Representations of Finite Groups and Applications

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Some history

Representation theory of finite groups started with the letter correspondence between Richard Dedekind and Ferdinand Georg Frobenius in April (12th, 17th, 26th) 1896.

Foundations of *complex* representation theory: developed by Frobenius, Dedekind, Burnside, Schur, Noether, . . .

Foundations of *modular* representation theory (over fields of char. $p > 0$ dividing $|G|$): laid out by Richard Brauer, started in 1935 and continued over the next few decades.

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Our favorite groups

- The *symmetric group* S_n and *alternating group* A_n
- *Finite groups of Lie type*:
  • Classical groups on $V = \mathbb{F}_q^n$, like $GL(V) = GL_n(q)$, $SL(V) = SL_n(q)$, $Sp(V)$, etc.
  • Their twisted and exceptional analogues.

Classification of Finite Simple Groups (CFSG): *The above groups, together with 26 sporadics, account for all finite non-abelian simple groups.*

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Example

Problem 1 (Main Problem)

Given a finite group G and a field $\mathbb{F}$, describe all irreducible representations of G over $\mathbb{F}$.

$G = S_n$

- $\mathbb{F} = \mathbb{C}$:

$$\{\text{Irreducible characters of } G\} \longleftrightarrow \{\text{Partitions of } n\}$$
$$\chi^\lambda \longleftarrow \lambda \vdash n$$

What is the *largest* degree $\max_{\lambda \vdash n} \chi^\lambda(1)$?

Asymptotic result: Vershik-Kerov and Logan-Shepp (1977).

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- $G = S_n$, $\mathbb{F} = \mathbb{F}_2$

$$\begin{array}{ccc} \{\text{Irreducible reps of } G\} & \longleftrightarrow & \{\text{Strict partitions of } n\} \\ \varphi^\lambda & \longleftarrow & \lambda = (\lambda_1 > \lambda_2 > \cdots) \vdash n \end{array}$$

What is $\varphi^\lambda(1)$, say for $n = 1000$ and $\lambda = (801, 121, 70, 8)$?

Same question for $G = GL_{1000}(2)$ and $\mathbb{F} = \mathbb{F}_3$?

What is the asymptotic of $\max_{\chi \in \mathrm{Irr}(GL_n(2))} \chi(1)$ when $n \to \infty$?

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Low-dimensional representations

Given G, $\mathbb{F} = \overline{\mathbb{F}}$ of characteristic p
$\mathfrak{d}_p(G)$ = smallest degree of *faithful* representations of G over $\mathbb{F}$

Problem 2

Given a simple group G and $\mathbb{F}$,
(i) *determine $\mathfrak{d}_p(G)$, and*
(ii) *classify irreducible $\mathbb{F}G$-representations of degree up to $\mathfrak{d}_p(G)^2$.*

Problem 2 looks like a special instance of Main Problem. But turns out to be important, and (its solution) efficient for many applications.

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Alperin weight conjecture (AWC)

If one cannot describe all irreps of G, can one at least find a good way to label them?

Jon Alperin: A p-weight of G is a pair (Q, δ), where

- Q is a p-subgroup of G,
- $\delta \in \mathrm{Irr}(N_G(Q)/Q)$ with $\delta(1)_p = |N_G(Q)/Q|_p$

Conjecture 3 (Alperin, 1986)

$|\{$Isomorphism classes of irreps of G over $\overline{\mathbb{F}}_p\}|$
$= |\{$G-conjugacy classes of p-weights of $G\}|$.

AWC implies: p-modular irreps of G could be classified by **weights**, as we do in Lie theory for (rational) irreps of reductive algebraic groups.

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Global-local principle

AWC is one of several fundamental conjectures in representation theory; all open until now.

Many of them follow *Global-local principle*:

Certain **global** *invariants of a finite group G can be determined* **locally** (in terms of its p-subgroups, eg. Sylow p-subgroups $P \in \mathrm{Syl}_p(G)$, their normalizers $N_G(P)$, etc.)

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McKay conjecture

Perhaps the easiest one to formulate among the global-local conjectures is:

Conjecture 4 (McKay, 1971)

G a finite group, p a prime, $P \in \mathrm{Syl}_p(G)$.
There exists a bijection
$\{\chi \in \mathrm{Irr}(G) \mid p \nmid \chi(1)\} \stackrel{\pi}{\longleftrightarrow} \{\psi \in \mathrm{Irr}(N_G(P)) \mid p \nmid \psi(1)\}$.

