Perfectoid spaces and the homological conjectures

Yves André

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An encounter between two domains

Commutative algebra $\dashrightarrow \dashrightarrow \dashrightarrow \dashrightarrow \dashrightarrow \dashrightarrow \dashrightarrow \dashrightarrow$ $p$-adic Hodge theory

(Hilbert, Krull, ...) (Tate, Fontaine, ...)

Noetherian world

(non-Noetherian world

(finite-dimensional rings, non-archimedean Banach algebras...)

Homological conjectures $\leftarrow$ perfectoid theory

(Peskine-Szpiro, Hochster...) (Faltings, Scholze, ...)

Yves André
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Direct summand conjecture

(1st instance of this encounter).

$R$: Noetherian commutative ring,

$R \subset S$: finite extension (of commutative rings)

$\Rightarrow$ exact sequence of finite $R$-modules:

$$0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0.$$

**Question:** does this sequence splits?

(equivalently: is $R$ a direct summand in $S$? Is there an $R$-linear map $S \rightarrow R$ which sends 1 to 1?)

*Ex.* Split if $R$ is a normal $\mathbb{Q}$-algebra (divide the trace by the degree).

*C.Ex.* Non-split if $R = \mathbb{Q}[x, y]$, $(xy)$ and $S =$ its normalization.

Non-split for some normal $\mathbb{F}_p$-algebras.
Hochster’s direct summand conjecture (1969):

*The sequence splits if \( R \) is regular.*

(motivated by the homological conjectures, see later).

\[ R \cong W(\mathbb{k})[[x_2, \ldots, x_d]], (x_1 = p) \]
Hochster’s direct summand conjecture (1969): 
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(motivated by the homological conjectures, see later).

- Hochster gave (short) proofs when $R$ contains a field.

Theorem 1 \[A. 2016\]

DSC holds: any finite extension of a regular ring splits (as a module).
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(motivated by the homological conjectures, see later).

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2. He reduced the problem to the unramified complete local case with perfect residue field $k$ of char. $p$:

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3. Heitmann (2002) gave a proof in dimension \( d \leq 3 \).
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1. a finite extension $R \subset S$ splits iff it is pure, i.e. universally injective.

2. If a composition $R \rightarrow S \rightarrow T$ is pure, so is $R \rightarrow S$.

3. Any faithfully flat $R \rightarrow T$ is pure.

4. Hence it suffices to construct an $S$-algebra $T$ (possibly big, i.e. non-Noetherian) which is $R$-faithfully flat.

5. If $T$ is $p$-torsion free and $p$-adically complete, this amounts to: $T/pT$ is $R/pR$-faithfully flat.
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The strategy works in char. $p$ (not the shortest way!):

- $R$: complete regular local domain of char. $p$, $S$ finite extension domain;
- $R^{1/p^\infty}$: perfect closure; $R$-faithfully flat (Kunz); may not contain $S$;
- $R^+$: absolute integral closure of $R$ (i.e. integral closure in an algebraic closure of the fraction field); contains $SR^{1/p^\infty}$ and is $R$-faithfully flat (Hochster-Huneke).

We now turn to the mixed characteristic case

$$R = W(k)[[x_2, \ldots, x_d]].$$

Replace $\overline{R^{1/p^\infty}}$ by introducing $p^{th}$-power roots of the system of parameters $x_1 = p, x_2, \ldots, x_d \leadsto \text{perfectoid world.}$
$K$: complete non-archimedean field
$K^0$: valuation ring, $K^{oo}$: valuation ideal.
Assume that the valuation is not discrete (equivalently: $K^{oo} = (K^{oo})^2$), and that the residue field $k$ is of char. $p > 0$.
Perfectoid notions

**Proposition [Gabber-Ramero]**

\[ K^0 / p \xrightarrow{\sim} K^0 / p \] is surjective iff for each finite separable \( L/K \), \( L^0 \) is almost etale over \( K^0 \), i.e. \( \Omega_{L^0/K^0} \) is killed by \( K^{oo} \).
Proposition [Gabber-Ramero]

$K^o/p \xrightarrow{\times_p} K^o/p$ is surjective iff for each finite separable $L/K$, $L^o$ is almost etale over $K^o$, i.e. $\Omega_{L^o/K^o}$ is killed by $K^{oo}$.

