Periods, cycles, and L-functions: a relative trace formula approach

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Two classical examples

The central theme of this talk

$$\begin{matrix}\text{Period integral} \\ \text{Algebraic cycle}\end{matrix} \longleftrightarrow \text{L-functions}$$

with an emphasis on the *relative trace formula* approach.
We first discuss two examples

- Dirichlet's solution to Pell's equation, and two formulas of Dirichlet.
- Heegner's solution to elliptic curve, and the formula of Gross–Zagier and of Birch–Swinnerton-Dyer.

Dirichlet's "explicit" solution to Pell's equation (1837)

Pell's equation

$$x^2 - dy^2 = \pm 1.$$

For simplicity, assume that $d = p \equiv 1 \mod 4$ is a prime. Dirichlet constructed an "explicit" triangular solution

$$\begin{aligned} x + y\sqrt{p} &= \theta_p \\ &= \frac{\prod_{a \not\equiv \square \mod p} \sin \frac{a\pi}{p}}{\prod_{b \equiv \square \mod p} \sin \frac{b\pi}{p}} \\ & 0 < a, b < p/2. \end{aligned}$$

Two formulas of Dirichlet

Let $\left(\frac{\cdot}{p}\right)$ denote the Legendre symbol for quadratic residues. Let

$$L\left(s,\left(\frac{\cdot}{p}\right)\right)=\sum_{n\geq 1,\, p\nmid n}\left(\frac{n}{p}\right)n^{-s}.$$

Dirichlet's first formula,

$$L'\left(0,\left(\frac{\cdot}{p}\right)\right)=\log\theta_p,$$

and the second formula

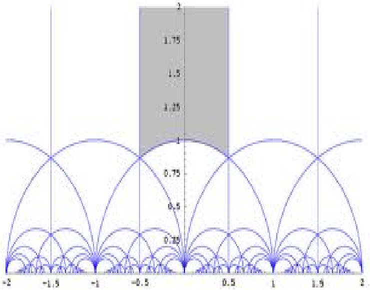
$$L'\left(0,\left(\frac{\cdot}{p}\right)\right)=h_p\log\epsilon_p,$$

where h_p is the class number and $\epsilon_p>1$ is the fundamental unit of $K=\mathbb{Q}[\sqrt{p}]$,

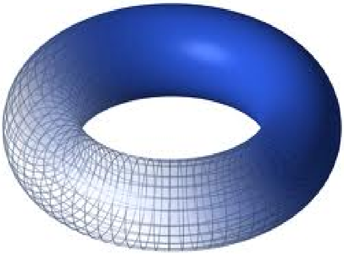
Modular parameterization of elliptic curves over $\mathbb{Q}$

- E: an elliptic curve over $\mathbb{Q}$.
- $\mathcal{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ the upper half plane.
- $\exists$ a modular parameterization

$$\varphi : \mathcal{H} \longrightarrow E_{\mathbb{C}}\ .$$



$\xrightarrow[\text{functions}]{\text{modular}}$



An example: Heegner (1950s), Birch(1960s-1970s)

The elliptic curve

$$E : y^2 = x^3 - 1728$$

is parameterized by (γ_2, γ_3):

$$\gamma_2(\tau) = \frac{E_4}{\eta^8} = \frac{1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n}{q^{1/3} \prod_{n=1}^{\infty} (1 - q^n)^8},$$

$$\gamma_3(\tau) = \frac{E_6}{\eta^{12}} = \frac{1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n}{q^{1/2} \prod_{n=1}^{\infty} (1 - q^n)^{12}},$$

where $q = e^{2\pi i \tau}, \tau \in \mathcal{H}$.

Modular solution: Heegner point

- $K = \mathbb{Q}[\sqrt{-d}] \subset \mathbb{C}$: a (suitable) imaginary quadratic number field.
- Heegner point: some of $\varphi(K \cap \mathcal{H})$ produces

$$\mathscr{P}_K \in E(K).$$

- $L(s, E/K)$: the Hasse–Weil L-function of E over K (centered at $s = 1$).

