Negative algebraic K-theory

Moritz Kerz

Fakultät für Mathematik
Universität Regensburg

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Overview

X noetherian scheme $\rightsquigarrow$
non-connective algebraic K-theory spectra $K(X)$ (Thomason)
$K_i(X) = \pi_i K(X)$

Theorem (Weibel's Conjecture (1980))

For $d = dim(X)$ we have

(i) $K_i(X) = 0$ *for* $i < -d$,

(ii) $K_i(X) \xrightarrow{\simeq} K_i(X[t_1, \dots t_r])$ *is an isomorphism for* $i \leq -d$.

Convention for talk

All (derived) schemes are noetherian.

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All (derived) schemes are noetherian.

Remarks

- For (commutative) ring A and $i < 0$ the group $K_i(A)$ was calculated by Bass for $\dim(A) \leq 1$ (1968)
- For $\dim(X) \leq 2$ conjecture was shown by Weibel using reductions of ideals (2001)
- For X alg. variety in characterstic zero conjecture was shown by Cortinas-Haesemeyer-Schlichting-Weibel (2008)
- For X alg. variety in positive characterstic part of conjecture shown assuming strong resolution of singularities by Geisser-Hesselholt, Krishna and up to p-torsion by Kelly
- General case K-Strunk-Tamme (2016)

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Examples

Example

For $R = \mathbb{C}[t,s]/(s^2 - (t^2 + s^3))$ we have $K_{-1}(R) = \mathbb{Z}$ and $K_n(R) = 0$ for $n < -1$.

More generally, for $\dim(X) \leq 1$ and X excellent we have $K_i(X) = 0$ for $i < -1$ and $K_{-1}(X) = \mathbb{Z}^\rho$ (Bass).

Example

For a field k and the normal surface $R = k[x,y,z]/(z^2 - x^3 - y^7)$ we have $K_{-1}(R) = k$ and $K_i(R) = 0$ for $i < -1$.

More generally, for a normal surface X we have $K_i(X) = 0$ for $i < -2$ and $K_{-2}(X) = \mathbb{Z}^\rho$ with ρ the number of loops in the exceptional divisor of a desingularization of X (Weibel).

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Cohomological properties of negative K-groups

- (Fundamental Theorem) We have canonically split exact sequence

$$0 \to K_i(X) \to K_i(X[t]) \times K_i(X[t^{-1}]) \to K_i(X[t,t^{-1}]) \to K_{i-1}(X) \to 0.$$

- (Zariski Mayer-Vietoris) For open covering $X = U \cup V$ we get exact sequence

$$\cdots \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(X) \to \cdots.$$

- (Excision) For ring homomorphism $A \to A'$, $I \subset A$ ideal mapping isomorphically onto ideal $I' \subset A'$, $i \leq 0$ we get isomorphism

$$K_i(A,I) \xrightarrow{\simeq} K_i(A',I').$$

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Elementary description of negative K-groups

Grothendieck/SGA 6

For X q-p (quasi-projective over noetherian ring)

$$K_0(X) = \mathbb{Z}\langle \mathcal{V} | \text{ loc. free, fin. gen. } \mathcal{O}_X\text{-module } \mathcal{V}\rangle / R$$

R generated by $[\mathcal{V}] - [\mathcal{V}'] - [\mathcal{V}'']$ for short exact sequences $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$.

Consequence of Fundamental Theorem (Bass)

For X q-p, $X' = X \times \mathbb{G}_m^j$

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R' contains $[\mathcal{W}|_{X'}]$ for with $\mathcal{W}$ loc. free, fin. gen. $\mathcal{O}_{X\times\mathbb{A}^j}$-module.

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Vanishing propositions

Vanishing Proposition I (via regularity)

For regular scheme X we have $K_i(X) = 0$ for $i < 0$.

Assume $X = \mathrm{Spec}(A)$, $i = -1$.
Let P be a f.g. projective $A[t, t^{-1}]$-module, then we show $[P] = 0 \in K_{-1}(A)$. Choose a f.g. $A[t]$-module P' with $P' \otimes_{A[t]} A[t, t^{-1}] \cong P$. Then choose finite resolution $P'_\bullet$ of P' by f.g. projective $A[t]$-modules. Then $\sum_i (-1)^i [P'_i] \in K_0(A[t])$ extends $[P] \in K_0(A[t, t^{-1}])$.

