Gaps between primes

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Introduction

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Theorem (Prime number theorem)

$$\#\{primes \le x\} \approx \frac{x}{\log x}.$$

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Question

Are prime gaps always this big?

Introduction II

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How small can prime gaps be?

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How small can prime gaps be?

- $(2, 3)$ is the only pair of primes which differ by 1.
  (One of n and $n + 1$ is a multiple of 2 for every integer n).
- There are lots of pairs of primes which differ by 2:
  $(3, 5)$, $(5, 7)$, $(11, 13)$, $\dots$, $(1031, 1033)$, $\dots$, $(1000037, 1000039)$, $\dots$, $(1000000007, 1000000009)$, $\dots$

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Conjecture (Twin prime conjecture)

There are infinitely many pairs of primes (p,p') which differ by 2.

This is one of the oldest problems in mathematics, and is very much open!

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- There is at most one pair of primes (p_1, p_2) with $p_1 - p_2 = h$ if h is odd.
- There are lots of pairs which differ by 2, 4, 6, 8, ...

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Conjecture (De Polignac's conjecture)

For every even h, there are infinitely many pairs (p_1, p_2) of primes such that $p_1 - p_2 = h$.

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Conjecture (De Polignac's conjecture)

For every even h, there are infinitely many pairs (p_1, p_2) of primes such that $p_1 - p_2 = h$.

More generally, we can look for triples (or more) of primes.

- $(2,3,5)$, $(2,3,7)$, $(2,5,7)$, $(3,5,7)$ are the only triples contained in an interval of length 5. (At least one of n, $n+2$, $n+4$ is a multiple of 3.)
- There are lots of triples of primes in an interval of length 6.

Prime k-tuples conjecture

We can look for general patterns $L_1(n), \dots, L_k(n)$ of linear functions of primes.

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Definition (admissibility)

*$\{L_1, \dots, L_k\}$ is **admissible** if $\prod_i L_i(n)$ has no fixed prime divisor.*

Conjecture (Prime k-tuple conjecture)

*Let $\{L_1, \dots, L_k\}$ be admissible. Then there are infinitely many integers n such that **all** of $L_1(n), \dots, L_k(n)$ are primes.*

Consequences of k-tuples

This has a huge number of other consequences!

- $L_1(n) = n$, $L_2(n) = n + 2$: Twin prime conjecture.
- $L_i(n) = n + h_i$: m primes in intervals of length $\approx m \log m$.
- $L_1(n) = n$, $L_2(n) = 2n - 1$: there are infinitely many Sophie Germain primes. (Weak FLT and Artin's conjecture on primitive roots.)
- $L_j(n) = n + j \times k!$: k-term arithmetic progressions of primes
- $L_1(n) = n$, $L_2(n) = 2N - n$: Goldbach's conjecture.
- $L_i(n) = n + h_i q$: Residue class containing many small primes.
- ...

Weak k-tuples

Theorem (M., Tao)

Let $\{L_1, \dots, L_k\}$ be admissible. Then there are infinitely many integers n such that $(1/4 + o(1)) \log k$ of $L_1(n), \dots, L_k(n)$ are primes.

Weak k-tuples

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Zhang proved that 2 of the L_i are simultaneously prime for k large enough by using a different method.

Both this result and Zhang's are based on earlier ideas of Goldston, Pintz and Yıldırım.

Independently discovered by Tao.

The underlying method is flexible and can work with subsets, uniformity etc.

Small gaps between primes

This automatically gives bounded gaps between primes.

Theorem (Zhang)

We have

$$\liminf_{n}(p_{n+1}-p_n)\le 70\,000\,000.$$

Theorem (M.)

We have

1. $\liminf_n(p_{n+m}-p_n)\le m^3e^{4m+8}$ *for all* $m\in\mathbb{N}$.
2. $\liminf_n(p_{n+1}-p_n)\le 600$.

Theorem (Polymath 8b)

We have

- $\liminf_n(p_{n+1}-p_n)\le 246$.

