Perfectoid spaces and the homological conjectures

Yves André

ICM, Rio de Janeiro, August 7, 2018

An encounter between two domains

Commutative algebra – – – – – – – – p-adic Hodge theory
(Hilbert, Krull, ...) (Tate, Fontaine, ...)
Noetherian world non-Noetherian world
(finite-dimensional rings, finite modules) (non-archimedean Banach algebras...)

Homological conjectures $\overset{new}{\longleftarrow}$ – – – – – perfectoid theory
(Peskine-Szpiro, Hochster...) (Faltings, Scholze, ...)

Direct summand conjecture

(1st instance of this encounter).

R: Noetherian commutative ring,
$R \subset S$: finite extension (of commutative rings)

$\rightsquigarrow$ exact sequence of finite R-modules:

$$0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0.$$

Question: does this sequence splits?

(equivalently: is R a direct summand in S? Is there an R-linear map $S \to R$ which sends 1 to 1?)

Ex. Split if R is a normal $\mathbb{Q}$-algebra (divide the trace by the degree).

C.Ex. Non-split if $R = \mathbb{Q}[x, y], (xy)$ and $S =$ its normalization.
Non-split for some normal $\mathbb{F}_p$-algebras.

Direct summand conjecture

Hochster's direct summand conjecture (1969):
The sequence splits if R is regular.

(motivated by the homological conjectures, see later).

Direct summand conjecture

Hochster's direct summand conjecture (1969):
The sequence splits if R is regular.

(motivated by the homological conjectures, see later).

1. Hochster gave (short) proofs when R contains a field.

Direct summand conjecture

Hochster's direct summand conjecture (1969):
The sequence splits if R is regular.

(motivated by the homological conjectures, see later).

1. Hochster gave (short) proofs when R contains a field.
2. He reduced the problem to the unramified complete local case with perfect residue field k of char. p:

$$R \cong W(k)[[x_2, \cdots, x_d]], \quad (x_1 = p).$$

Direct summand conjecture

Hochster's direct summand conjecture (1969):
The sequence splits if R is regular.

(motivated by the homological conjectures, see later).

1. Hochster gave (short) proofs when R contains a field.
2. He reduced the problem to the unramified complete local case with perfect residue field k of char. p:

$$R \cong W(k)[[x_2, \cdots, x_d]], \ \ (x_1 = p).$$

3. Heitmann (2002) gave a proof in dimension $d \leqslant 3$.

Direct summand conjecture

Hochster's direct summand conjecture (1969):
The sequence splits if R is regular.

(motivated by the homological conjectures, see later).

1. Hochster gave (short) proofs when R contains a field.
2. He reduced the problem to the unramified complete local case with perfect residue field k of char. p:

$$R \cong W(k)[[x_2, \cdots, x_d]], \quad (x_1 = p).$$

3. Heitmann (2002) gave a proof in dimension $d \leqslant 3$.

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

1. a finite extension $R \subset S$ splits *iff* it is *pure*, i.e. universally injective.

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

1. a finite extension $R \subset S$ splits *iff* it is *pure*, i.e. universally injective.
2. If a composition $R \to S \to T$ is pure, so is $R \to S$.

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

1. a finite extension $R \subset S$ splits *iff* it is *pure*, i.e. universally injective.
2. If a composition $R \to S \to T$ is pure, so is $R \to S$.
3. Any faithfully flat $R \to T$ is pure.

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

1. a finite extension $R \subset S$ splits *iff* it is *pure*, i.e. universally injective.
2. If a composition $R \to S \to T$ is pure, so is $R \to S$.
3. Any faithfully flat $R \to T$ is pure.
4. Hence it suffices to construct an S-algebra T (possibly big, i.e. non-Noetherian) which is R-faithfully flat.

Direct summand conjecture: strategy

Theorem 1 [A. 2016]

DSC holds: any finite extension of a regular ring splits (as a module).

Observations:

1. a finite extension $R \subset S$ splits *iff* it is *pure*, i.e. universally injective.
2. If a composition $R \to S \to T$ is pure, so is $R \to S$.
3. Any faithfully flat $R \to T$ is pure.
4. Hence it suffices to construct an S-algebra T (possibly big, i.e. non-Noetherian) which is R-faithfully flat.
5. If T is p-torsion free and p-adically complete, this amounts to: T/pT is R/pR-faithfully flat.