Vanishing Proposition II (via platification, K-Strunk)

For X q-p reduced scheme, $i < 0$ and $\gamma \in K_i(X)$ there exists a blow-up $q : \tilde{X} = \mathrm{Bl}_Z X \to X$ ($Z \subset X$ nowhere dense) such that $q^*(\gamma) = 0 \in K_i(\tilde{X})$.

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Topological blow-ups (motivation)

Consider cartesian "blow-up square" of "nice" topological spaces (CW-complexes ...):

$$\begin{array}{ccc} \tilde{X} & \longleftarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow \\ X & \underset{i}{\longleftarrow} & Y \end{array}$$

with q proper,
i closed immersion,
$\tilde{X} \setminus E \cong X \setminus Y$.

Topological descent theorem/Excision

For a generalized cohomology theory H^* we get a long exact descent sequence

$$\cdots \to H^n(X) \to H^n(Y) \oplus H^n(\tilde{X}) \to H^n(E) \to H^{n+1}(X) \to \cdots$$

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Heuristic proof of Weibel's conjecture (part (i))

Use induction on $d = \dim(X)$.
Let $\gamma \in K_i(X)$ with $i < -d$ ($X = X_{\mathrm{red}}$ q-p wlog)
By Vanishing Proposition there exists a blow-up $\tilde{X} = \mathrm{Bl}_Y X \xrightarrow{q} X$ with $q^*(\gamma) = 0$.
Assume we have an exact descent sequence

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As $\dim(Y), \dim(E) < d$ we have $K_{i+1}(E) = 0 = K_i(Y)$ (induction assumption), so $K_i(X) \overset{q^*}{\hookrightarrow} K_i(\tilde{X})$ is injective $\Rightarrow \gamma = 0$. □

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Descent for blow-ups

Mayer-Vietories induces the Zariski descent spectral sequence (for $\dim(X) < \infty$):

$$E_2^{pq} = H^p(X, K^{\sim}_{-q,X}) \Rightarrow K_{-p-q}(X)$$

$\rightsquigarrow$ wlog X local, reduced scheme in proof of Weibel's conjecture.

Descent Proposition for blow-ups

Consider local scheme X ($d = \dim(X)$), $Y \hookrightarrow X$, $q : \tilde{X} = \mathrm{Bl}_Y X \to X$ and $E = q^{-1}(Y)$ $\rightsquigarrow$ exact descent sequence

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Derived algebraic geometry

Definition (Toën-Vezzosi, Lurie)

A *derived scheme* is topological space $|\mathcal{X}|$ together with an ∞-sheaf $\mathcal{O}_\mathcal{X} : \mathrm{Opn}(|\mathcal{X}|)^{\mathrm{op}} \to$ **sRing**, where **sRing** $=$ (com. simplicial rings)/(weak eq.) is an $(\infty, 1)$-category, s.t.

- $t\mathcal{X} := (|\mathcal{X}|, \pi_0\mathcal{O}_\mathcal{X})$ is a scheme,
- $\pi_i\mathcal{O}_\mathcal{X}$ is a quasi-cohrent sheaf.

Note that t is a functor from derived schemes to schemes preserving finite limits.

Example

Commutative simplicial ring $A \rightsquigarrow$ affine derived scheme $\mathcal{X} = \mathrm{Spec}(A)$ with $t\mathcal{X} = \mathrm{Spec}(\pi_0 A)$ and $\Gamma(|\mathcal{X}|, \pi_i\mathcal{O}_\mathcal{X}) = \pi_i A$.

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K-theory of derived schemes

$\mathcal{X}$ derived scheme $\rightsquigarrow$ stable ∞-category $\mathrm{Perf}(\mathcal{X})$ of perfect $\mathcal{O}_{\mathcal{X}}$-modules $\rightsquigarrow$ $K(\mathcal{X}) = K(\mathrm{Perf}(\mathcal{X}))$ non-connective K-theory spectrum (via Waldhausen construction + delooping technique)

Proposition (Clausen-Mathew-Naumann-Noel)

K-theory of derived schemes satisfies Zariski Mayer-Vietoris.

Example

For $\mathcal{X}$ affine and $i \leq 1$ we have $K_i(\mathcal{X}) \xrightarrow{\simeq} K_i(t\mathcal{X})$.

Corollary

If $\mathcal{X}$ is separated and covered by $r+1$ affines then $K_i(\mathcal{X}) \xrightarrow{\simeq} K_i(t\mathcal{X})$ for $i \leq -r+1$.

