Functional Transcendence and Arithmetic Applications

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Lang's Conjecture: Polynomial Relations in Roots of Unity

Let $C \subset (\mathbb{C}^\times)^2$ be an algebraic curve, defined by (irreducible) $F(X, Y) = 0, F(X, Y) \in \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$.

Theorem (Lang 1965)

If C contains infinitely many points (ζ, η) with ζ, η roots of unity, then C is of the form

$$x^m y^n = \zeta,$$

with $n, m \in \mathbb{Z}, \zeta$ a root of unity.

$\therefore$ C - coset of a subgroup by a torsion point; we call C a **Torsion Coset**.

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Theorem (Raynaud, 1983)

For $V \subset (\mathbb{C}^\times)^n$, then V contains finitely many ***Maximal*** *torsion cosets.*

Manin-Mumford: Abelian Varieties

We replace $(\mathbb{C}^\times)^n$ by an abelian variety $A(\mathbb{C}) = \mathbb{C}^n/\Lambda$.

- $B \subset A$ - Abelian subvariety
- $\zeta \in A$ - Torsion point
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Theorem (Manin-Mumford Conjecture; Raynaud, 1983)

For $V \subset (\mathbb{C}^\times)^n$, then V contains finitely many ***Maximal*** *torsion cosets.*

André-Oort: Shimura Varieties

- S - Shimura variety. $S(\mathbb{C}) = \Gamma \backslash \mathcal{H}$
- S contains a discrete, dense set of **Special (CM) Points**
- S contains a countable set of **Special Subvarieties** T, which are distinguished "Shimura Subvarieties"

Conjecture (André)

*For $V \subset S$, then V contains finitely many **Maximal** Special Subvarieties.*

- André (1998): $S = Y(1)^2$
- Edixhoven, Klingler, Ullmo, Yafaev (2014) : True conditional on GRH.
- Pila, Pila-Zannier Strategy: $S = X(1)^n$
- T. (Pila-T): $S = \mathcal{A}_g$

Examples of Shimura Varieties: $Y(1)^2$

- $Y(1)$ - (coarse) moduli space of elliptic curves
- $Y(1)(\mathbb{C}) \cong \mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \pi : \mathbb{H} \to Y(1)$

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Special Points in $Y(1)$

- The CM points are classes $[E]$ of complex elliptic curves with $\mathbb{Z} \subsetneq \mathrm{End}(E)$
- $\pi^{-1}(Y(1)_{\mathrm{CM}}) = \{\tau \in \mathbb{H} \mid [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}$

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Special curves in $Y(1)^2$:

- $\{x\} \times Y(1)$, where x is a special point
- $Y(1) \times \{x\}$, where x is a special point
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Theorem (André, 1998)

A curve $C \in Y(1)^2$ containing infinitely many CM points is special.

Proved effectively by Kühne, Bilu-Masser Zannier, 2012/13.

Examples of Shimura Varieties: $\mathcal{A}_g$

- $\mathcal{A}_g$ - (coarse) moduli space of g-dimensional, principally polarized abelian varieties
- $\mathbb{H}_g = \{Z = X + iY, Y > 0, Z \in \mathrm{M}_g(\mathbb{C})^{\mathrm{sym}}\}$, $\mathcal{A}_g(\mathbb{C}) \cong \mathsf{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$.

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Special subvarieties in $\mathcal{A}_g$:

- $\mathrm{sym}^g X(1) \subset \mathcal{A}_g$
- F- totally real field, $[F : \mathbb{Q}] = g$, $Y_F \cong \mathsf{Sl}_2(\mathcal{O}_F)\backslash\mathbb{H}^g \subset \mathcal{A}_g$.
- Other Endomorphism structures, etc...

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- $\pi : (\mathbb{C}/\mathbb{Z})^2 \to (\mathbb{C}^\times)^2, (a, b) \to (e^{2\pi i a}, e^{2\pi i b})$.

$$\pi^{-1}(\mathbb{C}^{tor}) = \mathbb{Q}/\mathbb{Z}$$

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- **Lower Bound:** $\pi(\frac{a}{n}, \frac{b}{n}) \in Z \Rightarrow \forall (c, n) = 1, \pi(\frac{ca}{n}, \frac{cb}{n}) \in Z$. Z contains **1** point of order $n \Rightarrow Z$ contain $\phi(n)$ such points.

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  Z contains $O_\epsilon(n^\epsilon)$ points in $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^2$

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- $n^{1-\epsilon} \ll_\epsilon \phi(n)$, which is a contradiction.

Pila-Zannier Strategy:General case

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- $\pi^{-1}(V)$ cannot contain many rational points, unless $\pi^{-1}(V)$ **contains** algebraic subvarieties W, which contain most of the rational points in V.
- Functional transcendence: This only happens if V is a (weakly) special variety (Ax-Lindemann Theorem).

Transcendence: Pila-Zannier Strategy

3 fundamental Ingredients

- **Pila-Wilkie Theorem**: If a (definable) real analytic variety contains many rational points, it is algebraic.
- **Functional Transcendence**: There is no interaction between the algebraic structures on $\mathcal{H}$ and X, except that which is mandated by (weakly) special varieties.
- **Large Galois Orbits**: If $x \in X$ is a special point, then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x$ is big compared to the complexity of x.

