Sphere packing and Fourier interpolation

M. Viazovska

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Let $\mathbb{R}^d$ be Euclidean vector space. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$ we denote by $B_d(x, r)$ the ball in $\mathbb{R}^d$ with center $x$ and radius $r$. 
Let $\mathbb{R}^d$ be Euclidean vector space. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$ we denote by $B_d(x, r)$ the ball in $\mathbb{R}^d$ with center $x$ and radius $r$.

Let $X \subset \mathbb{R}^d$ be a set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union

$$\mathcal{P} = \bigcup_{x \in X} B_d(x, 1)$$

is a sphere packing.
The finite density of a packing $\mathcal{P}$ is defined as

$$
\Delta_{\mathcal{P}}(r) := \frac{\text{Vol}(\mathcal{P} \cap B_d(0, r))}{\text{Vol}(B_d(0, r))}, \quad r > 0.
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We define the *density* of a packing $\mathcal{P}$ as the limit superior

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$$\Delta_{\mathcal{P}} := \limsup_{r \to \infty} \Delta_{\mathcal{P}}(r).$$

The *sphere packing constant* is the supremum over all possible packing densities

$$\Delta_d := \sup_{\mathcal{P} \subset \mathbb{R}^d \text{ sphere packing}} \Delta_{\mathcal{P}}.$$
What is known about $\Delta_d$?

$\Delta_1 = 1$

$\Delta_2 = \frac{\pi}{\sqrt{12}} \approx 0.9069$

$\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$
What is known about $\Delta_d$

The best known density

Cohn-Elkies upper bound

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Sphere packing and Fourier interpolation
The $E_8$-lattice $\Lambda_8 \subset \mathbb{R}^8$ is given by

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The minimal distance between two points in $\Lambda_8$ is $\sqrt{2}$. 

Theorem (V. 2016)
No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the $E_8$-lattice packing. Therefore $\Delta_8 = \pi \frac{4}{384} \approx 0.25367$. 

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No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the $E_8$-lattice packing. Therefore $\Delta_8 = \frac{\pi^4}{384} \approx 0.25367$. 
The Leech lattice $\Lambda_{24}$ is an even, unimodular lattice of rank 24.

The minimal distance between two points in $\Lambda_{24}$ is 2. The $\Lambda_{24}$-lattice sphere packing is the packing of unit balls with centers at $\Lambda_{24}$.

Theorem (Cohn, Kumar, Miller, Radchenko, V. 2016) No packing of unit balls in Euclidean space $\mathbb{R}^{24}$ has density greater than that of the $\Lambda_{24}$-lattice packing. Therefore $\Delta_{24} = \frac{\pi}{12} \frac{1}{12!} \approx 0.00193$. 

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Theorem (Cohn, Elkies 2003)

Suppose that $r_0 > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function such that:

- $f(x) \leq 0$ for $\|x\| \geq r_0$
- $\hat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$
- $f(0) = \hat{f}(0) = 1$

Then

$$\Delta_d \leq \text{Vol}(B_d(0, r_0/2)).$$
Proof

Let $X \subset \mathbb{R}^d$ and $\|x - y\| \geq r_0$ for any distinct $x, y \in X$
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$$\geq \sum_{x \in X} \sum_{y \in X/L} f(x - y) = \sum_{x \in X/L} \sum_{y \in X/L} \sum_{l \in L} f(x - y + l)$$
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$$= \sum_{x \in X/L} \sum_{y \in X/L} \frac{1}{\text{vol}(\mathbb{R}^d/L)} \sum_{m \in L^*} \hat{f}(m) e^{2\pi im(x - y)}$$

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= \sum_{x \in X/L} \sum_{y \in X/L} \frac{1}{\text{vol}(\mathbb{R}^d/L)} \sum_{m \in L^*} \hat{f}(m) e^{2\pi i m(x - y)}
\]

\[
= \frac{1}{\text{vol}(\mathbb{R}^d/L)} \sum_{m \in L^*} \hat{f}(m) \cdot \left| \sum_{x \in X/L} e^{2\pi i m x} \right|^2 \geq \#(X/L) \cdot f(0).
\]