In this case, $K$ is a perfectoid field ($k$ is then perfect).

Ex. $K^o = W(k)[p^{1/p\infty}]$, $K = K^o[\frac{1}{p}]$ (basic perfectoid field in the sequel).
Here, **almost** is used in the sense of Almost Algebra: given a commutative ring $\mathcal{V}$ and an idempotent ideal $m$, “neglect" all $\mathcal{V}$-module killed by $m$.

Almost algebra (Faltings, Gabber-Ramero) goes much beyond mere categorical localization: notions of almost finite, almost flat, almost etale...

When $(\mathcal{V}, m) = (K^o, K^{ oo} = p^{1/p^\infty} K^o)$ as above, we say $p^{1/p^\infty}$-almost: “$p^{1/p^\infty}$-almost zero" means “killed by $p^{1/p^\infty}$".
Perfectoid notions

$A$: Banach $K$-algebra.

$A^o$: sub-$K^o$-algebra of power-bounded elements.

A is a perfectoid $K$-algebra if $A^o/p^{1/p} \xrightarrow{x \mapsto x^p} A^o/p$ is an isomorphism.

This implies that the norm of $A$ is equivalent to the spectral norm, and $A^o$ is the unit ball for the latter.

Ex. $A^o = R_{\infty} := \bigcup W(k)[p^{1/p}][[x_2^{p^i}, \cdots, x_d^{p^i}]], \ A = R_{\infty}[\frac{1}{p}].$
A: Banach $K$-algebra.
$A^0$: sub-$K^0$-algebra of power-bounded elements.

A is a perfectoid $K$-algebra if $A^0/p^{1/p} \xrightarrow{\phi} A^0/p$ is an isomorphism.

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Ex. $A^0 = R_\infty := \bigcup W(k)[1/p^j]\{[x_2^p, \ldots, x_d^p]\}, \quad A = R_\infty[1/p]$.

Theorem 2 (Almost purity [Faltings; Scholze, Kedlaya-Liu])

$A$: perfectoid $K$-algebra; $B$: finite etale $A$-algebra.
Then $B$ is perfectoid, and $B^0$ is an $p^{-1/p^\infty}$-almost finite etale $A^0$ algebra.
Perfectoid Abhyankar lemma

In the situation of DSC, $B = S \otimes_R R_{\infty}[\frac{1}{p}]$ may not be etale over $A = R_{\infty}[\frac{1}{p}]$, so Almost Purity does not apply (in fact, a finite extension of a perfectoid algebra need not be perfectoid).

*Abhyankar’s classical lemma*: under appropriate assumptions, one can achieve etaleness by adjoining roots of the discriminant.

We follow this strategy.

$g \in R = \mathcal{W}(k)[[x_2, \cdots, x_d]]$: a discriminant of $S[\frac{1}{p}]/R[\frac{1}{p}]$. We first note that adjoining $p^{th}$-power roots of $g$, in the (non-naive) sense of considering $R_{\infty}[\frac{1}{p}]\langle g^{1/p_{\infty}} \rangle^o$, is “harmless”:

**Theorem 3 [A.]**

$R_{\infty,g} := R_{\infty}[\frac{1}{p}]\langle g^{1/p_{\infty}} \rangle^o$ is $p^{1/p_{\infty}}$-almost faithfully flat over $R_{\infty}$. 

Yves André  Perfectoid spaces and homological conjectures
Perfectoid Abhyankar lemma

The following is both a perfectoid version of Abhyankar’s lemma, and a ramified version of the Almost Purity theorem.

**Theorem 4 [A.]**

- **A**: perfectoid $K$-algebra, containing a sequence of $p^{th}$-power roots of a non-zero divisor $g \in A^o$ (Ex: $A = R_{\infty,g[\frac{1}{p}]}$).
- **$B'$**: finite etale $A[\frac{1}{g}]$-algebra.
- **$B^o$**: integral closure of $A^o$ in $B'$ (hence $B^o[\frac{1}{pg}] = B'$).