Theorem (Gross–Zagier formula (1980s))

There is an explicit $c > 0$ such that

$$L'(1, E/K) = c \cdot \langle \mathscr{P}_K, \mathscr{P}_K \rangle_{\mathrm{NT}}$$

where the RHS is the Nerón–Tate height pairing.

Conjecture of Birch and Swinnerton-Dyer (1960s)

- The order $r = \mathrm{ord}_{s=1} L(s, E/\mathbb{Q})$ equals to $\mathrm{rank}\, E(\mathbb{Q})$.
- the leading term of the Taylor expansion

$$\frac{L^{(r)}(1, E/\mathbb{Q})}{r! \cdot c_E} = \#\text{Ш} \cdot \mathrm{Reg}(E)$$

where

- Ш : Tate–Shafarevich group.
- $\mathrm{Reg}(E)$ is the regulator ($\sim$ the “volume" of the abelian group $E(\mathbb{Q})$ in $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ w.r.t. the Néron–Tate metric).
- $c_E = \Omega_E \prod_{\ell \text{ prime}} c_\ell$, Ω_E is the real period, c_ℓ the number of connected components of the special fiber of Néron model at ℓ.

Theorem (Skinner, Z., $\sim$ '14)

If $\operatorname{ord}_{s=1} L(s, E/\mathbb{Q}) = 3$ *(or any odd integer* ≥ 3*), then either*

- $\#Ш = \infty$ *, or*
- $\operatorname{rank} E(\mathbb{Q}) \geq 3$.

1. Two classical examples
2. Automorphic period and L-values
3. Special cycles and L-derivatives
4. Higher Gross–Zagier formula
5. Relative trace formula and arithmetic fundamental lemma

Automorphic period integral

- G reductive group over a global field F, and (*spherical*) $\mathrm{H} \subset \mathrm{G}$.
- The *automorphic quotients* $[\mathrm{H}] := \mathrm{H}(F)\backslash\mathrm{H}(\mathbb{A}) \longrightarrow [\mathrm{G}]$.
- π: a (tempered) cuspidal automorphic repn. of G.
- Automorphic period integral

$$\mathscr{P}_{\mathrm{H}}(\phi) := \int_{[\mathrm{H}]} \phi(h)dh, \quad \phi \in \pi.$$

- Automorphic periods are often related to (special) values of L-functions, e.g. the Rankin–Selberg pair $(\mathrm{GL}_{n-1}, \mathrm{GL}_{n-1} \times \mathrm{GL}_n)$.

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Gan–Gross–Prasad pairs (H, G)

- F'/F: quadratic extension of number fields.
- W: F'/F-Hermitian space, $\dim_{F'} W = n$.
- $W^\flat \subset W$, codimension one, $\mathrm{U}(W^\flat) \subset \mathrm{U}(W)$.
- Diagonal embedding

$$\mathrm{H} = \mathrm{U}(W^\flat) \subset \mathrm{G} = \mathrm{U}(W^\flat) \times \mathrm{U}(W).$$

The pair (H, G) is called the *unitary Gan–Gross–Prasad pair*. Similar construction applies to *orthogonal* groups.

Global Gan–Gross–Prasad conjecture

- (H, G): the Gan–Gross–Prasad pair (unitary/orthogonal).
- π: a tempered cusp. automorphic repn. of G.
- $L(s, \pi, R)$: the Rankin–Selberg L-function for the endoscopic functoriality transfer of π.

Conjecture (Gan–Gross–Prasad)

The following are equivalent

1. *The automorphic* H*-period integral does not vanish on* π*, i.e.,* $\mathscr{P}_{\mathrm{H}}(\phi) \neq 0$ *for some* $\phi \in \pi$*.*
2. $L(\frac{1}{2}, \pi, R) \neq 0$ *(and* $\mathrm{Hom}_{\mathrm{H}(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$*).*

The unitary Gan–Gross–Prasad pair

Theorem

Let (H, G) be the unitary GGP pair. The conjecture holds if

there exists a place v of F split in F' where π_v is supercuspidal.

