Negative algebraic $K$-theory

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Overview

\( X \) noetherian scheme \( \leadsto \) non-connective algebraic \( K \)-theory spectra \( K(X) \) (Thomason)
\( K_i(X) = \pi_i K(X) \)

Theorem (Weibel’s Conjecture (1980))

For \( d = \text{dim}(X) \) we have

(i) \( K_i(X) = 0 \) for \( i < -d \),

(ii) \( K_i(X) \xrightarrow{\sim} K_i(X[t_1, \ldots, t_r]) \) is an isomorphism for \( i \leq -d \).

Convention for talk

All (derived) schemes are noetherian.
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**Convention for talk**

All (derived) schemes are noetherian.
Remarks

- For (commutative) ring $A$ and $i < 0$ the group $K_i(A)$ was calculated by Bass for $\dim(A) \leq 1$ (1968)
- For $\dim(X) \leq 2$ conjecture was shown by Weibel using reductions of ideals (2001)
- For $X$ alg. variety in characteristic zero conjecture was shown by Cortinas-Haesemeyer-Schlichting-Weibel (2008)
- For $X$ alg. variety in positive characteristic part of conjecture shown assuming strong resolution of singularities by Geisser-Hesselholt, Krishna and up to $p$-torsion by Kelly
- General case K-Strunk-Tamme (2016)
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Examples

Example

For $R = \mathbb{C}[t, s]/(s^2 - (t^2 + s^3))$ we have $K_{-1}(R) = \mathbb{Z}$ and $K_n(R) = 0$ for $n < -1$.

More generally, for $\dim(X) \leq 1$ and $X$ excellent we have $K_i(X) = 0$ for $i < -1$ and $K_{-1}(X) = \mathbb{Z}^\rho$ (Bass).

Example

For a field $k$ and the normal surface $R = k[x, y, z]/(z^2 - x^3 - y^7)$ we have $K_{-1}(R) = k$ and $K_i(R) = 0$ for $i < -1$.

More generally, for a normal surface $X$ we have $K_i(X) = 0$ for $i < -2$ and $K_{-2}(X) = \mathbb{Z}^\rho$ with $\rho$ the number of loops in the exceptional divisor of a desingularization of $X$ (Weibel).
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Cohomological properties of negative $K$-groups

■ (Fundamental Theorem) We have canonically split exact sequence

$$0 \to K_i(X) \to K_i(X[t]) \times K_i(X[t^{-1}]) \to K_i(X[t, t^{-1}]) \to K_{i-1}(X) \to 0.$$ 

■ (Zariski Mayer-Vietoris) For open covering $X = U \cup V$ we get exact sequence

$$\cdots \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(X) \to \cdots.$$ 

■ (Excision) For ring homomorphism $A \to A'$, $I \subset A$ ideal mapping isomorphically onto ideal $I' \subset A'$, $i \leq 0$ we get isomorphism

$$K_i(A, I) \xrightarrow{\sim} K_i(A', I').$$
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Grothendieck/SGA 6

For $X$ $q$-$p$ (quasi-projective over noetherian ring)

$$K_0(X) = \mathbb{Z}\langle \mathcal{V} | \text{loc. free, fin. gen. } \mathcal{O}_X \text{-module } \mathcal{V} \rangle / R$$

$R$ generated by $[\mathcal{V}] - [\mathcal{V}'] - [\mathcal{V}'']$ for short exact sequences $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$.

Consequence of Fundamental Theorem (Bass)

For $X$ $q$-$p$, $X' = X \times \mathbb{G}_m$

$$K_{-j}(X) = \mathbb{Z}\langle \mathcal{W} | \text{loc. free, fin. gen. } \mathcal{O}_{X'} \text{-module } \mathcal{W} \rangle / R'$$

$R'$ contains $[\mathcal{W}|_{X'}]$ for with $\mathcal{W}$ loc. free, fin. gen. $\mathcal{O}_{X \times \mathbb{A}^1}$-module.
Elementary description of negative $K$-groups

\[ K_0(X) = \mathbb{Z}\langle V \mid \text{loc. free, fin. gen. } \mathcal{O}_X\text{-module } V \rangle / R \]

$R$ generated by $[V] - [V'] - [V'']$ for short exact sequences $0 \to V' \to V \to V'' \to 0$.

