Tilting Cohen-Macaulay representations

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Part 1 : Derived category

Part 2 : Cluster category

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Throughout k is an algebraically closed field of characteristic zero for simplicity

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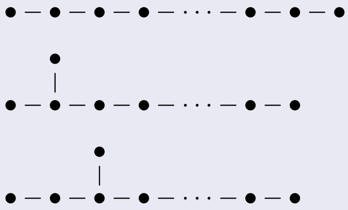
D_n $(n \geq 4)$

E_n $(n = 6, 7, 8)$



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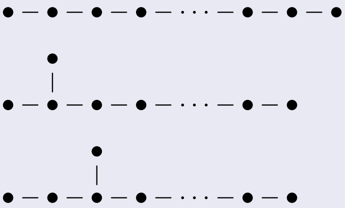

• In this case, $\exists$ a bijection between isoclasses of indecomposable kQ-modules and positive roots

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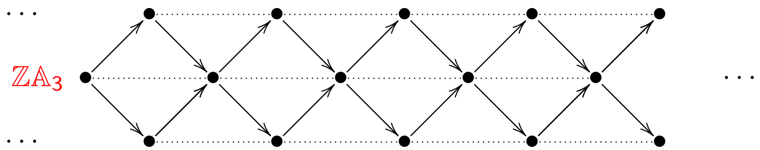

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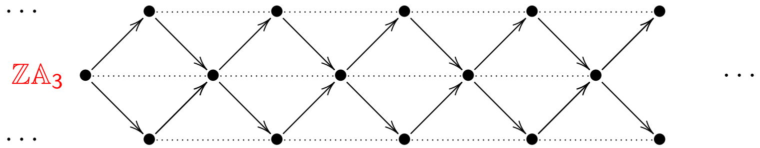

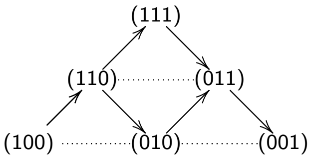

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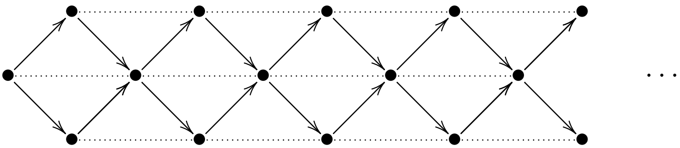

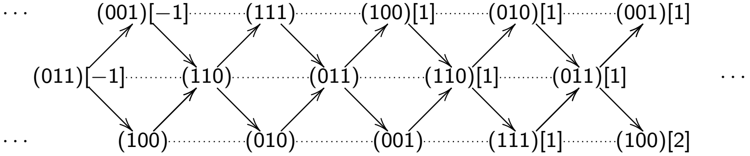

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- $\operatorname{Hom}_{\underline{\mathsf{CM}}\, R}(X, Y) := \operatorname{Hom}_R(X, Y)/P(X, Y)$, where $P(X, Y)$ consists of morphisms factoring through projective R-modules

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- Other properties also hold after suitable modifications

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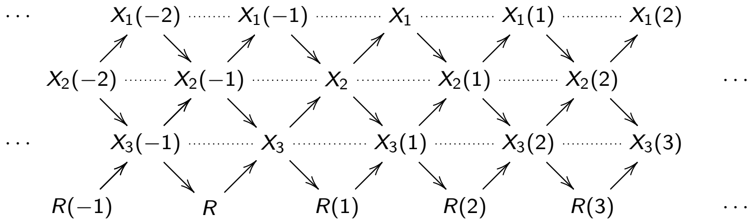

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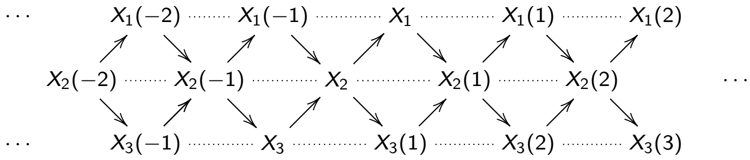

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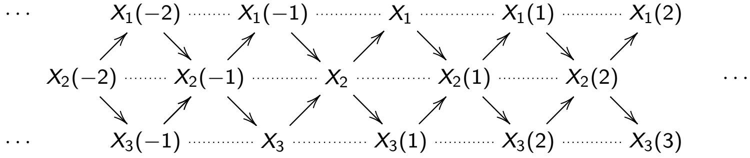