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$$= \frac{1}{\text{vol}(\mathbb{R}^d/L)} \sum_{m \in L^*} \hat{f}(m) \cdot \sum_{x \in X/L} |e^{2\pi imx}|^2 \geq \#(X/L)^2 \cdot \hat{f}(0).$$
Theorem (Cohn, Elkies 2003)

\[ \Delta_8 \leq 1.00016 \Delta_{E_8}, \]

\[ \Delta_{24} \leq 1.019 \Delta_{\Lambda_{24}}. \]
Theorem (V 2016)
There exists a radial Schwartz function $f_{E_8} : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:

\[
\begin{align*}
    f_{E_8}(x) &\leq 0 \text{ for } \|x\| \geq \sqrt{2} \\
    \hat{f}_{E_8}(x) &\geq 0 \text{ for all } x \in \mathbb{R}^8 \\
    f_{E_8}(0) &= \hat{f}_{E_8}(0) = 1.
\end{align*}
\]
Solution of the Cohn-Elkies linear programming problem

Theorem (Cohn, Kumar, Miller, Radchenko, V 2016)
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\[ f_{\Lambda_{24}}(0) = \hat{f}_{\Lambda_{24}}(0) = 1. \]
Plot of the “magic” function $f_{E_8}$ and its Fourier transform $\hat{f}_{E_8}$

$$f_{E_8}(x) e^{\pi \|x\|^2}$$

$$\hat{f}_{E_8}(y) e^{\pi \|y\|^2}$$
Remark 1
Without loss of generality we may assume that $f_{E_8}$ is radial.
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By the Poisson summation formula we have

$$f_{E_8}(0) \geq \sum_{\ell \in \Lambda_8} f_{E_8}(\ell) = \sum_{\ell \in \Lambda_8} \hat{f}_{E_8}(\ell) \geq \hat{f}_{E_8}(0).$$

This can happen only if

$$f_{E_8}(\sqrt{2n}) = \hat{f}_{E_8}(\sqrt{2n}) = 0 \text{ for all } n \in \mathbb{Z}_{>0}.$$
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By the Poisson summation formula we have
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This can happen only if $f_{E_8}(\sqrt{2n}) = \hat{f}_{E_8}(\sqrt{2n}) = 0$ for all $n \in \mathbb{Z}_{>0}$.

Remark 3
We have constructed the function $f_{E_8}$ in the form
$$f_{E_8}(r) = \sin(\pi r^2/2)^2 \int_0^\infty \varphi(it) e^{-\pi r^2 t} dt$$
where $\varphi$ is a holomorphic function on the upper half-plane.
Let $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} | \Im(z) > 0\}$. Consider the modular group $\Gamma_1 := \text{PSL}_2(\mathbb{Z})$. The group $\Gamma_1$ acts on $\mathbb{H}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}.$$
Fourier interpolation

The idea behind our construction of $f_{E_8}$ and $f_{\Lambda_{24}}$ is the hypothesis that a radial Schwartz function $p$ can be uniquely reconstructed from the values

$$\{p(\sqrt{2n}), p'(\sqrt{2n}), \hat{p}(\sqrt{2n}), \hat{p}'(\sqrt{2n})\}_{n=0}^{\infty}.$$
The idea behind our construction of $f_{E_8}$ and $f_{\Lambda_{24}}$ is the hypothesis that a radial Schwartz function $p$ can be uniquely reconstructed from the values 
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\{p(\sqrt{2n}), p'(\sqrt{2n}), \tilde{p}(\sqrt{2n}), \tilde{p}'(\sqrt{2n})\}_{n=0}^{\infty}.
\]

The proof of this statement is a goal an ongoing project of the author in collaboration with H. Cohn, A. Kumar, S. D. Miller, and D. Radchenko.
Theorem (Radchenko, V. 2017)

There exists a collection of Schwartz functions $c_0, a_n : \mathbb{R} \rightarrow \mathbb{R}$ with the property that for any Schwartz function $p : \mathbb{R} \rightarrow \mathbb{R}$ and any $x \in \mathbb{R}$ we have

$$p(x) = c_0(x) \ p'(0) + \sum_{n \in \mathbb{Z}} a_n(x) \ p(\text{sign}(n) \sqrt{|n|})$$

$$+ \hat{c}_0(x) \ \hat{p}'(0) + \sum_{n \in \mathbb{Z}} \hat{a}_n(x) \hat{p}(\text{sign}(n) \sqrt{|n|}),$$

where the right-hand side converges absolutely.
Interpolating basis functions

Figure: Plots of \( b_n(x) := a_n(x) + a_n(-x) \) and \( \hat{b}_n \) for \( n = 0, 1, 2 \).
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**Example**

