta(\tfrac{1}{2}+it) \ll t^{\varepsilon} \sup_{t^{\theta}<N\ll t^{\frac{1}{2}+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} + t^{\theta/2}$$

with

$$S(N) = \sum_{N\leq n<2N} n^{it} = \sum_{N\leq n<2N} e(t\log n/2\pi)$$

- Exponential sums: Weyl, van der Corput (exponent pairs), Vinogradov, Bombieri-Iwaniec
- Weyl-Hardy-Littlewood (1920): $\zeta(1/2+it) \ll t^{1/6+\varepsilon}$
  ... after 98 years we have $1/6 - 1/84 + \varepsilon$ (Bourgain).

- For $L(s,\chi)$: Functional equation $\implies$

$$L(\tfrac{1}{2},\chi) \rightsquigarrow S(N) = \sum_{N \le n < 2N} \chi(n)$$

with $M^{\theta} < N \ll M^{\frac{1}{2}+\varepsilon}$ (short character sums)

- Complete sum (Burgess' ingenious method)

$$S(N) \rightsquigarrow \sum_{x \bmod M} \chi((x+a)(x+b)\overline{(x+c)(x+d)})$$

where we have Weil bound (RH for curves)

- Burgess (1962): χ mod M (primitive)

$$L(\tfrac{1}{2},\chi) \ll M^{\frac{3}{16}+\varepsilon}.$$

Moment Method

- Start with the inequality

$$\zeta(\tfrac{1}{2}+it)^2 \ll \int_{t-G}^{t+G} |\zeta(\tfrac{1}{2}+i\tau)|^2 \mathrm{d}\tau = \int_0^{t+G} - \int_0^{t-G}$$

- Define

$$M_2(T) := \int_0^T |\zeta(\tfrac{1}{2}+i\tau)|^2 \mathrm{d}\tau$$

- $M_2(T) = TP_1(\log T) + E(T)$ with $E(T) \rightsquigarrow$ exp. sum
- The leading term of $(M_2(t+G) - M_2(t-G))$ is G
- For (k,ℓ) exponent pair $E(T) \ll T^{\frac{k+\ell}{2+2k}+\varepsilon}$
  Easy exponent pair $(\frac{1}{2},\frac{1}{2})$ yields $E(T) \ll T^{\frac{1}{3}+\varepsilon} \implies$ Weyl bound

Degree two L-functions

- Good (1982): f Hecke-cusp form of full level

$$M_2(T) := \int_0^T |L(\tfrac{1}{2} + i\tau, f)|^2 \mathrm{d}\tau = TQ_1(\log T) + E(T)$$

  with

$$E(T) \ll T^{\frac{2}{3}+\varepsilon}.$$

- It follows that

$$L(\tfrac{1}{2} + it, f) \ll t^{\frac{1}{3}+\varepsilon}$$

- $E(T) \rightsquigarrow \sum_n \lambda_f(n)\lambda_f(n+h)(n+h)^{-s}$ which is treated using spectral theory of the Laplacian
- Jutila: Treatment based on the Voronoi summation formula also yielding Weyl exponent

Moment Method

Subconvexity for $L(\frac{1}{2}, \pi_0)$: Get a 'family' $\mathcal{F}$ with $\pi_0 \in \mathcal{F}$. Estimate

$$M(\mathcal{F}) := \sum_{\pi \in \mathcal{F}} |L(\tfrac{1}{2}, \pi)|^2.$$

Positivity $\implies L(\frac{1}{2}, \pi_0) \ll M(\mathcal{F})^{\frac{1}{2}}$. At best we have $M(\mathcal{F}) \asymp |\mathcal{F}|$.
So for subconvexity we need $|\mathcal{F}| \ll q_{\pi_0}^{\frac{1}{2}-\delta}$.
On the other hand we need $\mathcal{F}$ big enough for orthogonality

$$\sum_{\pi \in \mathcal{F}} \lambda_\pi(n)\overline{\lambda_\pi(m)} = \begin{cases} |\mathcal{F}| & \text{if } n = m \\ \text{`small \& explicit'} & \text{otherwise.} \end{cases}$$

Amplification technique

'Often' it happens that $|\mathcal{F}| = q_{\pi_0}^{\frac{1}{2}}$. Then consider the amplified moment

$$M(\mathcal{F}) := \sum_{\pi \in \mathcal{F}} |L(\tfrac{1}{2}, \pi)|^2 |\mathcal{A}_\pi|^2.$$

The weights are chosen so that $\mathcal{A}_{\pi_0}$ is larger than the average size of $\mathcal{A}_\pi$.