Remark

- The same holds for a refined GGP conjecture of Ichino–Ikeda.
- $n = 2$ (i.e., $\mathrm{G} \simeq \mathrm{U}(1) \times \mathrm{U}(2)$): Waldspurger (1980s).
- $n > 2$: due to a series of work on Jacquet–Rallis relative trace formula by several authors: Yun, Beuzart-Plessis, Xue, and the author.
- Work in progress by Zydor and Chaudouard on the spectral side will remove the above local condition.

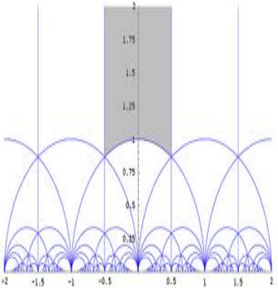
Shimura datum

Shimura datum: $(\mathrm{G}, X_{\mathrm{G}})$

- G: (connected) reductive group over $\mathbb{Q}$,
- $X_{\mathrm{G}} = \{h_{\mathrm{G}}\}$: a $\mathrm{G}(\mathbb{R})$-conjugacy class of $\mathbb{R}$-group homomorphisms $h_{\mathrm{G}} : \mathbb{C}^{\times} \to \mathrm{G}_{\mathbb{R}}$, satisfying Deligne's list of axioms (in particular, X_{G} is a *Hermitian symmetric domain*).

Examples of $(\mathrm{G}_{\mathbb{R}}, X_{\mathrm{G}})$

1. (Type A) $\mathrm{G}_{\mathbb{R}} = \mathrm{U}(r,s)$ (for $r+s=n$) and $X_{\mathrm{G}} = \frac{\mathrm{U}(r,s)}{\mathrm{U}(r)\times\mathrm{U}(s)}$. When $r=1$, $X_{\mathrm{G}} = D_{n-1} = \left\{z \in \mathbb{C}^{n-1} : z \cdot \overline{z} < 1\right\}$ is the unit ball.



$$\xrightarrow{\sim}$$

2. (Type B, D) Tube domains: $\mathrm{G}_{\mathbb{R}} = \mathrm{SO}(n,2)$, $X_{\mathrm{G}} = \frac{\mathrm{SO}(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}$.
3. (Type C) Siegel upper half space $\{z \in \mathrm{Symm}_{g\times g}(\mathbb{C}) : \mathrm{Im}(z) > 0\}$.

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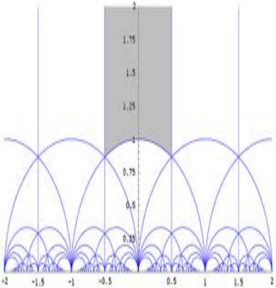

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Special pair of Shimura data

A *special pair* of Shimura data is a homomorphism

$$(\mathrm{H}, X_{\mathrm{H}}) \longrightarrow (\mathrm{G}, X_{\mathrm{G}})$$

such that

1. the pair (H, G) is *spherical*, and
2. the dimensions (as complex manifolds) satisfy

$$\dim_{\mathbb{C}} X_{\mathrm{H}} = \frac{\dim_{\mathbb{C}} X_{\mathrm{G}} - 1}{2}.$$

Example (Gross–Zagier pair)

Let $F = \mathbb{Q}[\sqrt{-d}]$ be an imaginary quadratic field. Let

$$\mathrm{H} = \mathrm{R}_{F/\mathbb{Q}}\mathbb{G}_m \subset \mathrm{G} = \mathrm{GL}_{2,\mathbb{Q}}.$$

Then $\dim X_{\mathrm{G}} = 1, \dim X_{\mathrm{H}} = 0$.