**Dirac comb**

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\mu_{\text{Dirac}} = \sum_{x \in \mathbb{Z}} \delta_x.
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Crystalline measures

Theorem (Lev, Olevskii 2013)

If $\mu$ is a crystalline measure, and $X$ and $Y$ are uniformly discrete, then $\mu$ is a generalized Dirac comb.

The interpolation formula implies that there exists a continuous family of exotic crystalline measures

$$\mu_x := \delta_x - \sum_{n=0}^{\infty} b_n(x) \delta_{\sqrt{n}}.$$
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Proof of the interpolation formula: explicit construction of the interpolation basis

We will separately consider the odd and even components of the Schwartz functions
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For $x \in \mathbb{R}$ and $\tau \in \mathbb{H}$ we define

$$G(x, \tau) := \sum_{n=0}^{\infty} b_n(x) e^{\pi in\tau}$$

$$\tilde{G}(x, \tau) := \sum_{n=0}^{\infty} \hat{b}_n(x) e^{\pi in\tau}.$$
The interpolation formula applied to the Gaussian $e^{\pi ix^2\tau}$ gives

$$e^{\pi ix^2\tau} = G(x, \tau) + \frac{1}{\sqrt{-i\tau}} \tilde{G}(x, \frac{-1}{\tau}).$$

We solve this functional equation explicitly using the Eichler cohomology and the theory of modular integrals.
Warning

⚠️ Work in progress

- Interpolation for the Goursat problem
  Joint work with Andrew Bakan, Haakan Hedenmalm, Alfonso Montes-Rodriguez, Danilo Radchenko

- Universal optimality of $E_8$ and $\Lambda_{24}$
  Joint work with Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko
Goursat problem

Find a function $U : \mathbb{R}^2 \to \mathbb{R}$ such that

$$U_{xy} + U = 0, \ x, y \in \mathbb{R}; \quad U(x, 0) = \varphi(x), \ U(0, y) = \psi(y), \ x, y \in \mathbb{R}, \quad (1)$$

for given $\varphi, \psi \in C(\mathbb{R})$ satisfying $\varphi(0) = \psi(0)$. 

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Sphere packing and Fourier interpolation
Interpolation for the Goursat problem

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**Growth condition**

$$|U(x, y)| \leq C \exp(\theta(|x| + |y|)), \quad \theta \in [0, 1).$$
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Growth condition

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Theorem (H. Hedenmalm, A. Montes-Rodriguez)
Suppose that $U(x, y) = \int_{\mathbb{R}} e^{ixt + iy/t} a(t)dt$ for some $a \in L_1(\mathbb{R})$. Then the equalities

$U(\pi n, 0) = U(0, -\pi n) = 0, \quad n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$

imply $U(x, -y) = 0$ for all $x, y \geq 0$ and therefore, such $U$ is uniquely determined by its values at the points $\{ (\pi n, 0) \}_{n \in \mathbb{N}_0}$ and $\{ (0, -\pi n) \}_{n \in \mathbb{N}}$. 

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Interpolation for the Goursat problem

Let $U$ be a real-valued continuous solution of the canonical telegraph PDE

$$U_{xy} + U = 0, \ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_-, \$$

satisfying

$$|U(x, y)| \leq C e^{\theta(|x| + |y|)}, \ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_-, \$$

for some $\theta \in [0, 1)$ and $C \in (0, \infty)$. Suppose that $U(x, y) = \int_{\mathbb{R}} e^{ixt + iy/t} a(t) dt$ for some $a \in L_1(\mathbb{R})$. 