Refinements (Alperin, Isaacs, Navarro, Turull, . . .): *There should exist such a π, which is compatible with*

- *distribution of $\mathrm{Irr}(G)$ into p-**blocks**,*
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Where do we stand?

The aforementioned conjectures have been proved to hold for various families of finite groups.

But: **none** has been proved to hold for **all** finite groups.

Possible approach: Use the *Classification of Finite Simple Groups* to reduce the conjecture to simple groups.

Already one of the main ideas in Dade's works in the 90s.

Caution: If a reduction is found, expect to have to prove **much more** about the simple groups, not just the original conjecture!

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A reduction theorem for McKay conjecture

Theorem 2.1 (Isaacs-Malle-Navarro, 2007)

Suppose that every finite non-abelian simple group S is McKay-good for a fixed prime p. Then McKay conjecture holds for arbitrary finite groups (for the prime p).

- McKay-goodness for the prime p is much more than just satisfying the McKay conjecture. It is in fact a long, complicated, list of conditions.
- Theorem 2.1 plays an important role in proving subsequent reduction theorems.
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AWC-goodness

Which simple groups S are proved to be AWC-good for p?

- $S \in Lie(p)$ (Navarro-T)
- $S \in Alt$ (Malle)
- $S \in Spor$ (An-Dietrich)
- It remains . . . to handle $S \in Lie(p')$ (like $PSL_n(q)$ with $p \nmid q$)

Further reduction theorems for other global-local conjectures:

- Späth
- Berger-Knörr, Navarro-Späth & Navarro-T

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Results on Problem 2

Problem 2 on Low-dimensional Representations

Given a simple group G and $\mathbb{F}$,
(i) *determine smallest nontrivial degree* $\mathfrak{d}_p(G)$, *and*
(ii) *classify irreducible* $\mathbb{F}G$*-representations of degree up to* $\mathfrak{d}_p(G)^2$.

There have been many results on this problem, due to Guralnick, Himstedt, Hiss, Hoffman, James, Kleshchev, Liebeck, Lübeck, Magaard, Malle, T, Zalesskii

Rely on: Deligne-Lusztig theory, Broué-Michel, Bonnafé-Rouquier, Branching laws, . . .

Two particular results

Theorem 2.3 (Kleshchev-Morotti-T)

The degree of the irreducible $\mathbb{F}S_n$-representation D^λ, labeled by $\lambda = (n-m, \lambda_2, \dots)$ with $n \geq p(m-1)$, is at least $(n-p)(n-2p)\dots(n-mp)/m!$.

Theorem 2.4 (Guralnick-T)

Suppose $p \nmid q$. Then

$$\mathfrak{d}_p(SL_n(q)) = \frac{q^n-1}{q-1} - \begin{cases} 1, & p \nmid \frac{q^n-1}{q-1} \\ 2, & p | \frac{q^n-1}{q-1}. \end{cases}$$

One irrep of degree $\mathfrak{d}_p$, $(q-1)_{p'} - 1$ of degree $(q^n-1)/(q-1)$. All other nontrivial irreps have degree at least "square" of it.

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Problem 5 (**Bounding character values**)

Let S be a finite simple group, $1 \neq g \in S$.

(i) *Find an explicit, and as small as possible, $0 < \gamma = \gamma(g) < 1$, such that $\frac{|\chi(g)|}{\chi(1)} \leq \gamma$, $\forall\, 1_S \neq \chi \in \mathrm{Irr}(S)$.*

(ii) *Better yet, find an explicit, and as small as possible, $0 < \alpha = \alpha(g) < 1$, such that $|\chi(g)| \leq \chi(1)^{\alpha}$, $\forall\, \chi \in \mathrm{Irr}(S)$.*

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Results on Problem 5(ii)

• Case of S_n:

Fomin-Lulov (1996): if all cycles of g have same size m, then

$$|\chi(g)| \leq \chi(1)^{1/m+o(1)}$$

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Led to results on: *Mixing times of random walks*, *Waring problem for* A_n, etc.

• What about (quasi)simple groups of Lie type?