- For t aspect or depth aspect: one amplifies by shortening the 'length of sum'
- For level aspect: one needs to use some sort of arithmetic

Example:

f : Hecke cuspform of level one, χ_0 mod M primitive character.
Then $f \otimes \chi_0 \in S_k(M^2, \chi_0^2)$ is a Hecke cuspform.
Convexity bound: $L(\frac{1}{2}, f \otimes \chi_0) \ll M^{\frac{1}{2}+\varepsilon}$.
Natural 'Family' $\mathcal{F} = \{L(\frac{1}{2}, f \otimes \chi) : \chi \text{ mod } M\}$, with $|\mathcal{F}| = M$.
Duke-Friedlander-Iwaniec:

$$M(\mathcal{F}) := \sum_{\chi \text{ mod } M} |L(\tfrac{1}{2}, f \otimes \chi)|^2 |\mathcal{A}_\chi|^2,$$

with

$$\mathcal{A}_\chi = \sum_{\ell \sim L} \bar{\chi_0}(\ell)\chi(\ell).$$

They established $L(\frac{1}{2}, f \otimes \chi_0) \ll M^{\frac{1}{2}-\frac{1}{22}+\varepsilon}$.

Results obtained via amplification method

Authors	Case
Friedlander-Iwaniec (1992)	$L(\frac{1}{2}, \chi)$
Iwaniec (1992) (conditional)	spectral aspect $L(\frac{1}{2}, f)$
DFI (1993), Blomer-Harcos (2008)	$L(\frac{1}{2}, f \otimes \chi)$
DFI (1994-2001)	level aspect $L(\frac{1}{2}, f)$
Sarnak (2001), Lau-Liu-Ye (2006)	weight aspect $L(\frac{1}{2}, f \otimes g)$
Kowalski-Michel-Vanderkam (2002) Michel(2004), Harcos-Michel(2006)	level aspect $L(\frac{1}{2}, f \otimes g)$

Period Approach

Michel-Venkatesh (2010) established:

$$L(\tfrac{1}{2}, \pi_1) \ll C(\pi_1)^{\frac{1}{4}-\delta}$$

and

$$L(\tfrac{1}{2}, \pi_1 \otimes \pi_2) \ll_{\pi_2} C(\pi_1, \pi_2)^{\frac{1}{4}-\delta}$$

for π_1 and π_2 automorphic forms for $GL_2(\mathbb{A}_F)$ over number field F.

- δ can be computed.
- Amplification plays a role.

Degree three L-functions

Conrey-Iwaniec (2000)

For χ mod M quadratic

$$L(\tfrac{1}{2}, \chi) \ll M^{\frac{1}{6}+\varepsilon}.$$

This follows from estimation of cubic moment

$$\sum_{|t_j|<R} L(\tfrac{1}{2}, f_j \otimes \chi)^3 + \int_{-R}^{R} |L(\frac{1}{2} + ir, \chi)|^6 \mathrm{d}r.$$

Crucial input $L(\frac{1}{2}, f \otimes \chi) \geq 0$.

- Li (2011) extends this to establish $L(\frac{1}{2} + it, \mathrm{Sym}^2 f) \ll t^{\frac{11}{16}+\varepsilon}$
- Blomer (2012): $L(\frac{1}{2}, \mathrm{Sym}^2 f \otimes \chi) \ll M^{\frac{5}{8}+\varepsilon}$, for χ quadratic.

Theorem 1. (M)

Let π be a $SL_3(\mathbb{Z})$ Hecke-Maass cusp form. Then

$$L(\tfrac{1}{2}+it,\pi) \ll t^{\frac{11}{16}+\varepsilon}.$$

Theorem 2. (M)

π a $SL_3(\mathbb{Z})$ Hecke-Maass cusp form, χ mod M primitive. Then

$$L(\tfrac{1}{2},\pi\otimes\chi) \ll M^{\frac{3}{4}-\delta}.$$

New Approach

- Functional equation $\implies$

$$L(\tfrac{1}{2}+it,\pi) \rightsquigarrow S(N) = \sum_{n\sim N} \lambda_\pi(n) n^{it}$$

with $N \asymp t^{3/2} = \sqrt{\text{conductor}}$

- **Separation of oscillation:**

$$S(N) = \sum\sum_{n,m\sim N} \lambda_\pi(n) m^{it} \delta(n,m)$$

- **Circle Method:** To detect event $n = m$

$$\delta(n,m) = 2\mathrm{Re}\int_0^1 \sum\sum_{\substack{1\le c\le C<a\le C+c\\(a,c)=1}} \frac{1}{ac} e\left(\frac{\bar{a}(n-m)}{c}\right) e\left(\frac{x(n-m)}{ac}\right)$$