Some more examples (over $\mathbb{R}$)

1. Gan–Gross–Prasad pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$\mathrm{U}(1, n-2) \times \mathrm{U}(1, n-1)$	$\mathrm{U}(1, n-2)$
orthogonal groups	$\mathrm{SO}(2, n-2) \times \mathrm{SO}(2, n-1)$	$\mathrm{SO}(2, n-2)$

2. Symmetric pairs

	$G_{\mathbb{R}}$	$H_{\mathbb{R}}$
unitary groups	$\mathrm{U}(1, 2n-1)$	$\mathrm{U}(1, n-1) \times \mathrm{U}(0, n)$
orthogonal groups	$\mathrm{SO}(2, 2n-1)$	$\mathrm{SO}(2, n-1) \times \mathrm{SO}(0, n)$

Arithmetic diagonal cycles

We now focus on the *unitary* GGP pair (H, G) that can be enhanced to a special pair of Shimura data.

- The *arithmetic diagonal cycle*

$$\mathrm{Sh}_{\mathrm{H}} \longrightarrow \mathrm{Sh}_{\mathrm{G}} \ ,$$

  (for certain level sugroups $K_{\mathrm{H}}^{\circ}, K_{\mathrm{G}}^{\circ}$).

- $\exists$ a *PEL* type variant of the GGP Shimura varieties, with *smooth* integral models Sh_{H} and Sh_{G} [Rapoport–Smithling–Z. '17].

Define

$$\mathrm{Int}(f) = \Big(f * [\mathrm{Sh}_{\mathrm{H}}],\ [\mathrm{Sh}_{\mathrm{H}}]\Big)_{\mathrm{Sh}_{\mathrm{G}}}, \quad f \in \mathscr{H}(\mathrm{G}, K_{\mathrm{G}}^{\circ}),$$

where the action is through the Hecke correspondence associated to certain f in the Hecke algebra $\mathscr{H}(\mathrm{G}, K_{\mathrm{G}}^{\circ})$.

One version of the arithmetic GGP conjecture

Conjecture

There is a decomposition

$$\mathrm{Int}(f) = \sum_{\pi} \mathrm{Int}_{\pi}(f), \quad \textit{for all } f \in \mathscr{H}\left(\mathrm{G}, K_{\mathrm{G}}^{\circ}\right),$$

- π : *cohomological automorphic repn. of* $\mathrm{G}(\mathbb{A})$,
- Int_{π} : *eigen-distribution for the spherical Hecke algebra* $\mathscr{H}^{S}(\widetilde{\mathrm{G}})$ *away from the set* S *of bad places, with eigen-character given by* π.

Moreover, if π *is tempered, the following are equivalent*

1. $\mathrm{Int}_{\pi} \neq 0$.
2. $L'(\frac{1}{2}, \pi, R) \neq 0$ *(and* $\mathrm{Hom}_{\mathrm{H}(\mathbb{A}^{\infty})}(\pi^{\infty}, \mathbb{C}) \neq 0$*).*

Theorem (Gross–Zagier '86, Yuan–S. Zhang–Z. '12)

When $n = 2$ (i.e., $\mathrm{G} = \mathrm{U}(1) \times \mathrm{U}(2)$), the conjecture holds.

Corollary

Let F be a totally real number field, and π a cusp. automorphic repn. of $\mathrm{PGL}_2(\mathbb{A}_F)$ with π_∞ parallel weight two. Then

$$\mathscr{L}'(1/2, \pi) \geq 0.$$

Question: What about $n \geq 3$, i.e., when the Shimura variety is of dimension higher than one?