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Character bounds for Lie-type groups. I

Theorem 2.6 (**Bezrukavnikov-Liebeck-Shalev-T**)

There is a constant $C = C(r)$ such that:
Let $S = \mathcal{G}^F$ be a finite (quasi)simple group of Lie type in **good** *characteristic p, $r = \mathrm{rank}(\mathcal{G})$. Let $g \in S$ be such that $C_S(g) \leq \mathcal{L}^F$, for an F-stable Levi subgroup $\mathcal{L}$ of a proper F-stable parabolic subgroup of $\mathcal{G}$. Define*

$$\alpha = \alpha(\mathcal{L}^F) := \max_{1 \neq u \in \mathcal{L}^F,\ u \text{ unipotent}} \frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}.$$

Then for all $\chi \in \mathrm{Irr}(S)$,

$$|\chi(g)| \leq C\chi(1)^{\alpha}.$$

Character bounds for Lie-type groups. II

- Relies on Lusztig's unipotent support and Kawanaka's wave front sets
- The exponent α in Theorem 2.6 is sharp in many cases.
- Removing the splitness condition on $\mathcal{L}$: work in progress with Bezrukavnikov-Malle-Taylor

Theorem 2.7 (Bezrukavnikov-Liebeck-Shalev-T)

Let $S = SL_n(q)$ and let $g \in S \smallsetminus Z(S)$. Then $|\chi(g)| \leq C(n)\chi(1)^{1-1/2n}$ for all $\chi \in \mathrm{Irr}(S)$.

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Character bounds for Lie-type groups. III

Theorem 2.8 (Guralnick-Larsen-T)

For any $0 < \varepsilon < 1$, *there is* $\delta > 0$ *such that the following holds.*
For any finite (quasi)simple group S *of Lie type,*
and for any $g \in S$ *with* $|C_S(g)| \le |S|^\delta$,
$|\chi(g)| \le \chi(1)^\varepsilon$ *for all* $\chi \in \mathrm{Irr}(S)$.

$S = SL_n(q)$ or $SU_n(q)$, $\varepsilon = 8/9$: one can take $\delta = 1/12$.

A modularity lifting theorem

A key ingredient of Wiles' proof of Fermat's Last Theorem:

Taylor-Wiles Modularity Lifting Theorem (1995)

Let $p > 2$, $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_p$, $\Phi : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ a Galois rep. such that
(i) *Φ "looks like" coming from a modular form;*
(ii) *The associated rep. $\overline{\Phi} : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is* **modular***;*
(iii) *$\overline{\Phi}(G_{\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})})$ is big.*
Then Φ is **modular***.*

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An automorphy lifting theorem

A key ingredient of the Harris-Shepherd-Barron-Taylor (2010) proof of the Sato-Tate conjecture (over totally real fields):

Clozel-Harris-Taylor Automorphy Lifting Theorem (2008)

Generalizes the Taylor-Wiles theorem, replacing GL_2 by GL_n and **modular** *by* **automorphic**.

Here bigness means irreducibility plus more, including one condition on the existence of a special element with a special multiplicity-one eigenvalue.

Adequate subgroups

Thorne (2012) generalizes further the Clozel-Harris-Taylor theorem, replacing bigness by adequacy:

Definition 3.1 (Thorne)

Let $\mathrm{char}(\mathbb{F}) = p$, $V = \mathbb{F}^n$, *and let* $G \leq GL(V)$ *be a finite irreducible subgroup. Suppose that*
(i) $H^1(G, \mathbb{F}) = 0$;
(ii) $H^1(G, \mathrm{End}(V)/\mathbb{F} \cdot 1_V) = 0$;
(iii) *The space* $\mathrm{End}(V)$ *is spanned by* p'*-elements* $x \in G$.
Then G is called adequate.

Adequacy

Can one prove that **all**, or **most**, finite irreducible subgroups of $GL_n(\mathbb{F})$ are **adequate**?

- If $p \nmid |G|$, then G is adequate.
- If $p \geq 2n + 2$, then G is adequate (Guralnick-Herzig-Taylor-Thorne, 2012). Already used to prove some new lifting theorems.

Theorem 3.2 (Guralnick-Herzig-T)

Let $G < GL(V)$ be a finite irreducible subgroup and let G^+ denote the subgroup of G generated by all p-elements $x \in G$. Suppose that the G^+-module V has a simple submodule of dimension $< p$. Then, aside from a few explicitly described examples, G is adequate.

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Tian's α-invariant

For a Kähler manifold X and a compact subgroup $G \leq \mathrm{Aut}(X)$, Tian (1987) defined an invariant $\alpha_G(X)$. He used it to prove the existence of a G-invariant Kähler-Einstein metric on X in some important cases.

Case of interest: a finite group $G < GL_n(\mathbb{C})$ acts on $\mathbb{P}^{n-1}$.

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Thompson's theorem

Theorem 3.3 (Thompson, 1981)

Suppose that $G < GL_n(\mathbb{C})$ is any finite group. Then $\alpha_G(\mathbb{P}^{n-1}) \leq 4n$.

$G < GL(V)$ is said to have a semi-invariant of degree k on V if $\mathrm{Sym}^k(V)$ contains a one-dimensional G-submodule.