Direct summand conjecture: strategy

• The strategy works in char. p (not the shortest way!):
R: complete regular local domain of char. p, S finite extension domain;
$R^{\frac{1}{p^\infty}}$: perfect closure; R-faithfully flat (Kunz); may not contain S;
R^+: absolute integral closure of R (i.e. integral closure in an algebraic closure of the fraction field); contains $SR^{\frac{1}{p^\infty}}$ and is R-faithfully flat (Hochster-Huneke).

• We now turn to the mixed characteristic case

$$R = W(k)[[x_2, \cdots, x_d]].$$

Replace "$R^{\frac{1}{p^\infty}}$" by introducing p^{th}-power roots of the system of parameters $x_1 = p, x_2, \cdots, x_d \rightsquigarrow$ *perfectoid world*.

Perfectoid notions

K: complete non-archimedean field
K^o: valuation ring, K^{oo}: valuation ideal.
Assume that the valuation is *not discrete* (equivalently: $K^{oo} = (K^{oo})^2$), and that the residue field k is of char. $p > 0$.

Perfectoid notions

K: complete non-archimedean field
K^o: valuation ring, K^{oo}: valuation ideal.
Assume that the valuation is *not discrete* (equivalently: $K^{oo} = (K^{oo})^2$), and that the residue field k is of char. $p > 0$.

Proposition [Gabber-Ramero]

$K^o/p \overset{x \mapsto x^p}{\to} K^o/p$ is surjective *iff* for each finite separable L/K, L^o is almost etale over K^o, i.e. Ω_{L^o/K^o} is killed by K^{oo}.

Perfectoid notions

Proposition [Gabber-Ramero]

$K^o/p \stackrel{x\mapsto x^p}{\to} K^o/p$ is surjective *iff* for each finite separable L/K, L^o is almost etale over K^o, i.e. Ω_{L^o/K^o} is killed by K^{oo}.

In this case, $\widehat{K \text{ is a}}$ perfectoid field (k is then perfect).

Ex. $K^o = W(k)[p^{\frac{1}{p^\infty}}]$, $K = K^o[\frac{1}{p}]$ (basic perfectoid field in the sequel).

Perfectoid notions

Here, almost is used in the sense of Almost Algebra: given a commutative ring $\mathfrak{V}$ and an idempotent ideal $\mathfrak{m}$, "neglect" all $\mathfrak{V}$-module killed by $\mathfrak{m}$.
Almost algebra (Faltings, Gabber-Ramero) goes much beyond mere categorical localization: notions of almost finite, almost flat, almost etale...

When $(\mathfrak{V}, \mathfrak{m}) = (K^o, K^{oo} = p^{\frac{1}{p^\infty}} K^o)$ as above, we say $p^{\frac{1}{p^\infty}}$-almost: "$p^{\frac{1}{p^\infty}}$-almost zero" means "killed by $p^{\frac{1}{p^\infty}}$".

Perfectoid notions

A: Banach K-algebra.
A^o: sub-K^o-algebra of power-bounded elements.

A is a perfectoid K-algebra if $A^o/p^{1/p} \overset{x \mapsto x^p}{\to} A^o/p$ is an isomorphism.

This implies that the norm of A is equivalent to the spectral norm, and A^o is the unit ball for the latter.

Ex. $A^o = R_\infty := \cup W(k)[p^{\frac{1}{p^i}}]\widehat{[[x_2^{\frac{1}{p^i}}, \cdots, x_d^{\frac{1}{p^i}}]]}, \; A = R_\infty[\frac{1}{p}]$.

Perfectoid notions

A: Banach K-algebra.
A^o: sub-K^o-algebra of power-bounded elements.

A is a perfectoid K-algebra if $A^o/p^{1/p} \stackrel{x \mapsto x^p}{\to} A^o/p$ is an isomorphism.

This implies that the norm of A is equivalent to the spectral norm, and A^o is the unit ball for the latter.

$$Ex.\ A^o = R_\infty := \cup \widehat{W(k)[p^{\frac{1}{p^i}}][[x_2^{\frac{1}{p^i}}, \cdots, x_d^{\frac{1}{p^i}}]]},\ A = R_\infty[\tfrac{1}{p}].$$

Theorem 2 (Almost purity [Faltings; Scholze, Kedlaya-Liu])

A: perfectoid K-algebra; B: finite etale A-algebra.

Then B is perfectoid, and B^o is an $p^{\frac{1}{p^\infty}}$-almost finite etale A^o algebra.

Perfectoid Abhyankar lemma

In the situation of DSC, $B = S \otimes_R R_\infty[\frac{1}{p}]$ may not be etale over $A = R_\infty[\frac{1}{p}]$, so Almost Purity does not apply (in fact, a finite extension of a perfectoid algebra need not be perfectoid).