GGP, and Arithmetic GGP

Central value	1^{st} central derivative
Waldspurger	Gross–Zagier
↓	↓
GGP Ichino–Ikeda	Arithmetic GGP

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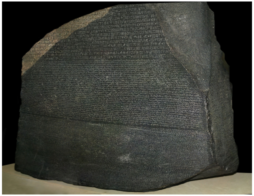
Higher Gross–Zagier formula (in positive equal char. case)

$\left\{ \begin{array}{c} \text{Number fields} \\ [F : \mathbb{Q}] < \infty \end{array} \right\}$ ⟷ $\left\{ \begin{array}{c} \text{Function fields} \\ [F : \mathbb{F}_q(t)] < \infty \end{array} \right\}$



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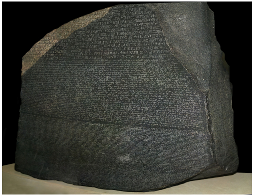
Higher Gross–Zagier formula (in positive equal char. case)

{ Arithmetic curves } ⟷ { Algebraic curves $/\mathbb{F}_q$ }



{ Riemann surfaces }

Drinfeld Shtukas

- $k = \mathbb{F}_q$, and X/k a curve.
- Shtukas of rank n with r-legs: for S over $\mathrm{Spec} k$

$$\mathrm{Sht}^r_{\mathrm{GL}_n, X}(S) = \left\{ \begin{array}{c} \text{vector bundles } \mathcal{E} \text{ of rank } n \text{ on } X \times S \\ + \text{simple modification } \mathcal{E} \to (\mathrm{id} \times \mathrm{Frob}_S)^* \mathcal{E} \\ \text{at } r\text{-marked points } x_i : S \to X, 1 \leq i \leq r \end{array} \right\}$$

$$\begin{array}{c} \mathrm{Sht}^r_{\mathrm{GL}_n, X} \\ \downarrow \\ X^r = \underbrace{X \times_{\mathrm{Spec} k} \cdots \times_{\mathrm{Spec} k} X}_{r \text{ times}} \end{array}$$

The special case $r = 0$, $\mathrm{G} = \mathrm{GL}_n$

$\mathrm{Sht}_{\mathrm{G}}^{r=0} = \mathrm{Bun}_{\mathrm{G}}(k)$

$\updownarrow$

{ rank n vector bundles over X}

$\updownarrow$ Weil

$[\mathrm{G}] = \mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A})$

$H_c^0(\mathrm{Sht}_{\mathrm{G}}^{r=0})$

$\updownarrow$

$C_c^\infty([\mathrm{G}])$

Heegner–Drinfeld cycle

Fix an étale double covering $X' \to X$. We have a natural morphism

$$\mathrm{Sht}^r_{\mathrm{GL}_{n/2},X'} \longrightarrow \mathrm{Sht}^r_{\mathrm{GL}_n,X}.$$

They have dimensions

$$\frac{nr}{2}, \qquad nr.$$

A technical simplification: we pass to PGL_n, then take base change to $(X')^r$:

$$\theta^r : \mathrm{Sht}^r_{\mathrm{H}} \longrightarrow \mathrm{Sht}'^r_{\mathrm{G}} := \mathrm{Sht}^r_{\mathrm{G}} \times_{X^r} (X')^r$$

where

$$\mathrm{H} = \mathrm{R}_{X'/X}(\mathrm{GL}_{n/2})/\mathbb{G}_{m,X} \subset \mathrm{G} = \mathrm{PGL}_{n,X}.$$

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Higher Gross–Zagier formula, $n = 2$

- Now $\mathrm{G} = \mathrm{PGL}_2$, and $\mathrm{Sht}'^{r}_{\mathrm{G}}$, for even integer $r \geq 0$.
- $V_r = H_c^{2r}\left(\mathrm{Sht}'^{r}_{\mathrm{PGL}_2} \otimes_k \overline{k}, \overline{\mathbb{Q}}_\ell\right)$ has a spectral decomposition

$$V_r = \left(\bigoplus_{\pi} V_{r,\pi}\right) \oplus \text{“Eisenstein part''},$$

π : unramified cusp. automorphic repn. of $\mathrm{PGL}_2(\mathbb{A})$.

- $L(s, \pi_{X'})$: the (normalized) base change L-function.

Theorem (Yun–Z.)