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Sphere packing and Fourier interpolation
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**Theorem (Bakan, Hedenmalm, Montes-Rodriguez, Radchenko, V. 2018+)**

1. There exist continuous solutions \( \{ R_n \} \) of the canonical telegraph PDE such that each \( R_n \) is uniformly bounded on \( \mathbb{R}_+ \times \mathbb{R}_- \),

\[
(a) \quad R_0(\pi m, 0) = \delta_{0m}, \quad R_0(0, -\pi m) = \delta_{0m}, \quad m \geq 0,
\]

\[
(b) \quad R_n(\pi m, 0) = \delta_{nm}, \quad R_n(0, -\pi m) = 0, \quad m \geq 0, \quad n \geq 1,
\]

and the functions \( R_n(t, -y), R_n(x, -t) \) of \( t \) belong to \( S(\mathbb{R}_+) \) for every \( n \geq 0 \) and \( x, y \geq 0 \).

2. If \( \sum_{n \geq 1} \sqrt{n} (|U(\pi n, 0)| + |U(0, -\pi n)|) < +\infty \) then for arbitrary \( (x, y) \in \mathbb{R}_+ \times \mathbb{R}_- \) we have the following absolutely convergent expansion

\[
U(x, y) = U(0, 0) R_0(x, y) + \sum_{n \geq 1} \left[ U(\pi n, 0) R_n(x, y) + U(0, -\pi n) R_n(-y, -x) \right].
\]
Potential energy

Given a potential function $p: (0, \infty) \to \mathbb{R}$, we define the potential energy of a finite subset $C$ of $\mathbb{R}^d$ to be

$$\frac{1}{|C|} \sum_{x,y \in C, \ x \neq y} p(|x - y|).$$
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Let $C$ be a discrete, closed subset of $\mathbb{R}^d$. We say $C$ has density $\rho$ if

$$\lim_{r \to \infty} \frac{|C \cap B_d(0, r)|}{\text{vol}(B_d(0, r))} = \rho.$$
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The lower $p$-energy of a nonempty, discrete, closed subset $C$ of $\mathbb{R}^d$ is

$$E_p(C) := \liminf_{r \to \infty} \frac{1}{|C \cap B_d(0, r)|} \sum_{x,y \in C \cap B_d(0, r), x \neq y} p(|x - y|).$$

If the limit of the above quantity exists then we call $E_p(C)$ the $p$-energy of $C$. 

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Sphere packing and Fourier interpolation
Let $C$ be a discrete subset of $\mathbb{R}^d$ with density $\rho$, where $\rho > 0$, and let $p : (0, \infty) \to \mathbb{R}$ be any function. We say that $C$ minimizes energy for $p$ if its $p$-energy $E_p(C)$ exists and every configuration in $\mathbb{R}^d$ of density $\rho$ has lower $p$-energy at least $E_p(C)$. We also call $C$ a ground state for $p$. 
Let $C$ be a discrete subset of $\mathbb{R}^d$ with density $\rho$, where $\rho > 0$. We say $C$ is *universally optimal* if it minimizes $p$-energy whenever $p: (0, \infty) \to \mathbb{R}$ is a completely monotonic function of squared distance.
Let $\mathcal{C}$ be a discrete subset of $\mathbb{R}^d$ with density $\rho$, where $\rho > 0$. We say $\mathcal{C}$ is universally optimal if it minimizes $p$-energy whenever $p: (0, \infty) \to \mathbb{R}$ is a completely monotonic function of squared distance.

**Conjecture (Cohn, Kumar)**
The lattices $\mathbb{Z}$, $A_2$, $\Lambda_8$ and $\Lambda_{24}$ are universally optimal.
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The lattice $\mathbb{Z}$ is universally optimal.

**Theorem (Cohn, Kumar, Miller, Radchenko, V 2018+)**
The lattices $\Lambda_8$ and $\Lambda_{24}$ are universally optimal.
Idea of the proof: linear programming

Theorem (Cohn, Kumar)

Let $p : (0, \infty) \to \mathbb{R}$ be any function, and suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function. If $f(x) \leq p(|x|)$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^d$, then every subset of $\mathbb{R}^d$ with density $\rho$ has lower $p$-energy at least $\rho \hat{f}(0) - f(0)$. 
Theorem (2018+)