Abhyankar's classical lemma: under appropriate assumptions, one can achieve etaleness by adjoining roots of the discriminant.

We follow this strategy.
$g \in R = W(k)[[x_2, \cdots, x_d]]$: a discriminant of $S[\frac{1}{p}]/R[\frac{1}{p}]$. We first note that adjoining p^{th}-power roots of g, in the (non-naive) sense of considering $R_\infty[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}}\rangle^o$, is “harmless":

Theorem 3 [A.]

$R_{\infty,g} := R_\infty[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}}\rangle^o$ is $p^{\frac{1}{p^\infty}}$-almost faithfully flat over R_∞.

Perfectoid Abhyankar lemma

The following is both a perfectoid version of Abhyankar's lemma, and a ramified version of the Almost Purity theorem.

Theorem 4 [A.]

A: perfectoid K-algebra, containing a sequence of p^{th}-power roots of a non-zero divisor $g \in A^o$ (*Ex:* $A = R_{\infty,g}[\frac{1}{p}]$).

B': finite etale $A[\frac{1}{g}]$-algebra.

B^o: integral closure of A^o in B' (hence $B^o[\frac{1}{pg}] = B'$).

Then for any n, B^o/p^n is an $(pg)^{\frac{1}{p^\infty}}$-almost finite etale A^o/p^n algebra.

Perfectoid Abhyankar lemma

The proofs of both theorems use deformation arguments of perfectoid spaces.

For Th. 3:

Theorem 3 [A.]

$R_{\infty,g} := R_\infty[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}}\rangle^o$ is $p^{\frac{1}{p^\infty}}$-almost faithfully flat over R_∞.

Spread out the perfectoid space X attached to $R_\infty = \mathcal{O}^+(X)$.
$X^{<\varepsilon}$: (perfectoid) ε-tubular neighborhood of X.

- $R_{\infty,g}$ is $p^{\frac{1}{p^\infty}}$-almost equal to $\widehat{\text{colim}_\varepsilon}\ \mathcal{O}^+(X^{<\varepsilon})$.
- Using Scholze's (almost) description of $\mathcal{O}^+(X^{<\varepsilon})/p^\varepsilon$ in terms of "Puiseux-like" series with coefficients in $O^+(X)/p^\varepsilon$, show that $\mathcal{O}^+(X^{<\varepsilon})/p^\varepsilon$ is almost faithfully flat over $\mathcal{O}^+(X)/p^\varepsilon$.

Perfectoid Abhyankar lemma

For Th. 4:

X: perfectoid space X attached to $A^o = \mathcal{O}^+(X)$,
$X_{>\varepsilon}$: (perfectoid) complement of ε-tubular neighborhood of the discriminant locus $g = 0$.

• $\mathcal{O}^+(X)$ is $(pg)^{\frac{1}{p^\infty}}$-almost equal to $\lim_\varepsilon \mathcal{O}^+(X_{>\varepsilon})$ (by Scholze's perfectoid Riemann extension theorem).

• By almost purity over $X_{>\varepsilon}$, $(B'\mathcal{O}(X_{>\varepsilon}))^+$ is almost finite etale over $\mathcal{O}^+(X_{>\varepsilon})$.

• Pass to the limit (main step).

Back to DSC

$R = W(k)[[x_2, \cdots, x_d]],\ S$ finite extension, etale outside $pg = 0$.

$A = R_{\infty,g}[\frac{1}{p}],\ B' = S \otimes_R A[\frac{1}{g}] \rightsquigarrow S$-algebra B^o sitting on top of a tower

$$R \xrightarrow{\alpha} R_\infty \xrightarrow{\beta} R_{\infty,g} \xrightarrow{\gamma} B^o$$

where α is faithfully flat,
β is $p^{\frac{1}{p^\infty}}$-almost faithfully flat,
γ is $(pg)^{\frac{1}{p^\infty}}$-almost faithfully flat mod. p.

Thus B^o is "almost" our wanted T.

How to get rid of "almost"?

Big Cohen-Macaulay algebras

(detour through Cohen-Macaulay notions)

S: Noetherian local ring, T: (possibly big) extension.

T is a (big) Cohen-Macaulay S-algebra if any sequence of parameters $x_1, \ldots, x_d$ of S becomes regular in T
(i.e. x_1 is non-zero-divisor in T, x_2 is non-zero divisor in T/x_1 etc, and $T \neq (x_1, \ldots, x_d)T$).

