p-Adic variation of automorphic sheaves

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Langlands-Clozel-Fontaine-Mazur conjecture

Let K be a number field and $n \in \mathbb{Z}_{\geq 1}$. Let $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$.

Conjecture

There is a bijection preserving L-functions between :

$$\{\textbf{Cuspidal algebraic automorphic forms on } \mathrm{GL}_n/K\}$$

and

$$\{\textbf{Irreducible geometric representations } Gal(\bar{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)\ \}$$

$\{$**Cuspidal algebraic automorphic forms on** $\mathrm{GL}_n/K\} \rightsquigarrow$

smooth functions $\mathrm{GL}_n(\mathbb{A}_K)/\mathbb{GL}_n(K) \to \mathbb{C}$.

$\{$**Irreducible geometric representations** $Gal(\bar{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)\ \} \rightsquigarrow$

p-adic étale cohomology of algebraic varieties over K.

This conjecture would imply the **Hasse-Weil conjecture on Zeta functions of proper smooth algebraic varieties** X over K (analytic continuation and functional equation).

$$\begin{aligned} Z(X)(s) &= \prod_{x\in|\mathfrak{X}|} \frac{1}{1-\sharp k(x)^{-s}} \times \text{finite number of factors} \\ &= \prod_{i=0}^{2\dim X} L(H^i_{et}(X_{\bar{K}},\mathbb{Q}_p))^{(-1)^i} \text{ Cohomological interpretation} \\ &\stackrel{?}{=} \prod_j L(\pi_j,s)^{\epsilon_j}, \quad \pi_j \text{ on } \mathrm{GL}_{n_j}/K, \quad \epsilon_j\in\mathbb{Z} \end{aligned}$$

For example, for $K = \mathbb{Q}$ and $X/\mathbb{Q}$ a genus 1 curve we have :

Theorem (Wiles, Breuil-Conrad-Diamond-Taylor)

$$Z(X)(s) = \frac{\zeta(s)\zeta(s-1)}{L(\pi_f, s)}$$

where f is a weight 2 modular form and π_f the corresponding automorphic form on $\mathrm{GL}_2/\mathbb{Q}$.

The p-adic topology

A fundamental and classical idea is to introduce a p-**adic topology** on both sides.
The set :

$$\{ \textbf{Galois representations } Gal(\bar{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p) \}$$

has a natural p-adic topology.

The p-adic topology : Automorphic side

We now consider the second set :

$$\{\textbf{Cuspidal algebraic automorphic forms on } \mathrm{GL}_n/K\ \}$$

1. **Algebraicity** : Find an incarnation in the **coherent cohomology** of an algebraic variety over $\mathbb{Q}$ or the **Betti cohomology** of any variety.
2. Study the p-adic variation of this realization.

Coherent cohomology realization

Let *Sh* be a **Shimura variety** defined over a number field.

- It carries "**automorphic vector bundles**" $\mathcal{V}$.
- The cohomology groups $\mathrm{H}^i(Sh, \mathcal{V}) \otimes \mathbb{C}$ can be computed using automorphic forms (Harris...).
- $\mathrm{H}^i(Sh, \mathcal{V})$ are **finite dimensional** $\mathbb{Q}$-vector spaces : **Algebraicity**.

Remark

There are many interesting algebraic automorphic forms which don't admit a coherent cohomology realization (Maass forms...).

Aim : Study the *p*-Adic properties of the cohomology $\mathrm{R}\Gamma(Sh, \mathcal{V})$**.**

A fundamental example : Siegel Shimura varieties

We take $Sh =$

moduli of Abelian varieties of dimension g + level + polarization

Over Sh we have :

- the **universal abelian variety** A
- the **locally free sheaf** of rank g of invariant differential forms ω_A.

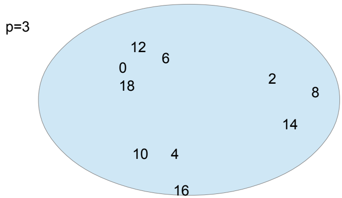
We get a functor :

$$\begin{array}{rcl} \text{Representations of } \mathrm{GL}_g & \to & \text{Automorphic sheaves}/\mathrm{Sh} \\ V & \mapsto & \omega_A \times^{\mathrm{GL}_g} V = \mathcal{V} \end{array}$$

$$\begin{array}{rcl} \text{Dominant characters } \mathbb{G}_m^g \to \mathbb{G}_m & \to & \text{Automorphic sheaves}/\mathrm{Sh} \\ \kappa = (k_1, \cdots, k_g),\ k_1 \geq \cdots \geq k_g & \mapsto & \mathcal{V}(\kappa) \end{array}$$