Let \((d, n_0)\) be \((8, 1)\) or \((24, 2)\). There exists a collection of radial Schwartz functions \(a_n, b_n, \tilde{a}_n, \tilde{b}_n : \mathbb{R}^d \to \mathbb{R}\) such that for every \(f \in S_{\text{rad}}(\mathbb{R}^d)\) and \(x \in \mathbb{R}^d\),

\[
f(x) = \sum_{n=n_0}^{\infty} f(\sqrt{2n}) a_n(x) + \sum_{n=n_0}^{\infty} f'(\sqrt{2n}) b_n(x)
+ \sum_{n=n_0}^{\infty} \hat{f}(\sqrt{2n}) \tilde{a}_n(x) + \sum_{n=n_0}^{\infty} \hat{f}'(\sqrt{2n}) \tilde{b}_n(x),
\]

and these series converge absolutely.
Construction of “Magic functions”

Let $p : (0, \infty) \to \mathbb{R}$ be a strictly monotonic potential function. The only possible “magic” function $f$ that could prove a sharp bound for $E_8$ or the Leech lattice under a potential $p$:

$$f(x) = \sum_{n=n_0}^{\infty} p(\sqrt{2n}) a_n(x) + \sum_{n=n_0}^{\infty} p'(\sqrt{2n}) b_n(x).$$  \hfill (2)
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In order to prove that $E_8$ or the Leech lattice minimize the $p$-energy, it suffices to show that $f(x) \leq p(|x|)$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\hat{f}(y) \geq 0$ for all $y \in \mathbb{R}^d$. 
Let \( p : (0, \infty) \to \mathbb{R} \) be a strictly monotonic potential function. The only possible “magic” function \( f \) that could prove a sharp bound for \( E_8 \) or the Leech lattice under a potential \( p \):

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In order to prove that \( E_8 \) or the Leech lattice minimize the \( p \)-energy, it suffices to show that \( f(x) \leq p(|x|) \) for all \( x \in \mathbb{R}^d \setminus \{0\} \) and \( \hat{f}(y) \geq 0 \) for all \( y \in \mathbb{R}^d \).

If a configuration is a ground state for every Gaussian \( r \mapsto e^{-\alpha r^2} \), then the same is true for every completely monotonic function of squared distance.
Consider the generating functions

\[ F(\tau, x) = \sum_{n \geq n_0} a_n(x) e^{2\pi i n \tau} + 2\pi i \tau \sum_{n \geq n_0} \sqrt{2n} b_n(x) e^{2\pi i n \tau} \]

and

\[ \tilde{F}(\tau, x) = \sum_{n \geq n_0} \tilde{a}_n(x) e^{2\pi i n \tau} + 2\pi i \tau \sum_{n \geq n_0} \sqrt{2n} \tilde{b}_n(x) e^{2\pi i n \tau}, \]

The interpolation formula for the complex Gaussian \( x \mapsto e^{\pi i \tau |x|^2} \) is equivalent to

\[ F(\tau, x) + \left( i/\tau \right)^{d/2} \tilde{F}(-1/\tau, x) = e^{\pi i \tau |x|^2}. \]
Solving the functional equation

Using the methods developed in the theory of automorphic forms and by improving these techniques we can explicitly solve the functional equation

\[
F(\tau + 2, x) - 2F(\tau + 1, x) + F(\tau, x) = 0
\]
\[
\tilde{F}(\tau + 2, x) - 2\tilde{F}(\tau + 1, x) + \tilde{F}(\tau, x) = 0
\]
\[
F(\tau, x) + (i/\tau)^{d/2}\tilde{F}(-1/\tau, x) = e^{\pi i |x|^2}
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Moderate growth of \( F \) implies the interpolation formula
Solving the functional equation

Using the methods developed in the theory of automorphic forms and by improving these techniques we can \textit{explicitly} solve the functional equation

\begin{align*}
F(\tau + 2, x) - 2F(\tau + 1, x) + F(\tau, x) &= 0 \\
\tilde{F}(\tau + 2, x) - 2\tilde{F}(\tau + 1, x) + \tilde{F}(\tau, x) &= 0 \\
F(\tau, x) + (i/\tau)^{d/2}\tilde{F}(-1/\tau, x) &= e^{\pi i |x|^2}
\end{align*}

Moderate growth of $F$ implies the interpolation formula

The inequality $F(it, x) > 0$ for $t \in (0, \infty)$ implies the universal optimality of $\Lambda_8$ and $\Lambda_{24}$.

M. Viazovska  
Sphere packing and Fourier interpolation
Thank you for your attention.
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Please ask questions.