Hochster conjectured that *such a T always exists*.

Ex. If S complete of char. p, S^+ is a big Cohen-Macaulay S-algebra (Hochster-Huneke).

Lemma

Assume S is a complete local domain, hence a finite extension of complete regular local ring R (Cohen).
Then T is a (big) Cohen-Macaulay S-algebra *iff* T is R-faithfully flat.

Back to DSC: our B^o is only an *almost* Cohen-Macaulay S-algebra.
But one can pass from an Cohen-Macaulay S-algebra to a *genuine* Cohen-Macaulay S-algebra T using either
- Hochster's (classical) technique of algebra modifications, or
- Gabber's ultraproduct (new) technique.

This is the last step in the proof of DSC and of Hochster's conjectured existence of big CM algebras.

Big Cohen-Macaulay algebras

Main Theorem 5 [A. 2016, 2018]

• Any Noetherian local ring S admits a big CM algebra T.
• For any local morphism $S \to S'$ of Noetherian complete local domains, there is morphism of respective big CM algebras $T \to T'$.

Big Cohen-Macaulay algebras

Main Theorem 5 [A. 2016, 2018]

• Any Noetherian local ring S admits a big CM algebra T.
• For any local morphism $S \to S'$ of Noetherian complete local domains, there is morphism of respective big CM algebras $T \to T'$.

Corollary [Heitmann-Ma; A.]

Any subring of a regular ring, which is a direct summand as a module, is Cohen-Macaulay.

Big Cohen-Macaulay algebras

Theorem 6

• [A. , Shimomoto] Any Noetherian complete local domain S of char. $(0, p)$ admits an (integral) perfectoid CM algebra T.
• [Ma-Schwede] Any two such T are dominated by a third.

This allows Ma and Schwede to develop
• an analog of tight closure theory (Hochster-Huneke) in mixed characteristic,
• in-depth study of singularities in mixed characteristic, where perfectoid CM algebras play somehow the role of resolution of singularities in equal char. 0.

Kunz' theorem in mixed characteristic

• R: Noetherian ring of char. p.

Kunz' classical theorem:

R is regular *iff* $R \overset{x \mapsto x^p}{\to} R$ is flat,

iff there exists a perfect, faithfully flat R-algebra.

• R: Noetherian p-adically complete ring.

Theorem 7 [Bhatt-Iyengar-Ma 2018]

R is regular *iff* there exists an integral perfectoid, faithfully flat R-algebra.

Applications to singularities

S: local domain, essentially of finite type over $\mathbb{C}$.
Rational singularity? (i.e. $R \cong \mathrm{R}\Gamma\mathcal{O}_Y$ for a log-resolution Y)

Criteria by reduction mod. p after spreading out.

S rational singularity
$\Leftrightarrow$ (S mod. p) F-rational singularity for $p >> 0$ (i.e. local cohomology = simple Frob.-module - checkable on Macaulay2)
$\overset{M.-S.}{\Leftrightarrow}$ (S mod. p) F-rational singularity for some p.

$\rightsquigarrow$ algorithm. Ma-Schwede use a perfectoid avatar of rational singularity in mixed char. as link between char. p and char. 0.

Applications to the homological conjectures

Homological turn in commutative algebra in the 60's:

study of noetherian rings and their ideals (Krull, Zariski...) $\rightsquigarrow$ homological properties of their modules (Auslander, Serre...)

Ex. R (local) is regular iff *every finite R-module has a finite free resolution.*

Peskine-Szpiro: reduction techniques to char. p + extension of Kunz' theorem: Frobenius preserves finite free resolutions.

Ex. of application [P.-S., Roberts]. *S (local) is Cohen-Macaulay* iff *there is an S-module of finite length with a finite free resolution.*

Homological conjectures

S: Noetherian local ring. M: finite S-module with a finite free resolution.

Theorem 8 [Evans-Griffiths; Hochster, Ogoma]

(Under DSC)
Let $0 \to S^{b_s} \to S^{b_{s-1}} \to \ldots \to S^{b_0} \to M \to 0$ be a minimal free resolution of M.
Then $b_i \geqslant 2i + 1$ if $i < s - 1$, and $b_{s-1} \geqslant s$.

Thanks to the perfectoid techniques, these optimal bounds are now *unconditional*.

In fact, all homological conjectures which were standing for a while on Hochster's list are now solved, as consequences of the above theorems – but Hochster recently added a few more to his list...

– – –