Then for any $n$, $B^o/p^n$ is an $(pg)^{\cdot\frac{1}{p^\infty}}$-almost finite etale $A^o/p^n$ algebra.
Perfectoid Abhyankar lemma

The proofs of both theorems use deformation arguments of perfectoid spaces.

For Th. 3:

**Theorem 3 [A.]**

\[ R_{\infty}g := R_{\infty}[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}} \rangle^o \text{ is } p\frac{1}{p^\infty} \text{-almost faithfully flat over } R_{\infty}. \]

Spread out the perfectoid space \( X \) attached to \( R_{\infty} = \mathcal{O}^+(X) \).

\( X^{<\varepsilon} \): (perfectoid) \( \varepsilon \)-tubular neighborhood of \( X \).

- \( R_{\infty}g \) is \( p\frac{1}{p^\infty} \)-almost equal to \( \varprojlim_{\varepsilon} \mathcal{O}^+(X^{<\varepsilon}) \).

- Using Scholze’s (almost) description of \( \mathcal{O}^+(X^{<\varepsilon})/p^\varepsilon \) in terms of "Puiseux-like" series with coefficients in \( \mathcal{O}^+(X)/p^\varepsilon \), show that \( \mathcal{O}^+(X^{<\varepsilon})/p^\varepsilon \) is almost faithfully flat over \( \mathcal{O}^+(X)/p^\varepsilon \).
Perfectoid Abhyankar lemma

For Th. 4:
$X$: perfectoid space $X$ attached to $A^0 = \mathcal{O}^+(X)$,
$X_{>\varepsilon}$: (perfectoid) complement of $\varepsilon$-tubular neighborhood of the discriminant locus $g = 0$.

- $\mathcal{O}^+(X)$ is $(pg)^{1/\varphi}\overline{\mathcal{O}^{\infty}}$-almost equal to $\lim_{\varepsilon} \mathcal{O}^+(X_{>\varepsilon})$ (by Scholze’s perfectoid Riemann extension theorem).
- By almost purity over $X_{>\varepsilon}$, $(B'(\mathcal{O}(X_{>\varepsilon})))^+$ is almost finite etale over $\mathcal{O}^+(X_{>\varepsilon})$.
- Pass to the limit (main step).
\( R = W(k)[[x_2, \cdots, x_d]], \) \( S \) finite extension, etale outside \( pg = 0. \)

\( A = R_{\infty, g}[\frac{1}{p}], B' = S \otimes_R A[\frac{1}{g}] \leadsto S\)-algebra \( B^0 \) sitting on top of a tower

\[
R \xrightarrow{\alpha} R_{\infty} \xrightarrow{\beta} R_{\infty, g} \xrightarrow{\gamma} B^0
\]

where \( \alpha \) is faithfully flat,
\( \beta \) is \( p^{p^{\infty}} \)-almost faithfully flat,
\( \gamma \) is \((pg)^{p^{\infty}}\)-almost faithfully flat mod. \( p \).

Thus \( B^0 \) is "almost" our wanted \( T \).

How to get rid of "almost"?
(detour through Cohen-Macaulay notions)

$S$: Noetherian local ring, $T$: (possibly big) extension.

$T$ is a (big) Cohen-Macaulay $S$-algebra if any sequence of parameters $x_1, \ldots, x_d$ of $S$ becomes regular in $T$

(i.e. $x_1$ is non-zero-divisor in $T$, $x_2$ is non-zero divisor in $T/x_1$ etc, and $T \neq (x_1, \ldots, x_d)T$).

Hochster conjectured that such a $T$ always exists.

Ex. If $S$ complete of char. $p$, $S^+$ is a big Cohen-Macaulay $S$-algebra (Hochster-Huneke).
Lemma

Assume $S$ is a complete local domain, hence a finite extension of complete regular local ring $R$ (Cohen).
Then $T$ is a (big) Cohen-Macaulay $S$-algebra iff $T$ is $R$-faithfully flat.