$d(G) := \min\{k \geq 1 \mid G \text{ has a semi-invariant of degree } k \text{ on } V\}$.

The connection (Cheltsov-Shramov):
$\alpha_G(\mathbb{P}V) \leq d(G)/\dim V$.
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Conjecture 6 (Thompson, 1981)

There exists a constant $C > 0$ such that
(i) $\alpha_G(\mathbb{P}^{n-1}) \leq C$, *and*
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for all finite subgroups $G < GL_n(\mathbb{C})$.

Theorem 3.4 (T)

Thompson's conjecture holds, with $C = 1184036$.

Corollary 3.5

Any finite subgroup $G < GL_n(\mathbb{C})$ admits an invariant of degree $\leq Cn \cdot |G/G'|$, with $C = 1184036$.

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Word maps

F_d the free group on $x_1, \ldots, x_d$, $1 \neq w \in F_d$, G any group

Word map $w : G^d \to G$,

$w(G) := w(G^d) = \{w(g_1, \ldots, g_d) \in g_i \in G\}$

Problem 7 (Non-commutative Waring Problem)

Let $1 \neq w \in F_d$ and let G be simple.

(i) *How large is $w(G)$?*

(ii) *Assuming $w(G) \neq 1$, what is the width of w: $c(w) := \min\{k > 0 \mid w(G)^k = G\}$*

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Theorem 3.6 (Larsen-Shalev-T)

Let $w = w_1 w_2$ be a product of disjoint words $1 \neq w_1, w_2 \in F_d$ and let G be simple.

(i) *If $|G|$ is large enough, then $w(G) = G$.*

(ii) *If $|G|$ is large enough, then any nontrivial word map w_1 has width 2.*

(iii) *When $|G| \to \infty$, $p_{w,G}$ induces an almost uniform distribution on G:*

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Word maps that are surjective on all simple groups:

(i) $w(x, y) = xyx^{-1}y^{-1}$ (Ore conjecture, Liebeck-O'Brien-Shalev-T)

(ii) $w(x, y) = x^N y^N$, $N = p^a q^b$ any product of two prime powers (Guralnick-Liebeck-O'Brien-Shalev-T)

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(ii) and (iii) are sharp.

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Random walks on finite groups

$G = \langle S \rangle$ a finite group

Cayley graph $\Gamma = \Gamma(G, S)$
Vertices: G, Edges: $\{(g, gs) \mid g \in G, s \in S\}$

Random walk on Γ: Start from 1. At each step choose the next edge uniformly with probability $1/|S|$.

Mixing time $T = T(G, S)$: the minimal number of steps required to reach an almost uniform distribution on G

$$T := \min\left\{k > 0 : ||P^k - U||_{L^1} := \sum_{x \in G} |P^k(x) - U(x)| < \frac{1}{e}\right\}.$$

Random walks on simple groups

Diaconis-Shashahani (1981): $G = S_n$, $S = \{1, 2\text{-cycles}\}$
Hildebrand (1992): $G = SL_n(q)$, $S = \{\text{transpositions}\}$, $T \approx n$

Theorem 3.7 (Bezrukavnikov-Liebeck-Shalev-T)

Let $G = SL_n(q)$ and let $x \in G \setminus Z(G)$. The random walk on $\Gamma = \Gamma(G, x^G)$ has mixing time $\leq 2n+3$, if q is large.

Theorem 3.8 (Guralnick-Larsen-T)

There is $\gamma > 0$ such that:
Let G be a finite simple group of Lie type over $\mathbb{F}_q$, $x \in G$, $\Gamma = \Gamma(G, x^G)$. If $|C_G(x)| \leq |G|^\gamma$ and q is large, then
(i) *Γ has diameter ≤ 3, and*
(ii) *the random walk on Γ has mixing time 2.*

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Random walks on group representations

G a finite group, α a fixed faithful character

$\Gamma(G,\alpha)$: Vertices $\mathrm{Irr}(G)$, Edges $\{(\chi,\varphi) \mid [\varphi,\chi\alpha]_G > 0\}$

Random walk on Γ: Start at 1_G. At each vertex χ choose the next edge φ with probability $\dfrac{[\varphi,\chi\alpha]_G \cdot \varphi(1)}{\chi(1)\alpha(1)}$.

- Random walks on **complex** representations: Fulman, Diaconis
- Random walks on **modular** representations: Benkart-Diaconis-Liebeck-T
- Covering number, Diameter of $\Gamma(G,\alpha)$: Liebeck-Shalev-T

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