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$$
\begin{array}{ccccccccccccc}
\cdots & & X_1(-2) & \cdots & X_1(-1) & \cdots & X_1 & \cdots & X_1(1) & \cdots & X_1(2) & & \\
 & \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & \searrow\ \nearrow & & & \\
X_2(-2) & \cdots & & X_2(-1) & \cdots & X_2 & \cdots & X_2(1) & \cdots & X_2(2) & & & \cdots \\
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• R is CM-finite (i.e. there are only finitely many isoclasses of indecomposable objects in CM R) $\iff$ R is a simple singularity

$$R = k[[x, y, z_2 \dots, z_d]]/(f),\ f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2 & A_n \\ x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2 & D_n \\ x^4 + y^3 + z_2^2 + \cdots + z_d^2 & E_6 \\ x^3y + y^3 + z_2^2 + \cdots + z_d^2 & E_7 \\ x^5 + y^3 + z_2^2 + \cdots + z_d^2 & E_8. \end{cases}$$

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R and Q	$A_n : x^{n+1} - yz_2$	D_n	E_6	E_7	E_8
$\deg x$	1	2	3	4	6
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$\Longrightarrow$ $\exists$ a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D^b}(\mathrm{mod}\, kQ)$

Theorem [Buchweitz-Greuel-Schreyer '87]

R : a commutative complete local Gorenstein k-algebra

• R is CM-finite (i.e. there are only finitely many isoclasses of indecomposable objects in CM R) $\iff$ R is a simple singularity

$$R = k[[x, y, z_2 \ldots, z_d]]/(f),\ f = \begin{cases} x^{n+1} + y^2 + z_2^2 + \cdots + z_d^2 & A_n \\ x^{n-1} + xy^2 + z_2^2 + \cdots + z_d^2 & D_n \\ x^4 + y^3 + z_2^2 + \cdots + z_d^2 & E_6 \\ x^3y + y^3 + z_2^2 + \cdots + z_d^2 & E_7 \\ x^5 + y^3 + z_2^2 + \cdots + z_d^2 & E_8. \end{cases}$$

Theorem [Geigle-Lenzing '91, Kajiura-Saito-Takahashi '07]

$R = k[x, y, z_2]/(f)$: a simple singularity with $\dim R = 2$ and $\mathbb{Z}$-grading given by

R and Q	$A_n : x^{n+1} - yz_2$	D_n	E_6	E_7	E_8
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Gabriel's Theorem $\longleftrightarrow$ Buchweitz-Greuel-Schreyer's Theorem

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$\underline{\mathrm{mod}}^{\mathbb{Z}}(k[x]/(x^n)) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, k\mathbb{A}_{n-1})$ and $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, kQ)$

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Answer : Use tilting theory.

Tilting theory

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- If moreover $\mathrm{gl.dim}\,\Lambda < \infty$, then $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ and we have a triangle equivalence $\mathcal{T} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$

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[joint work with Herschend, Minamoto, Oppermann]

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- $R := C[X_1, \dots, X_n]/(X_i^{p_i} - \ell_i \mid 1 \leq i \leq n)$: a ring
- $\mathbb{L} := \langle \vec{c}, \vec{x}_1, \dots, \vec{x}_n \rangle / \langle p_i \vec{x}_i - \vec{c} \mid 1 \leq i \leq n \rangle$: an abelian group

Properties

- R is $\mathbb{L}$-graded by $\deg T_j := \vec{c}$ and $\deg X_i := \vec{x}_i$
- $\mathbb{L}$ is an abelian group of rank one
- R is a complete intersection ring with $\dim R = d + 1$
- R has the a-invariant $\vec{\omega} := (n - d - 1)\vec{c} - \sum_{i=1}^{n} \vec{x}_i \in \mathbb{L}$

i.e. $\mathrm{Ext}_R^{d+1}(k, R(\vec{\omega})) \simeq k$ in $\mathrm{mod}^{\mathbb{L}} R$

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$(p_1, \ldots, p_n)$	$n \leq 2$	$(2,2,p)$	$(2,3,3)$	$(2,3,4)$	$(2,3,5)$
$R^{(\vec{\omega})}$	$A_{p_1+\cdots+p_n-1}$	D_{p+2}	E_6	E_7	E_8

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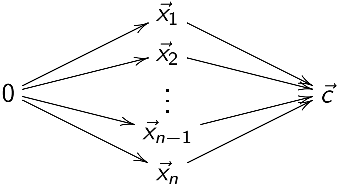

Part 1 : Derived category $\Longleftrightarrow$ graded Cohen-Macaulay modules

Part 2 : Cluster category

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where the composition is defined naturally

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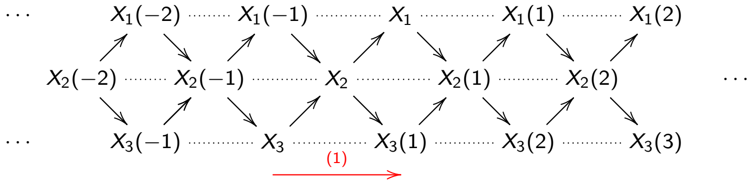

