On the classification of fusion categories

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CIEM–CONICET

ICM 2018
Rio de Janeiro, August 8, 2018.

A *monoidal category* is a collection $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$, where

$\mathcal{C}$ is a category

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor

$\mathbf{1} \in \mathcal{C}$

Associativity constraint:

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z),$$

Unit constraints:

$$l_X : \mathbf{1} \otimes X \xrightarrow{\cong} X, \quad r_X : X \otimes \mathbf{1} \xrightarrow{\cong} X$$

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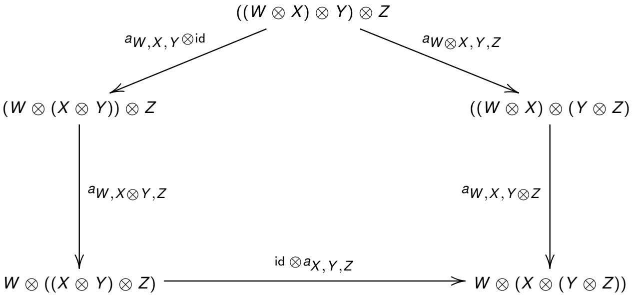

Triangle diagram:

$$\begin{array}{ccc} (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a} & X \otimes (\mathbf{1} \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id} \otimes l_Y} & \\ & X \otimes Y. & \end{array}$$

$\mathcal{C}$ is called *rigid* if every object X has duals X^*, ${}^*X$:

$$\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}, \quad \mathrm{coev}_X : \mathbf{1} \to X \otimes X^*,$$

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Let $\mathcal{C}$, $\mathcal{D}$ be monoidal categories.

A *monoidal functor* $\mathcal{C} \to \mathcal{D}$ is a triple (F, F^2, F^0), where

$F : \mathcal{C} \to \mathcal{D}$ is a functor

$F^2_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\cong} F(X \otimes Y)$

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Obeying appropriate conditions.

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Obeying appropriate conditions.

k : algebraically closed field of characteristic zero.

A *tensor category over k* is a k-linear abelian rigid monoidal category $\mathcal{C}$ where:

The monoidal product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is k-bilinear.

Hom spaces are fin. dim. and objects have finite length.

The unit object $\mathbf{1}$ is simple.

A *fusion category over k* is a semisimple finite tensor category.

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Problem: Classify fusion categories over $k = \mathbb{C}$.

Solution: out of reach for the moment.

Consider special classes:

Assumptions on dimensions

Assumptions on rank (= number of simple objects up to iso.)

Assumptions on fusion rules, etc.

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Example (Pointed fusion categories)

G finite group, $\omega : G \times G \times G \to k^\times$ a 3-cocycle:

$$\omega(xy, z, t)\omega(x, y, zt) = \omega(x, y, z)\omega(x, yz, t)\omega(y, z, t).$$

$\mathcal{C}(G, \omega)$: fin. dim. G-graded vector spaces $V = \oplus_{g \in G} V_g$

$$a_{X,Y,Z} : (x \otimes y) \otimes z \mapsto \omega(a, b, c)\; x \otimes (y \otimes z),$$

$x \in X_a$, $y \in Y_b$, $z \in Z_c$.

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Let k be a field, G a finite group.

$$\rho : G \to \mathrm{GL}(V) \quad k\text{-linear representation of } G.$$

Rep G = Rep$_k$ G: fin. dim. k-linear representations of G

$$\otimes = \otimes_k, \quad \mathbf{1} = k, \quad V^* = \mathrm{Hom}_k(V, k), \ldots$$

$$(V \otimes_k W) \otimes_k U \cong V \otimes_k (W \otimes_k U), \quad k \otimes_k V \cong V \cong V \otimes_k k$$

$$\tau : V \otimes_k W \to W \otimes_k V, \quad x \otimes y \mapsto y \otimes x$$

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A *symmetric monoidal category* is a monoidal category $\mathcal{C}$ endowed with a *braiding*:

$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X,$$

subject to certain (so called *hexagon*) axioms s.t.

$$c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y}, \quad X, Y \in \mathcal{C}.$$

Rep G is a *rigid symmetric monoidal category*.

$$F : \mathrm{Rep}\, G \to \mathrm{Vec}_k, \quad G \cong \mathrm{Aut}_{\otimes} F.$$

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More generally: $u \in Z(G)$ s.t. $u^2 = 1$

$\mathrm{Rep}(G, u)$: f.d. representations of G on super-vector spaces where u acts as the parity operator.