p-Adic interpolation : the weight space

- We replace $\mathbb{G}_m^g$ by $(\mathbb{Z}_p^\times)^g$.
- We define $\mathcal{W}$ = **The space of continuous p-adic characters** of $(\mathbb{Z}_p^\times)^g$.
- $\mathcal{W}$ is a finite union of g-**dimensional open polydiscs of radius** 1 over $\mathbb{Q}_p$.
- Each algebraic dominant character $\kappa : \mathbb{G}_m^g \to \mathbb{G}_m$ restricts to $\kappa : (\mathbb{Z}_p^\times)^g \to \mathbb{Z}_p^\times$ and a defines **classical weight** on $\mathcal{W}$.
- Classical weights are **Zariski dense** in $\mathcal{W}$.



p-Adic interpolation : the automorphic sheaves

Let $\mathcal{S}h$ be the **analytic space** over $\mathbb{Q}_p$ associated with Sh. Let $\mathcal{S}h^{ord} \hookrightarrow \mathcal{S}h$ be the **ordinary** locus. Let $\mathcal{S}h^{ord,\dagger}$ be the **overconvergent neighborhood** of $\mathcal{S}h^{ord}$.

Theorem (AIP)

The automorphic sheaves $\mathcal{V}(\kappa)|_{\mathcal{S}h^{ord,\dagger}}$ can be interpolated over $\mathcal{W}$. There is a map :

$$\begin{array}{rcl} \mathcal{W} & \to & p\text{-}adic\ Automorphic\ sheaves/\mathcal{S}h^{ord,\dagger} \\ \kappa & \mapsto & \mathcal{V}^{\dagger}(\kappa) \end{array}$$

- There is a **universal family of p-adic automorphic sheaves** over $\mathcal{S}h^{ord,\dagger} \times \mathcal{W}$.
- For each **classical weight** κ, there is a canonical map

$$\mathcal{V}(\kappa)|_{\mathcal{S}h^{ord,\dagger}} \hookrightarrow \mathcal{V}^{\dagger}(\kappa)$$

  (inclusion of algebraic in analytic induction).
- The sheaves $\mathcal{V}^{\dagger}(\kappa)$ are not coherent unless $g = 1$.

An idea of the proof when $g = 1$

Let $E/\mathbb{C}_p$ be an elliptic curve. We have a **Hodge-Tate period map** :

$$T_pE \to \omega_E$$

where

- $T_pE = \lim_n E[p^n] \simeq \mathbb{Z}_p^2$,
- $\omega_E \simeq \mathbb{C}_p$.

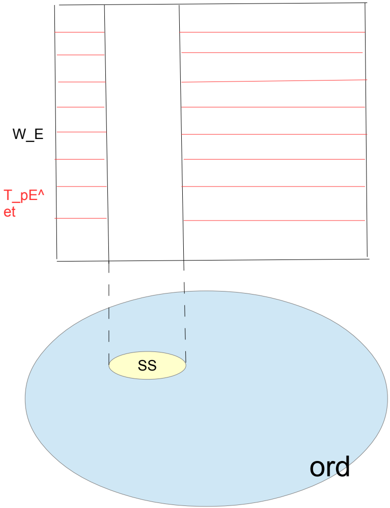
Assume that E has **ordinary** reduction. Then T_pE is filtered :

$$0 \to T_pE^m \to T_pE \to T_pE^{et} \to 0$$

and the **Hodge-Tate period map factors into an isomorphism** :

$$T_pE^{et} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to \omega_E$$

The GL_1-torsor ω_E has a $\mathbb{Z}_p^\times$-reduction !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We get a commutative diagram :

$$\begin{array}{ccc} \text{Characters of } \mathrm{GL}_1 & \longrightarrow & \text{Automorphic sheaves}/\mathcal{S}h \\ \downarrow & & \downarrow \\ \text{Characters of } \mathbb{Z}_p^\times & \longrightarrow & p\text{-adic automorphic sheaves}/\mathcal{S}h^{ord} \end{array}$$