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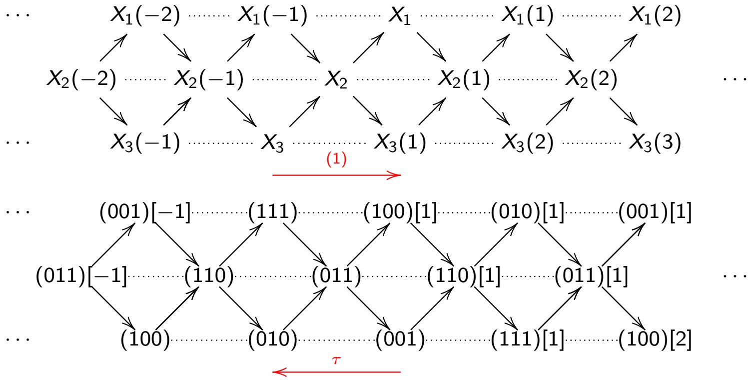

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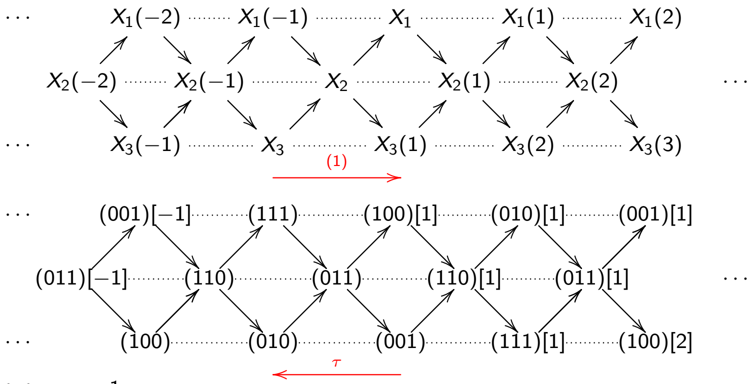

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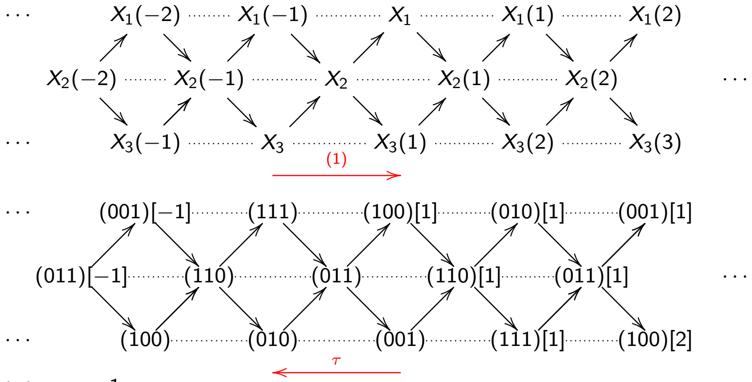

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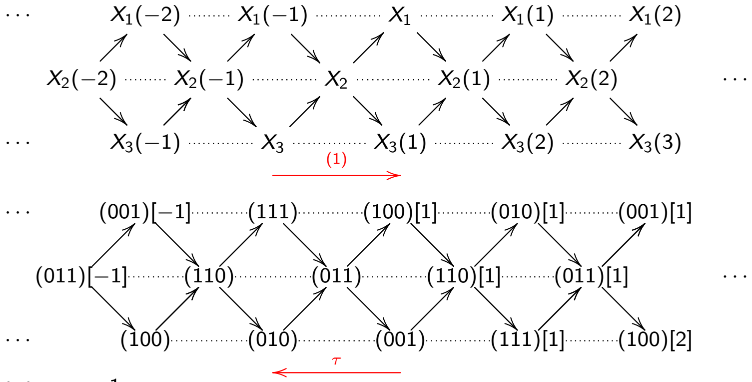

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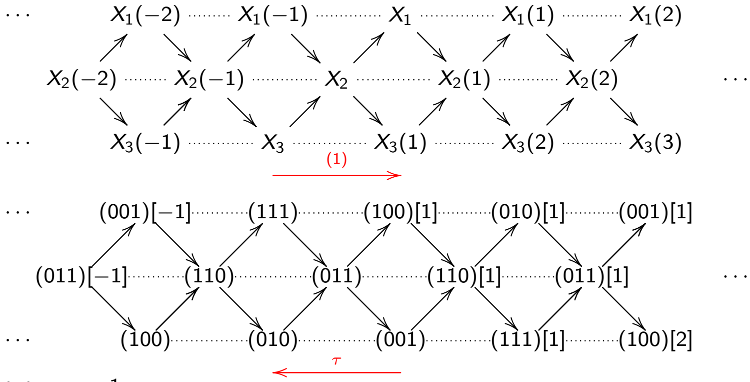

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- If $\mathrm{C}_d(\Lambda)$ is Hom-finite, then it is d-Calabi-Yau

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Answer : Yes, there are many. We will see examples given by preprojective algebras.