Upper bounds

- $\gcd(a, b) = 1, H(\frac{a}{b}) = \mathrm{Max}(|a|, |b|)$
  $H(z_1, .., z_m) = Max_{i=1}^{m} H(z_i)$

Pila-Wilkie: Counting Rational Points

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- $N(S,X):=\#\{z\in\mathbb{Q}^m\mid H(z)\leq X, z\in S\}$

Theorem (Pila-Wilkie,'04; Bombieri-Pila)

$N(Y^{tran};X)=O_\epsilon(X^\epsilon)$

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Generalizations: $\mathbb{Q} \Rightarrow$ Points of bounded degree, compact, real analytic $\Rightarrow$ Definable in an o-minimal structure $(\mathbb{R}_{an,exp})$.

Largeness of Galois Orbits

- **The $(\mathbb{C}^\times)^n$ case**
  - $x \in (\mathbb{C}^\times)^n_{\mathrm{tor}}$
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- $x \in \mathcal{A}_g$ Corresponds to PPAV A_x
- $\mathrm{End}(A_x) = R \subset K, [K : \mathbb{Q}] = 2g$

Theorem (T)

$\exists \delta_g > 0$ *with*

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Corollary

For any $g \geq 1$, there are finitely many CM points in $\mathcal{A}_g$ over $\mathbb{Q}$.

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- Not enough low degree isogenies exist.

Functional Transcendence: Bi-algebraic sets

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- X - $(\mathbb{C}^\times)^n$ **or** $A(\mathbb{C})$ - V bi-algebraic = coset of an algebraic subgroup
- X - **Shimura variety.** V bi-algebraic = weakly special = "translate of special subvariety" (Ullmo-Yafaev)
- E.g. $X = Y(1)^2, V = \{x\} \times Y(1)$.

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- E.g. $X = Y(1)^2, V = \{x\} \times Y(1)$.
- **Heuristic: No algebraic interaction besides weakly specials**

Functional Transcendence: Conjectures

Setup: $V \subset X, W \subset \mathcal{H}$ algebraic subvarieties.

Conjecture (Ax-Lindmenann)

If $\pi(W) \subset V, \exists$ *weakly special* S *such that* $W \subset S, \pi(S) \subset V$.

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Conjecture (Ax-Schanuel)

Let $U \subset W \cap \pi^{-1}(V)$ be an analytic component. If $dim U > dim W + dim V - dim X$ then $\exists$ weakly special S such that $U \subset S$ and $dim U = dim W \cap S + dim V \cap S - dim S$.

Functional Transcendence: Results

- AS for $X = \mathbb{C}^n$ or $A(\mathbb{C})$, (Ax, 1971)
- AL for $X = Y(1)^n$, (Pila, 2011)
- AL for all Shimura Varieties (Klingler-Ullmo-Yafaev, 2015)
- AL for Mixed Shimura Varieties (Gao, 2017)
- AL for non-arithmetic rank 1 quotients (Mok, 2018)
- AS for $X = Y(1)^n$, (Pila-T, 2016)
- AS for all Shimura varieties, (Mok-Pila-T, 2018)
- AS for Variations of Hodge Structures, (Bakker-T, 2018)

Functional Trancendence: Proof ingredients

- Pila-Wilkie (Again!)
- o-minimality, Definable Chow Theorem (Peterzil-Starchenko)
- Hyperbolic Geometry (Hwang-To)

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- $\Gamma \backslash (\mathrm{Hilb}_W \mid \mathcal{H})$ can be given a canonical algebraic structure over $S \cong \Gamma \backslash \mathcal{H}$.
- For $V \subset S, \{p \in V \mid \exists W, \pi^{-1}(p) \in W, \dim(W \cap V)_p = k\}$ is algebraic.

Functional Trancendence: Descending Hilbert Schemes

- $\mathcal{H} \subset \mathcal{H}^\vee$ embedding in the compact dual
- For $W \subset \mathcal{H}^\vee$ algebraic, let Hilb_W denote the pointed Hilbert scheme containing W, consisting of pairs $(W', p), p \in W'$.
- $\Gamma \backslash (\mathrm{Hilb}_W \mid \mathcal{H})$ can be given a canonical algebraic structure over $S \cong \Gamma \backslash \mathcal{H}$.
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- **Punchline** One counterexample (V, W) to Ax-Schanuel leads to many counterexamples (V, W').

Further Applications: Zilber-Pink Conjectures

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Conjecture (Bombieri-Masser-Zannier, Zilber-Pink)

Assume V is not contained in a proper torus coset. Then the locus of atypical points in V is not Zariski-dense.

Known for Curves (Maurin, bombier-Masser-Zannier)

Further Applications: Zilber-Pink Conjectures

- S - Shimura Variety
- $V \subset S$ - subvariety not contained in a proper special subvariety
- If $T \subset S$ is special, and $\dim V + \dim T < \dim S + \dim V \cap T$, say $V \cap T$ is *atypical*

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Pila-Zannier Strategy needs Ax-Schanuel

Known for curves in $X(1)^n$ assuming Assymetric degrees (Habbeger-Pila)

Shavarevich theorems: Lawrence-Venkatesh

- $Y \to X$ projective, smooth variation over $\mathbb{Z}$. Suppose that the period map $x \to H^*(Y_x)$ is injective. Then $X(\mathbb{Z})$ *should* be finite.

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- Contradicts p-adic Ax-schanuel for period maps, which follows formally from usual Ax-Schanuel.

Thank You!