$\mathrm{Rep}(G, u)$ is a symmetric fusion category.

Theorem

(Deligne, 2002.) k alg. closed of char 0. Let $\mathcal{C}$ be a symmetric fusion category. Then

$$\mathcal{C} \cong \mathrm{Rep}(G, u)$$

for certain finite group G and $u \in G$ as above.

Doplicher–Roberts: C^*-algebraic version.

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$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X \quad + \quad \text{hexagon axioms.}$$

The braiding c obeys the *Braid Equation*:

$$(c_{Y,Z} \otimes \mathrm{id}_X)(\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) = (\mathrm{id}_Z \otimes c_{X,Y})(c_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z}).$$

Opposite to symmetric $\rightsquigarrow$ *modular categories*.

Monoidal category $\mathcal{C} \rightsquigarrow$ braided monoidal category $\mathcal{Z}(\mathcal{C})$: the *Drinfeld center of* $\mathcal{C}$.

$$U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}.$$

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Example

Let H be a Hopf algebra over a field k.

$$\Delta : H \to H \otimes H, \quad \epsilon : H \to k, \quad \mathcal{S} : H \xrightarrow{\cong} H^{op}$$

Rep H is a rigid monoidal category ($\text{Rep}_k G = \text{Rep}\, kG$):

- $V \otimes W = V \otimes_k W$, H-action induced by Δ.
- $\mathbf{1} = k$, H-action induced by ϵ.
- $V^* = \text{Hom}_k(V, k) = {}^*V$, H-action induced by $\mathcal{S}$ ($\mathcal{S}^{-1}$).

Other related examples: H quasi-Hopf algebra, weak Hopf algebra, Hopf monad, etc.

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Other related examples: H quasi-Hopf algebra, weak Hopf algebra, Hopf monad, etc.

Example

Let H be a Hopf algebra over a field k.

$$\Delta : H \to H \otimes H, \quad \epsilon : H \to k, \quad \mathcal{S} : H \xrightarrow{\cong} H^{op}$$

Rep H is a rigid monoidal category ($\mathrm{Rep}_k\, G = \mathrm{Rep}\, kG$):

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Other related examples: H quasi-Hopf algebra, weak Hopf algebra, Hopf monad, etc.

Theorem

(Hayashi, 2000; Ostrik, 2003.) Every (multi)fusion category is equivalent to the category of fin. dim. representations of a fin. dim. semisimple weak Hopf algebra.

Example

Let H be a Hopf algebra. Then *Forget* $= F : \operatorname{Rep} H \to \mathrm{Vec}_k$ is a monoidal functor.

Suppose H, L are finite-dimensional Hopf algebras.

$\operatorname{Rep} H \cong \operatorname{Rep} L$ if and only if $L \cong H^J$, $J \in H \otimes H$ *twist*:

$H^J = H$ as algebras

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Module categories and Morita equivalence

(Right) *module category* over $\mathcal{C}$:

$$\otimes : \mathcal{M} \times \mathcal{C} \to \mathcal{M} \quad + \text{axioms.}$$

$\mathcal{C}^*_{\mathcal{M}} = \mathcal{C}$-module endofunctors of $\mathcal{M} \rightsquigarrow$ (multi-)fusion category.

Let $A \in \mathcal{C}$ an (indecomposable) separable algebra. Then $\mathcal{M} = \mathcal{C}_A$ is an (indecomposable) $\mathcal{C}$-module category.