- The magnitude of analytic oscillation $D = N/C^2$

- If $C = \sqrt{N}$ (minimize CD) then essentially

$$S(N) \rightsquigarrow \sum_{\substack{1 \leq c \leq C < a \leq C+c \\ (a,c)=1}} \sum \frac{1}{ac} \sum_{n \sim N} \lambda_\pi(n) e\left(\frac{\bar{a}n}{c}\right) \sum_{m \sim N} m^{it} e\left(-\frac{\bar{a}m}{c}\right)$$

-

$$\langle \{\lambda_\pi(n)\}, \{n^{it}\} \rangle \rightsquigarrow \sum_{\mathbf{v} \in \mathcal{B}} \langle \{\lambda_\pi(n)\}, \{\mathbf{v}_n\} \rangle \langle \{\mathbf{v}_n\}, \{n^{it}\} \rangle$$

- **Summation Formula:** Apply Poisson summation and $GL(3)$ Voronoi summation formula
- **Cauchy Inequality + Poisson:** Apply Cauchy to get rid of Fourier coefficients/break involution, followed by Poisson

Conductor lowering

Direct application of circle method does not work. One needs to synchronize the harmonics to lower the conductor. This is a crucial ingredient

Instead of picking $C = \sqrt{\text{size of eqn}} = N^{1/2}$, we pick $C = N^{1/2-\theta}$ as the analytic oscillation m^{it} is already present.

'Triple Kloosterman refinement'

We obtain cancellation in the sum over a, c (on average) and the integral over x.

$GL(2)$ δ-method

Kloosterman's formula, δ method of Duke-Friedlander-Iwaniec & Heath-Brown, involve only trigonometric func. ($GL(1)$ harmonics)

Petersson Formula

$$\sum_{f \in H_k(q,\psi)} w_f^{-1} \lambda_f(n) \overline{\lambda_f(m)} = \delta(n,m) + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S_\psi(m,n;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{cq}\right)$$

Taking average over ψ mod q, $q \in \mathcal{Q}$, $k \sim K$:

$$\delta(n,m) = \{GL(1) \text{ harmonics}\} + \{GL(2) \text{ Fourier coeff.}\}$$

$$\{GL(1)\} \rightsquigarrow \sum_{q \sim Q} \sum_{c < C = N/QK^2} \sum_{a \bmod c} e\left(\frac{(a-1)m\bar{q} + (\bar{a}-1)n\bar{q}}{c}\right) \times e\left(\frac{(\sqrt{m}-\sqrt{n})^2}{cq}\right)$$

- The q sum takes the place of x integral
- Subconvexity of $L(1/2, \pi \otimes \chi)$ follows from this form of δ-method
- Again a conductor lowering mechanism is required. This is achieved by taking $f \in H_k(qM, \psi)$ in the circle method, where χ mod M

Further Results via $GL(2)$ δ-method

Theorem 3. (M.)

For f a Hecke cuspform of level M

$$L(\tfrac{1}{2}, \mathrm{Sym}^2 f) \ll M^{\frac{1}{2}-\delta}$$

Theorem 4. (M.& Nelson)

Arithmetic QUE

Theorem 5. (M.)

Burgess bound for $GL(1)$ and $GL(2)$:

$$L(\tfrac{1}{2},\chi) \ll M^{\frac{3}{16}+\varepsilon} \quad \& \quad L(\tfrac{1}{2}, f\otimes\chi) \ll M^{\frac{3}{8}+\varepsilon}$$

Theorem 6. (M.)

Sub-Weyl for $GL(2)$:

$$L(\tfrac{1}{2}+it, f) \ll t^{\frac{1}{3}-\delta}$$

Remarks

1) 'Conductor drop' is a bad news for amplification technique but is a goods new for circle method.
2) Holowinsky-Nelson simplification bypasses the $GL(2)$ δ-method in some instances.

What Lies Ahead?

Correlation

This method can be applied to any sums of the form

$$\sum_{n\sim N} A(n)B(n)$$

where at least one of $\{A(n)\}$, $\{B(n)\}$ is 'automorphic'.

- $\sum_{n\leq x} d(n)e(a\sqrt{nx^3})$: error term in the second moment of $\zeta(s)$ & the Dirichlet divisor problem.
- $\sum_{n\leq x} \lambda_f(n)e(a\sqrt{nx^3})$: Voronoi barrier $\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{3}+\varepsilon}$
- $\sum_{n\leq x} \lambda_\pi(n)e(an^{2/3}) \ll x^{\frac{5}{6}-\varepsilon}$: Hardy's thm for $L(s, \mathrm{Sym}^2 f)$

Thank You!