Probabilistic method

We can view these results as an application of the probabilistic method.

1. Choose $n \in [X, 2X]$ randomly (according to some probability measure).
2. Calculate the expected number of $L_i(n)$ which are prime.
3. If this expectation is $> m$ for all large X, then there are infinitely many n such that at least m of the $L_i(n)$ are prime.

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How to choose the probability measure to be concentrated on n when 'many' of the $L_i(n)$ prime?

Sieve methods

The choice of probability measure is inspired by sieve methods.

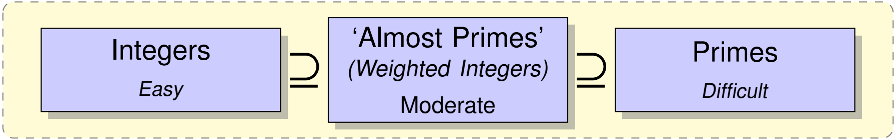
One way to view sieve methods is the study of ‘almost-primes’.

Integers	$\supseteq$	‘Almost Primes’	$\supseteq$	Primes
Easy		*(Weighted Integers)*		*Difficult*
		Moderate		

Sieve methods

The choice of probability measure is inspired by sieve methods.

One way to view sieve methods is the study of ‘almost-primes’.



- **The primes have positive density in the almost-primes**
- **We can solve additive problems for almost-primes if we know solutions in aritmetic progressions**

We understand integers and primes in arithmetic progressions, so we can calculate the expected number of L_i which are prime.

This expectation depends on the definition of ‘almost-prime’.

Optimised GPY method

Question

How do we choose the weights w_n which define almost-primes?

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We choose w_n to mimic 'Selberg sieve' weights.

1. **Standard choice**: Gives expectation $\approx 1/2$. Fails to prove bounded gaps.

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4. **New choice**: Get expectation $(1/4+o(1))\log k$. Gives m primes if $k > Ce^{(4+\epsilon)m}$.

Overview

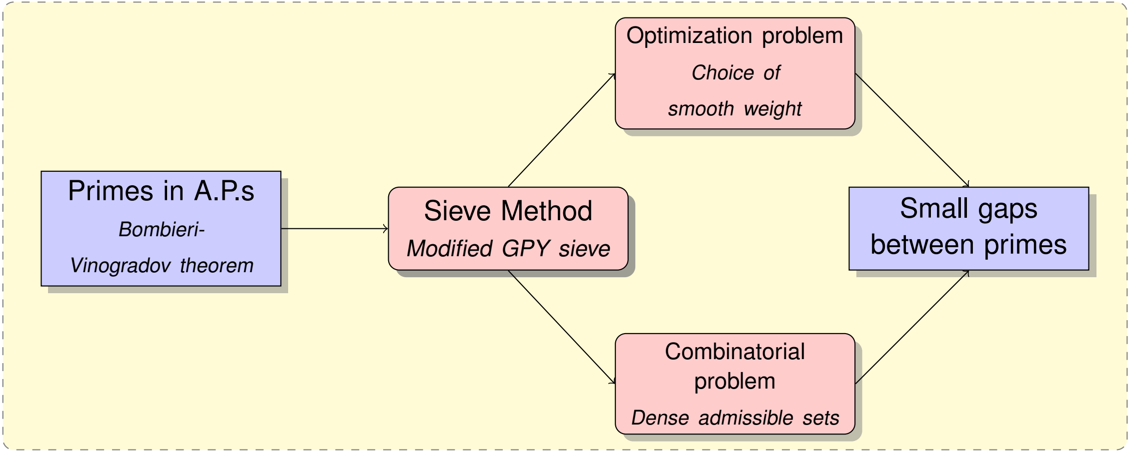

Figure: Outline of steps to prove small gaps between primes

Large gaps between primes

Question

What about large gaps? How big is $G(X) = \sup_{p_n \le X}(p_{n+1} - p_n)$*?*

$G(X) \ge (1 + o(1)) \log X$ by prime number theorem.