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$\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if $\mathcal{D}^{op} \cong \mathcal{C}^*_{\mathcal{M}}$ for some $\mathcal{M}$.

Theorem

(Etingof-Nikshych-Ostrik, 2011.)
The fusion categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent $\Longleftrightarrow$ $\mathcal{Z}(\mathcal{C}) \cong \mathcal{Z}(\mathcal{D})$.

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Fusion rules and Frobenius-Perron dimension

Let $\mathcal{G}(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$.

$\mathrm{Irr}(\mathcal{C})$ = isomorphism classes of simple objects of $\mathcal{C}$.

$\mathcal{G}(\mathcal{C})$ ring: $[X][Y] = [X \otimes Y]$, $X, Y \in \mathcal{C}$, $1 = [\mathbf{1}]$.

$X, Y \in \mathrm{Irr}(\mathcal{C})$:

$$[X][Y] = \sum_{Z \in \mathrm{Irr}(\mathcal{C})} N_{X,Y}^{Z} [Z],$$

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$\mathrm{FPdim}\, X$ = Frobenius-Perron eigenvalue of $(N_{X,Y}^{Z})_{Z,Y \in \mathrm{Irr}(\mathcal{C})}$.

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Theorem

(Etingof-Nikshych-Ostrik, 2005.) Up to equivalence, there is a finite number of fusion categories with a given Grothendieck ring.

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Example

A fusion category $\mathcal{C}$ is called a *Tambara-Yamagami* fusion category if $\mathrm{Irr}(\mathcal{C}) = G \cup \{X\}$ with

$$g \otimes h = gh, \quad X \otimes X = \sum_{g \in G} g, \quad g, h \in G.$$

$\mathrm{FPdim}\, \mathcal{C} = 2|G|$.

(Tambara-Yamagami, 1998.) G must be an abelian group and $\mathcal{C}$ is classified by a non-degenerate symmetric bilinear form

$$\chi : G \times G \to k$$

and $\tau \in k$ s.t. $\tau^2 = |G|^{-1}$.

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(Calegary-Morrison-Snyder, 2011.) Let X be an object in a fusion category such that $\mathrm{FPdim}\, X$ *belongs to the interval* $(2, 76/33]$*. Then* $\mathrm{FPdim}\, X$ *is equal to one of the following:*

$$\frac{\sqrt{7}+\sqrt{3}}{2}, \quad \sqrt{5}, \quad 1+2\cos(\frac{2\pi}{7}), \quad \frac{1+\sqrt{5}}{\sqrt{2}}, \quad \frac{1+\sqrt{13}}{2}.$$

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Question

(Ostrik, 2003.) Are there finitely many equivalence classes of fusion categories with a given finite rank?

Affirmative answers:

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Examples of tensor categories obtained as extensions

Let H be a fin. dim. Hopf algebra over k.

$K \subseteq H$ a normal Hopf subalgebra:

$$h_{(1)} a \mathcal{S}(h_{(2)}) \in K, \ \forall h \in H, a \in K, \quad \Leftrightarrow HK^+ = K^+ H.$$

Exact sequence of Hopf algebras:

$$k \to K \to H \to H/HK^+ \to k.$$

In this case: $H \cong K^\tau \#_\sigma \overline{H}$.

H is *simple* if it contains no proper normal Hopf subalgebra.

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Definition

(Andruskiewitsch-Müller, 2015.) A *composition series* of H is a sequence of simple Hopf algebras

$$\mathfrak{H}_1, \dots, \mathfrak{H}_n,$$

defined as:

• If H is simple: $n = 1$, $\mathfrak{H}_1 = H$.