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Derived blow-ups

Given an (ordinary) affine scheme $X = \mathrm{Spec}(A)$ and a sequence $\mathbf{a} = (a_0, \ldots, a_r) \in A^{r+1}$

$\rightsquigarrow$ commutative diagram of derived schemes (*derived blow-up square*)

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Properties:

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K-theory descent for derived blow-up squares

Blow-up Theorem (Thomason,K-Strunk-Tamme)

K-theory satisfies descent for derived blow-up squares, i.e. we get a long exact sequence

$$\cdots \to K_{i+1}(\mathcal{E}) \to K_i(X) \to K_i(\mathcal{Y}) \oplus K_i(\tilde{\mathcal{X}}) \to K_i(\mathcal{E}) \to \cdots$$

or equivalently $K(X,\mathcal{Y}) \simeq K(\tilde{\mathcal{X}},\mathcal{E})$.

Idea of proof: The perfect modules on $\mathcal{E} = \mathbb{P}^r_{\mathcal{Y}}$ have a filtration

$$\mathrm{Perf}_l(\mathcal{E}) = \langle \mathcal{O}_{\mathcal{E}}(0), \ldots, \mathcal{O}_{\mathcal{E}}(-l) \rangle, \quad l \in \{0, \ldots, r\}$$

with graded pieces $\mathrm{gr}_l(\mathcal{E}) \cong \mathrm{Perf}(\mathcal{Y})$.
Similar filtration for perfect modules on $\tilde{\mathcal{X}}$ with $\mathrm{gr}_0(\tilde{\mathcal{X}}) \cong \mathrm{Perf}(X)$ and $\mathrm{gr}_l(\tilde{\mathcal{X}}) \cong \mathrm{Perf}(\mathcal{Y})$ for $l > 0$. □

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Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A/I)$. Choose reduction $(a_0, \dots, a_r) \subset I$ with $r < d = \dim(X)$ $\rightsquigarrow$ derived blow-up square $\rightsquigarrow$ $K(X, \mathcal{Y}) \simeq K(\tilde{\mathcal{X}}, \mathcal{E})$. We get isomorphisms

$$K_i(X, Y = t\mathcal{Y}) \overset{\text{for } i \le 0}{\cong} K_i(X, \mathcal{Y}) \overset{\text{b-u thm.}}{\cong} K_i(\tilde{\mathcal{X}}, \mathcal{E}) \overset{\text{cor.}, i \le -r}{\cong}$$

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cdh-cohomology

The cdh-topology of a scheme X is generated Zariski covers + covers $\tilde{X} \coprod Y \to X$ with $Y \hookrightarrow X$ closed immersion and with $\tilde{X} \to X$ proper, isomorphism away from Y (Voevodsky).

Theorem (K-Strunk-Tamme, Cortinas-Haesemeyer-Schlichting-Weibel)

For $d = \dim(X) < \infty$ there is an isomorphism $K_{-d}(X) \cong H^d(X_{\mathrm{cdh}}, \mathbb{Z})$.

Example

If X has only an isolated singularity x with desingularization $q : \tilde{X} \to X$, $E = q^{-1}(x)$ then $K_{-d}(X) \cong H^{d-1}(\mathrm{Com}(E), \mathbb{Z})$.
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Continuous K-theory of affinoid algebras I

k non-archimedean valued field, A/k affinoid algebra

Definition (Karoubi-Villamayor, Calvo)

- $K_0^{\mathrm{cont}}(A) = K_0(A)$,
- $K_i^{\mathrm{cont}}(A) = \mathrm{coker}(K_{i+1}^{\mathrm{cont}}(A\langle t\rangle) \times K_{i+1}^{\mathrm{cont}}(A\langle t^{-1}\rangle) \to K_{i+1}^{\mathrm{cont}}(A\langle t, t^{-1}\rangle))$ for $i < 0$.

Here $A\langle t\rangle \subset A[\![t]\!]$ consists of power series $a_0 + a_1 t + \cdots$ with $\lim_i |a_i| = 0$ etc.

Remark

In order to extend K^{cont} in a sensible way to positive degrees one has to work with pro-abelian groups (Morrow, K-Saito-Tamme).

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If k is discretely valued and $d = dim(X)$ we have

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An analog of the cdh-topology calculation would be:

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