Let $Z_r \in V_r$ be the cycle class of Heegner–Drinfeld cycle, and $Z_{r,\pi} \in V_{r,\pi}$. Then

$$L^{(r)}(1/2, \pi_{X'}) = c \cdot \left(Z_{r,\pi}, Z_{r,\pi}\right),$$

where $(\cdot,\cdot)$ is the intersection pairing, and $c > 0$ is explicit.

A comparison with the number field case

1. When $r = 0$, the automorphic quotient space (versus $\mathrm{Bun}_n(\mathbb{F}_q)$)

$$[\mathrm{G}] = \mathrm{G}(F)\backslash\mathrm{G}(\mathbb{A}).$$

2. When $r = 1$, Shimura variety (versus moduli of Shtukas)

$$\begin{array}{ccc} \mathrm{Sh}_{\mathrm{G}} & \qquad & \mathrm{Sht}^{r}_{\mathrm{GL}_n} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z} & \qquad & X^r = \underbrace{X \times_{\mathrm{Spec}k} \cdots \times_{\mathrm{Spec}k} X}_{r \text{ times}} \end{array}$$

An indirect example: Faltings heights of CM abelian varieties

Kronecker limit formula for an imaginary quadratic field $K = \mathbb{Q}[\sqrt{-d}]$:

$$h_{\mathrm{Fal}}(E_d) = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2}\log|d|,$$

where E_d is an elliptic curve with complex multiplication by O_K.
Colmez conjecture generalizes the identity to CM abelian varieties.

Faltings heights of CM abelian varieties	$\longleftrightarrow$	$d\log$ of L-functions totally negative Artin repn. of $\mathrm{Gal}_{\mathbb{Q}}$

An *averaged* version is recently proved by Yuan–S. Zhang and by Andreatta–Goren–Howard–Madapusi-Pera.

A summary

Central value	1^{st} derivative	r^{th} derivative
Waldspurger	Gross–Zagier	Higher G-Z
↓	↓	↓
GGP Ichino–Ikeda	Arithmetic GGP	$* * *$

Relative trace formula (RTF)

The basic strategy is to compare two relative trace formulas:

- one for the “geometric" side (intersection numbers of algebraic cycles),
- the other for the “analytic" side (L-values).

Below we consider the two cases

- Higher Gross–Zagier formula.
- GGP and its arithmetic version.

Geometric RTF (over function fields)

Geometric side: Let f be an element in the spherical Hecke algebra $\mathscr{H}$. Set

$$\mathrm{Int}_r(f) := \left(f * [\mathrm{Sht}^r_{\mathrm{H}}], \quad [\mathrm{Sht}^r_{\mathrm{H}}] \right)_{\mathrm{Sht}'^r_{\mathrm{G}}}.$$

Analytic side: consider the triple $(\mathrm{G}', \mathrm{H}'_1, \mathrm{H}'_2)$ where $\mathrm{G}' = \mathrm{G} = \mathrm{PGL}_2$ and $\mathrm{H}'_1 = \mathrm{H}'_2$ are the diagonal torus A of PGL_2.

$$\mathbb{J}(f, s) := \int_{[\mathrm{H}'_1]} \int_{[\mathrm{H}'_2]} K_f(h_1, h_2)\, |h_1 h_2|^s\, \eta(h_2)\, dh_1\, dh_2, \quad s \in \mathbb{C}$$

where $\eta_{F'/F}$ is a quadratic character, and

$$K_f(x, y) := \sum_{\gamma \in \mathrm{G}'(F)} f(x^{-1}\gamma y), \quad x, y \in \mathrm{G}'(\mathbb{A}), f \in \mathscr{C}_c^\infty(\mathrm{G}'(\mathbb{A})).$$

Note that this is a weighted version of

$$\left(f * [\mathrm{Sht}^0_{\mathrm{H}'_1}], [\mathrm{Sht}^0_{\mathrm{H}'_2}] \right)_{\mathrm{Sht}^0_{\mathrm{G}}} = \left(f * [\mathrm{Bun}_A(k)], [\mathrm{Bun}_A(k)] \right)_{\mathrm{Bun}_{\mathrm{G}}(k)}.$$

Geometric RTF (over function fields)

Let

$$\mathbb{J}_r(f) = \frac{d^r}{ds^r}\Big|_{s=0} \mathbb{J}(f, s).$$

The following *key identity*, which we may call a *geometric RTF* (in contrast to the arithmetic intersection numbers in the number field case (AGGP) below).