Overconvergence

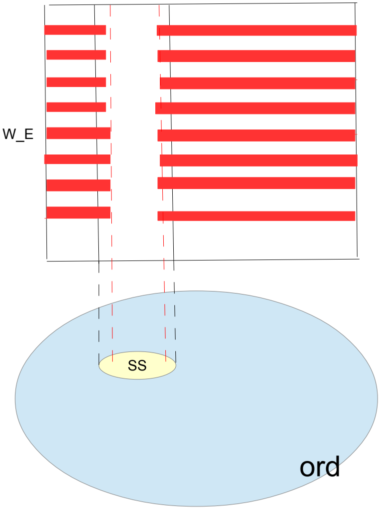
Actually this can be improved to

$$\begin{array}{ccc} \text{Characters of } \mathrm{GL}_1 & \longrightarrow & \text{Automorphic sheaves}/\mathcal{S}h \\ \downarrow & & \downarrow \\ \text{Characters of } \mathbb{Z}_p^\times & \longrightarrow & p\text{-adic automorphic sheaves}/\mathcal{S}h^{ord,\dagger} \end{array}$$

For any $n \geq 0$, there is a **strict neighborhood** $\mathcal{S}h^{ord} \subset \mathcal{U}$ such that ω_E has a reduction to a **thickening** of $\mathbb{Z}_p^\times$:

$$\mathbb{Z}_p^\times(1 + p^n \mathcal{O}_\mathcal{U}^+)$$

$$0 \to 1 + p^n \mathcal{O}_\mathcal{U}^+ \to \mathbb{Z}_p^\times(1 + p^n \mathcal{O}_\mathcal{U}^+) \to (\mathbb{Z}/p^n\mathbb{Z})^\times \to 0$$



Sh

Overconvergent and classical modular forms

We have a restriction map $\mathrm{H}^0(\mathcal{S}h, \mathcal{V}(\kappa)) \to \mathrm{H}^0(\mathcal{S}h^{ord,\dagger}, \mathcal{V}(\kappa)^\dagger)$ for **classical weight** κ.

1. The *LHS* is a **finite dimensional** $\mathbb{Q}_p$-vector spaces.
2. The *RHS* is an **infinite dimensional** (inductive limit of) Banach spaces.
3. Both sides are equipped with and action of **Hecke operators**.
4. We have a natural **compact operator** $\mathcal{U}_p$: spectral decomposition, notion of slope (p-adic valuation of the spectrum).

Theorem (Coleman, Kassaei, Bijakowski-P-Stroh)

The above map induces a quasi-isomorphism on the small slope part : $\mathrm{H}^0(\mathcal{S}h, \mathcal{V}(\kappa))^{s-slope} \xrightarrow{\sim} \mathrm{H}^0(\mathcal{S}h^{ord,\dagger}, \mathcal{V}^\dagger(\kappa))^{s-slope}$.

The small slope condition is a **regularity** condition on κ.

Application : eigenvarieties

The eigenvariety is $\mathcal{E}$ = "**Spectrum of Hecke algebra**" acting on $\mathrm{H}^0(\mathcal{S}h^{ord,\dagger}, \mathcal{V}^\dagger(\kappa))$ for varying $\kappa \in \mathcal{W}$.

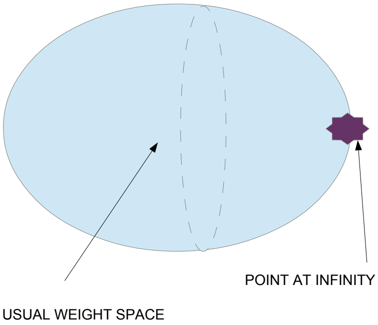
Theorem (Coleman-Mazur, AIP)

1. *The eigenvariety is equidimensional of dimension g.*
2. *The weight map :*
$$\mathcal{E} \to \mathcal{W}$$
has discrete fibers.
3. *Classical automorphic points are Zariski dense in $\mathcal{E}$.*

Any automorphic point x on $\mathcal{E}$ is a limit of automorphic points x_n of very regular weight κ_n.

On the geometry of the eigencurve : the halo

The weight space (= union of open unit disc) has a **compactification** in the category of analytic adic spaces obtained by adding a **finite number of points at infinity** corresponding to continuous characters $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times[[T]]$ (with infinite image).

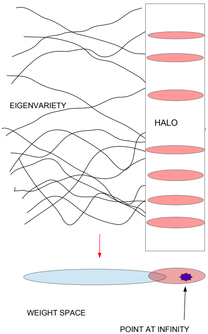


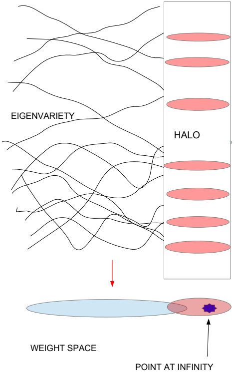
Theorem (AIP)

The sheaves $\mathcal{V}^\dagger(\kappa)$ and the eigencurve extend over the compactified weight space.