• If $k \subsetneq A \subsetneq H$ normal, and $\mathfrak{A}_1, \dots, \mathfrak{A}_m$, $\mathfrak{B}_1, \dots, \mathfrak{B}_l$, c.s. of A and $B = H/HA^+$, respectively: $n = m + l$ and

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Analogue of Jordan-Hölder theorem:

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(N., 2014) Let $\mathfrak{H}_1, \dots, \mathfrak{H}_n$ and $\mathfrak{H}'_1, \dots, \mathfrak{H}'_m$ be two composition series of H. Then there exists a bijection $f : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $\mathfrak{H}_i \cong \mathfrak{H}'_{f(i)}$ as Hopf algebras.

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(N., 2014) Let $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ and $\mathfrak{H}'_1, \ldots, \mathfrak{H}'_m$ be two composition series of H. Then there exists a bijection $f : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $\mathfrak{H}_i \cong \mathfrak{H}'_{f(i)}$ as Hopf algebras.

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Extensions of tensor categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

F is *dominant* if any $Y \in \mathcal{D}$ is a subobject of $F(X)$, $X \in \mathcal{C}$.

$\mathfrak{Ker}_F := F^{-1}(\langle \mathbf{1} \rangle) \subseteq \mathcal{C}$.

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(Bruguières-N.) A tensor functor $F : \mathcal{C} \to \mathcal{D}$ is *normal* if for any object X of $\mathcal{C}$, there exists a subobject $X_0 \subset X$, such that $F(X_0)$ is the largest trivial subobject of $F(X)$.

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is an *exact sequence of tensor categories* if the following hold:

F is dominant and normal;

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Suppose $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ exact $\Rightarrow \mathcal{C}' \simeq$ comod-H.

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Extensions with respect to a module category

$\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{C}$ finite tensor categories.

$\mathcal{M}$ an exact indecomposable left $\mathcal{A}$-module category.

$\mathrm{End}(\mathcal{M}) =$ k-linear right exact endofunctors of $\mathcal{M}$.

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(Etingof-Gelaki, 2017.) An exact sequence *with respect to* $\mathcal{M}$ is a sequence of tensor functors

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A *G-grading* on $\mathcal{C}$ is a decomposition: $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$, s.t.

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Weakly group-theoretical fusion categories

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(Gelaki-Nikshych, 2008.) $\mathcal{C}$ is *nilpotent* if there exists a series of fusion subcategories

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s.t. $\mathcal{C}_i$ is a G_i-extension of $\mathcal{C}_{i-1}$, $\forall i$, $G_1, \ldots, G_n$, finite groups.

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Weakly group-theoretical fusion categories

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(Etingof-Nikshych-Ostrik, 2011.) $\mathcal{C}$ is weakly group-theoretical $\iff \exists$ series of f. c.

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and a series of finite groups

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s.t. $\forall i$, $\mathrm{Rep}\, G_i \cong_{\mathrm{symm}} \mathcal{E}_i \subseteq \mathcal{Z}(\mathcal{C}_i) \quad \& \quad (\mathcal{E}_i')_{G_i} \cong \mathcal{Z}(\mathcal{C}_{i-1})$.

Here:

$Y \in \mathcal{D}' \iff c_{Y,X} c_{X,Y} = \mathrm{id} : X \otimes Y \to Y \otimes X$.

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Core of a weakly group-theoretical BFC

(Drinfeld-Gelaki-Nikshych-Ostrik, 2010.) Ising category: non-pointed fusion categories of FP-dimension 4.

Theorem

(N., 2017.) Let $\mathcal{C}$ be a weakly group-theoretical BFC. Then

$$Core(\mathcal{C}) \cong \mathcal{B} \boxtimes \mathcal{D},$$

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Question

(Etingof-Nikshych-Ostrik, 2011.) Is every fusion category with integer FP-dimension weakly group-theoretical?

Some known partial answers:

- (Etingof-Nikshych-Ostrik.) If $\mathrm{FPdim}\,\mathcal{C} = p^n \implies \mathcal{C}$ is nilpotent.
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Suppose C is (braided) non-degenerate. Then:

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