Back to DSC: our $B^0$ is only an almost Cohen-Macaulay $S$-algebra. But one can pass from an Cohen-Macaulay $S$-algebra to a genuine Cohen-Macaulay $S$-algebra $T$ using either
- Hochster’s (classical) technique of algebra modifications, or
- Gabber’s ultraproduct (new) technique.
This is the last step in the proof of DSC and of Hochster’s conjectured existence of big CM algebras.
Main Theorem 5 [A. 2016, 2018]

- Any Noetherian local ring $S$ admits a big CM algebra $T$.
- For any local morphism $S \rightarrow S'$ of Noetherian complete local domains, there is morphism of respective big CM algebras $T \rightarrow T'$. 
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Corollary [Heitmann-Ma; A.]

Any subring of a regular ring, which is a direct summand as a module, is Cohen-Macaulay.
Big Cohen-Macaulay algebras

Theorem 6

- [A., Shimomoto] Any Noetherian complete local domain $S$ of char. $(0, p)$ admits an (integral) **perfectoid** CM algebra $T$.
- [Ma-Schwede] Any two such $T$ are dominated by a third.

This allows Ma and Schwede to develop
- an analog of **tight closure theory** (Hochster-Huneke) in mixed characteristic,
- in-depth study of **singularities in mixed characteristic**, where perfectoid CM algebras play somehow the role of resolution of singularities in equal char. 0.
Kunz’ theorem in mixed characteristic

- $R$: Noetherian ring of char. $p$.  
  Kunz’ classical theorem:  
  $R$ is regular iff $R \xrightarrow{x \mapsto x^p} R$ is flat,  
  iff there exists a perfect, faithfully flat $R$-algebra.

- $R$: Noetherian $p$-adically complete ring.

**Theorem 7 [Bhatt-Iyengar-Ma 2018]**

$R$ is regular iff there exists an integral perfectoid, faithfully flat $R$-algebra.
Applications to singularities

\( S \): local domain, essentially of finite type over \( \mathbb{C} \).

Rational singularity? (i.e. \( R \cong R\Gamma_{\mathcal{O}_Y} \) for a log-resolution \( Y \))

Criteria by reduction mod. \( p \) after spreading out.

\( S \) rational singularity
\( \iff (S \mod. p) \text{ } F\text{-rational singularity for } p \gg 0 \) (i.e. local cohomology = simple Frob.-module - checkable on Macaulay2)

\( M\text{-}S. \iff (S \mod. p) \text{ } F\text{-rational singularity for some } p. \)

\( \leadsto \) algorithm. Ma-Schwede use a perfectoid avatar of rational singularity in mixed char. as link between char. \( p \) and char. 0.
Applications to the homological conjectures

Homological turn in commutative algebra in the 60’s:
study of noetherian rings and their ideals (Krull, Zariski...) →
homological properties of their modules (Auslander, Serre...)

*Ex.* $R$ (local) is regular iff every finite $R$-module has a finite free resolution.

Peskine-Szpiro: reduction techniques to char. $p$ + extension of Kunz’ theorem: Frobenius preserves finite free resolutions.

*Ex.* of application [P.-S., Roberts]. $S$ (local) is Cohen-Macaulay iff there is an $S$-module of finite length with a finite free resolution.
Homological conjectures

\( S \): Noetherian local ring. \( M \): finite \( S \)-module with a finite free resolution.

**Theorem 8 [Evans-Griffiths; Hochster, Ogoma]**

(Under DSC)

Let \( 0 \rightarrow S^{b_s} \rightarrow S^{b_{s-1}} \rightarrow \ldots \rightarrow S^{b_0} \rightarrow M \rightarrow 0 \) be a minimal free resolution of \( M \).

Then \( b_i \geq 2i + 1 \) if \( i < s - 1 \), and \( b_{s-1} \geq s \).

Thanks to the perfectoid techniques, these optimal bounds are now *unconditional*.

In fact, all homological conjectures which were standing for a while on Hochster’s list are now solved, as consequences of the above theorems — but Hochster recently added a few more to his list...