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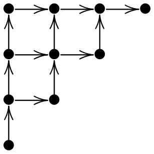

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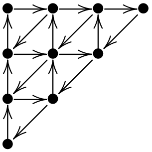

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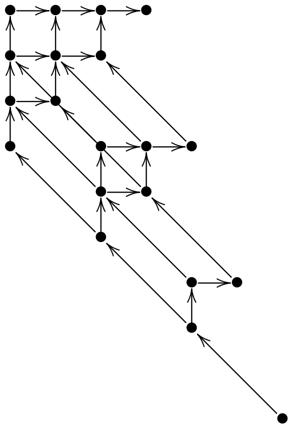

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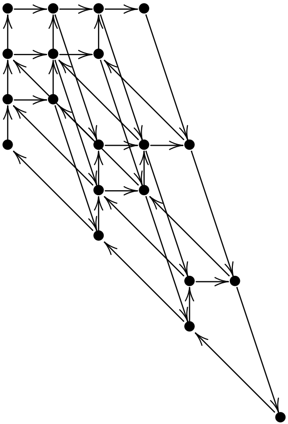

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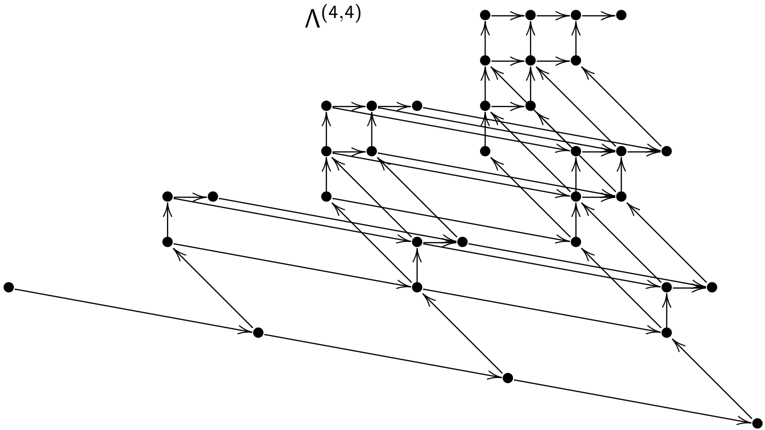

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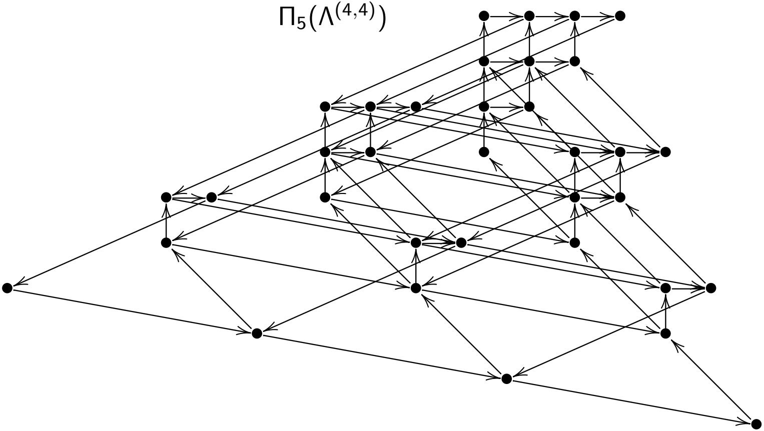

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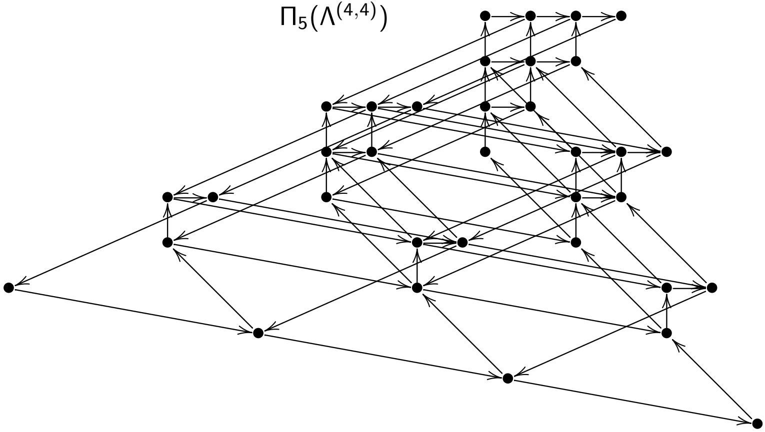

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