Theorem (Yun–Z.)

Let $f \in \mathscr{H}$. Then

$$\mathbb{I}_r(f) = (-\log q)^{-r} \mathbb{J}_r(f).$$

Informally we may state the identity as

$$\left(f * [\mathrm{Sht}^r_{\mathrm{H}}], [\mathrm{Sht}^r_{\mathrm{H}}]\right)_{\mathrm{Sht}'^r_{\mathrm{G}}} \text{``} = \text{''} \frac{d^r}{ds^r}\Big|_{s=0} \left(f_{s,\eta} * [\mathrm{Sht}^0_A], [\mathrm{Sht}^0_A]\right)_{\mathrm{Sht}^0_{\mathrm{G}}}.$$

Jacquet–Rallis RTF: analytic side

We now move to the number field case. Similarly, we define linear functionals on Hecke algebras:

- $\mathbb{I}(f)$ for the unitary GGP triple $(\mathrm{G}, \mathrm{H}, \mathrm{H})$, and
- $\mathbb{J}(f', s)$ for the Jacquet–Rallis triple $(\mathrm{G}', \mathrm{H}'_1, \mathrm{H}'_2)$ where

$$\begin{aligned} \mathrm{G}' &= \mathrm{R}_{F'/F}(\mathrm{GL}_{n-1} \times \mathrm{GL}_n) \\ \mathrm{H}'_1 &= \mathrm{R}_{F'/F}\mathrm{GL}_{n-1}, \quad \mathrm{H}'_2 = \mathrm{GL}_{n-1} \times \mathrm{GL}_n. \end{aligned}$$

Then we have an analogous RTF identity

Theorem

There is a natural correspondence (for nice test functions) $f \leftrightarrow f'$ such that

$$\mathbb{I}(f) = \mathbb{J}(f', 0).$$

An arithmetic intersection conjecture

Let

$$\partial\mathbb{J}(f') = \frac{d}{ds}\Big|_{s=0}\mathbb{J}(f', s).$$

Recall we have defined an arithmetic intersection number

$$\mathrm{Int}(f) = \Big(f * [\mathrm{Sh_H}],\ [\mathrm{Sh_H}]\Big)_{\mathrm{Sh_G}}, \quad f \in \mathscr{H}\left(\mathrm{G}, K_{\mathrm{G}}^{\circ}\right).$$

Conjecture (Z. '12, Rapoport–Smithling–Z. '17)

There is a natural correspondence (for nice test functions) $f \leftrightarrow f'$ such that

$$\mathrm{Int}(f) = -\partial\mathbb{J}(f').$$

Connection to L-functions

For nice f', we have a decomposition as a sum of *relative characters* for the triple $(\mathrm{G}', \mathrm{H}'_1, \mathrm{H}'_2)$

$$\mathbb{J}(f', s) = \sum_{\Pi} \mathbb{J}_{\Pi}(f', s),$$

and, for cuspidal Π, a factorization into certain *local relative characters*

$$\mathbb{J}_{\Pi}(f', s) = 2^{-2} \mathscr{L}(s + 1/2, \pi) \prod_{v} \mathbb{J}_{\Pi_v}(f'_v, s).$$

Passing to the local situation

Number fields $[F : \mathbb{Q}] < \infty$ ⇠ ⇢ Function fields $[F : \mathbb{F}_q(t)] < \infty$

⇣ ⇣

p-adic fields $[F : \mathbb{Q}_p] < \infty$ ⇠ ⇢ local function fields $[F : \mathbb{F}_q((t))] < \infty$