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Theorem (Rankin, 1938)

$$G(X) \gg \frac{(\log X)(\log\log X)(\log\log\log\log X)}{(\log\log\log X)^2}$$

Question (Erdős, $10,000)

Can we improve on Rankin's result by an arbitrarily large constant?

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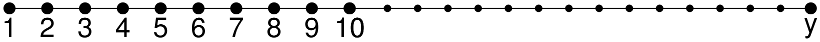
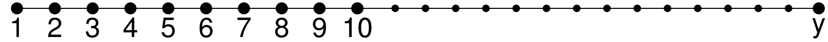
Visual representation

Take the integers between 1 and y:

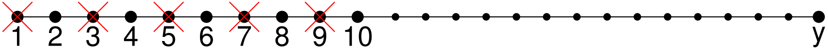


Visual representation

Take the integers between 1 and y:



Step 1: Cross out all integers which are congruent to $a_2 \pmod 2$:



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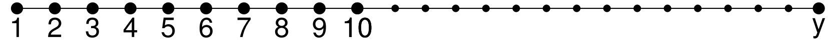

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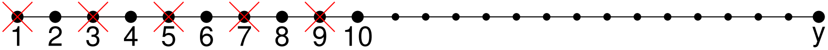
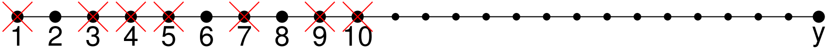


Step 2: Cross out all integers which are congruent to $a_3 \pmod 3$:



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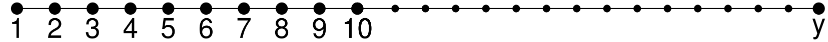

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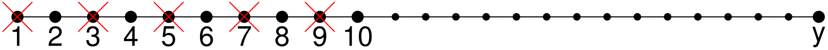

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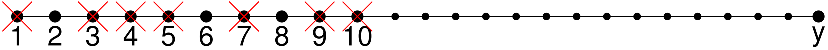

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Choosing $U \equiv -a_p \pmod p$ $\forall p$ gives $U + n$ composite for $n \leq y$.

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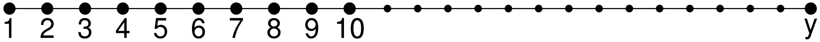

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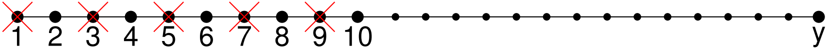

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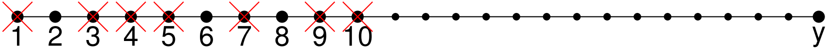

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What is the minimal number of steps?

If the minimal number of steps is small, then we can find long gaps between primes.

Idea behind large gaps

Erdős-Rankin method: Every integer $n \leq y$ satisfies $n \equiv a_p \pmod{p}$ for some $p \leq (y \log_2 y)/(\log y \cdot \log_3 y)$ where

$$a_p = \begin{cases} 0, & p \text{ 'medium'}, \\ \text{(chosen greedily)} & p \text{ 'small' and 'large'}. \end{cases}$$

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But this we solved using weak prime k-tuples with $L_i(n) = n + h_i p$!

Questions

Thank you for listening.

Conditional result

Recall that the quality of our ‘almost-primes’ depends on results about primes in arithmetic progressions. What if we assume the most optimistic conjectures of this type?

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Theorem (Polymath 8b)

Assume 'GEH'. Then we have,

$$\liminf_n (p_{n+1} - p_n) \leq 6.$$

Unfortunately, this is a hard limit of our methods.

Conditional Result II

This has a second consequence:

Theorem (Polymath 8b)

Assume 'GEH'. Then at least one of the following is true.

1. *There are infinitely many twin primes.*
2. *Every large even number is within 2 of a number which is the sum of two primes.*

Of course we expect both to be true!

Questions

Thank you for listening.