Conjecture (Coleman-Mazur-Buzzard-Kilford)

There is a neighborhood of infinity in the weight space such that its preimage in the eigencurve is a disjoint union of finite flat covers.





Theorem (Liu-Wan-Xiao)

1. *This conjecture is known for quaternionic eigencurves.*
2. *Automorphic points of weight 2 are Zariski dense in quaternionic eigencurves.*

Higher coherent cohomology and irregular weight

The classicity theorem

$$\mathrm{H}^0(\mathcal{S}h, \mathcal{V}(\kappa))^{s-slope} \xrightarrow{\sim} \mathrm{H}^0(\mathcal{S}h^{ord,\dagger}, \mathcal{V}^{\dagger}(\kappa))^{s-slope}$$

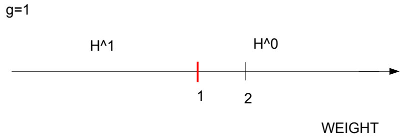
is vacuous at irregular weights (corresponding to irregular motives).

- At irregular weights we have several consecutive cohomology degree occuring.
- It reflects that we are dealing with **motives of irregular weights** (Calegari-Geraghty, Venkatesh...).

The case $g = 1$

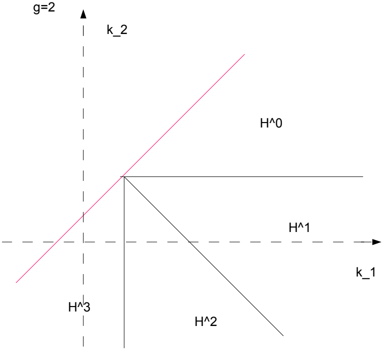
For $g = 1$, we have

1. an interpolation theory of H^0 with classicity theorem for **large positive** weights.
2. In **weight one** there is cohomology in H^0 and H^1 (corresponds to **irregular** motives : Artin representations !).



For $g = 2$:

1. We have interpolated H^0 with classicity in regular weight.
2. There is an **entire wall of irregular weight** with cohomology in H^0 and H^1.



The theory interpolating simultaneously H^0 **and** H^1 **was recently constructed** :

1. the theory is "supported" over a **one dimensional wall** of the weight space.
2. One interpolates a **perfect complexes over the one dimensional weight space of amplitude** $[0,1]$ but the actual cohomology groups may have **zero dimensional support**.

This plays an important role in the following (potential) modularity theorem in irregular weight :

Theorem (Boxer-Calegari-Gee-P)

1. *Genus 2 curves over $\mathbb{Q}$ are potentially automorphic.*
2. *The Hasse-Weil conjecture holds for genus two curves over $\mathbb{Q}$.*

1. Irregularity : A genus two curve has hodge numbers $h^{0,1} = h^{1,0} = 2$.
2. The relevant automorphic forms on $\mathrm{GSp}_4/\mathbb{Q}$ have weight $(2,2)$.

More interpolation : the de Rham cohomology and Gauss-Manin connection

The automorphic vector bundle are naturally endowed with **differential operators**.
On the modular curve, we have the relative de Rham cohomology :

$$(\mathcal{H}^1_{dR}(E), \mathrm{Fil}, \nabla)$$

equipped with the Hodge filtration Fil :

$$0 \to \omega_E \to \mathcal{H}^1_{dR}(E) \to \omega_E^{-1} \to 0$$

and the Gauss Manin connection :

$$\nabla : \mathcal{H}^1_{dR}(E) \to \mathcal{H}^1_{dR}(E) \otimes \Omega^1_{Sh}$$

For any $k \in \mathbb{Z}_{\geq 0}$, we may consider $\mathrm{Sym}^k(\mathcal{H}^1_{dR}(E), \mathrm{Fil}, \nabla)$.

Theorem (AI)

There is a "universal symmetric power" $(\mathbb{W}, \mathrm{Fil}, \nabla)$ *of* $(\mathcal{H}^1_{dR}(E), \mathrm{Fil}, \nabla)$ *over* $\mathcal{S}h^{ord,\dagger} \times \mathcal{W}$.

Application :

1. A p-adic theory of Shimura's **nearly holomorphic modular forms** (Urban...).
2. p-adic interpolation of powers of the connection ∇ on the infinite slope part.
3. **Construction of p-adic L-functions** (Garrett-Rankin triple product, Katz's p-adic L-function for CM-fields, Bertolini-Darmon-Prasana).