Unitary Rapoport–Zink space

- F'/F : an unramified quadratic extension of p-adic fields.
- $\mathbb{X}_n$: n-dim'l *Hermitian supersingular formal* $O_{F'}$*-modules of signature* $(1, n-1)$ (unique up to isogeny).
- $\mathcal{N}_n$: the unitary Rapoport–Zink formal moduli space over $\mathrm{Spf}(O_{\breve{F}})$ (parameterizing "deformations" of $\mathbb{X}_n$).
- The group $\mathrm{Aut}^0(\mathbb{X}_n)$ is a unitary group in n-variable and acts on $\mathcal{N}_n$.
- The $\mathcal{N}_n$'s are non-archimedean analogs of Hermitian symmetric domains. They have a "skeleton" given by a union of Deligne–Lusztig varieties for unitary groups over finite fields.

Local intersection numbers

- A natural closed embedding $\delta : \mathcal{N}_{n-1} \to \mathcal{N}_n$, and its graph

$$\Delta : \mathcal{N}_{n-1} \longrightarrow \mathcal{N}_{n-1,n} = \mathcal{N}_{n-1} \times_{\mathrm{Spf} O_{\breve{F}}} \mathcal{N}_n.$$

  Denote by $\Delta_{\mathcal{N}_{n-1}}$ the image of Δ.

- The group $\mathrm{G}(F) := \mathrm{Aut}^0(\mathbb{X}_{n-1}) \times \mathrm{Aut}^0(\mathbb{X}_n)$ acts on $\mathcal{N}_{n-1,n}$. For (nice) $g \in \mathrm{G}(F)$, we define the intersection number

$$\begin{aligned} \mathrm{Int}(g) &= \left(\Delta_{\mathcal{N}_{n-1}}, g \cdot \Delta_{\mathcal{N}_{n-1}}\right)_{\mathcal{N}_{n-1,n}} \\ &:= \chi\left(\mathcal{N}_{n-1,n}, \mathcal{O}_{\Delta_{\mathcal{N}_{n-1}}} \otimes^{\mathbb{L}} \mathcal{O}_{g \cdot \Delta_{\mathcal{N}_{n-1}}}\right). \end{aligned}$$

The arithmetic fundamental lemma (AFL) conjecture

Define a family of (weighted) orbital integrals:

$$\mathrm{Orb}\left(\gamma, \mathbf{1}_{\mathfrak{gl}_n(O_F)}, s\right) = \int_{\mathrm{GL}_{n-1}(F)} \mathbf{1}_{\mathfrak{gl}_n(O_F)}(h^{-1}\gamma h) \left| \det(h) \right|^s (-1)^{\mathrm{val}(\det(h))} \, dh.$$

This serves as the local version of the analytic RTF. Then the local version of the global “arithmetic intersection conjecture" is

Conjecture (Z. '12)

Let $\gamma \in \mathfrak{gl}_n(F)$ *match an element* $g \in \mathrm{G}(F)$*. Then*

$$\pm \frac{d}{ds}\Big|_{s=0} \mathrm{Orb}\left(\gamma, \mathbf{1}_{\mathfrak{gl}_n(O_F)}, s\right) = -\mathrm{Int}(g) \cdot \log q.$$

The status

Theorem (Z. '12)

The AFL conjecture holds when $n \leq 3$.

A simplified proof when $p \geq 5$ is given by Mihatsch.
For $n > 3$, we only have some partial results.

Theorem (Rapoport–Terstiege–Z. '13)

When $p \geq \frac{n}{2} + 1$, the AFL conjecture holds for minuscule elements $g \in \mathrm{G}(F)$.

A simplified proof is given by Chao Li and Yihang Zhu.

Ramified quadratic extension F'/F

- Non-archimedean ramified F'/F (Rapoport–Smithling–Z. '15, '16): an *arithmetic transfer* (AT) conjecture, and the case $n \leq 3$ is proved.
- Question: what about archimedean F'/F?

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Thank you!

Periods, cycles, and L-functions: a relative trace formula approach

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