Consequence of Fundamental Theorem (Bass)

For $X$ q-p, $X' = X \times \mathbb{G}_m^j$

\[ K_{-j}(X) = \mathbb{Z}\langle V \mid \text{loc. free, fin. gen. } \mathcal{O}_{X'}\text{-module } V \rangle / R' \]

$R'$ contains $[W|_{X'}]$ for with $W$ loc. free, fin. gen. $\mathcal{O}_{X \times \mathbb{A}^j}$-module.
Vanishing propositions

Vanishing Proposition I (via regularity)

For regular scheme $X$ we have $K_i(X) = 0$ for $i < 0$.

Assume $X = \text{Spec}(A)$, $i = -1$.

Let $P$ be a f.g. projective $A[t, t^{-1}]$-module, then we show $[P] = 0 \in K_{-1}(A)$. Choose a f.g. $A[t]$-module $P'$ with $P' \otimes_{A[t]} A[t, t^{-1}] \cong P$. Then choose finite resolution $P'_\bullet$ of $P'$ by f.g. projective $A[t]$-modules. Then $\sum_i (-1)^i [P'_i] \in K_0(A[t])$ extends $[P] \in K_0(A[t, t^{-1}])$.

Vanishing Proposition II (via platification, K-Strunk)

For $X$ q-p reduced scheme, $i < 0$ and $\gamma \in K_i(X)$ there exists a blow-up $q : \tilde{X} = \text{Bl}_Z X \to X$ ($Z \subset X$ nowhere dense) such that $q^*(\gamma) = 0 \in K_i(\tilde{X})$.

Note: This is clear if there exists a desingularization $q : \tilde{X} \to X$. 

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Topological blow-ups (motivation)

Consider cartesian “blow-up square” of “nice” topological spaces (CW-complexes ...):

\[
\begin{array}{ccc}
\tilde{X} & \leftarrow & E \\
\downarrow & & \downarrow \\
q & & i \\
\downarrow & & \downarrow \\
X & \leftarrow & Y
\end{array}
\]

with \( q \) proper, \( i \) closed immersion,

\[ \tilde{X} \setminus E \cong X \setminus Y. \]

Topological descent theorem/Excision

For a generalized cohomology theory \( H^* \) we get a long exact descent sequence

\[
\cdots \rightarrow H^n(X) \rightarrow H^n(Y) \oplus H^n(\tilde{X}) \rightarrow H^n(E) \rightarrow H^{n+1}(X) \rightarrow \cdots
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Use induction on \( d = \dim(X) \).
Let \( \gamma \in K_i(X) \) with \( i < -d \) \((X = X_{\text{red}} \text{ q-p wlog})\)
By Vanishing Proposition there exists a blow-up \( \tilde{X} = \text{Bl}_Y X \rightarrow X \)
with \( q^*(\gamma) = 0 \).
Assume we have an exact descent sequence
\[
\cdots \rightarrow K_{i+1}(E) \rightarrow K_i(X) \rightarrow K_i(Y) \oplus K_i(\tilde{X}) \rightarrow \cdots.
\]
As \( \dim(Y), \dim(E) < d \) we have \( K_{i+1}(E) = 0 = K_i(Y) \) (induction assumption), so \( K_i(X) \xrightarrow{q^*} K_i(\tilde{X}) \) is injective \( \Rightarrow \gamma = 0 \). \( \square \)
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Use induction on $d = \dim(X)$. Let $\gamma \in K_i(X)$ with $i < -d$ ($X = X_{\text{red}}$ q-p wlog). By Vanishing Proposition there exists a blow-up $\tilde{X} = \text{Bl}_Y X \xrightarrow{q} X$ with $q^*(\gamma) = 0$. Assume we have an exact descent sequence

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As $\dim(Y), \dim(E) < d$ we have $K_{i+1}(E) = 0 = K_i(Y)$ (induction assumption), so $K_i(X) \xrightarrow{q^*} K_i(\tilde{X})$ is injective $\Rightarrow \gamma = 0$. $\square$
Heuristic proof of Weibel’s conjecture (part (i))

Use induction on $d = \dim(X)$.

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Mayer-Vietories induces the Zariski descent spectral sequence (for \( \dim(X) < \infty \)):

\[
E_2^{pq} = H^p(X, K_{\sim -q, X}) \Rightarrow K_{-p-q}(X)
\]

\( \rightsquigarrow \) wlog \( X \) local, reduced scheme in proof of Weibel’s conjecture.

**Descent Proposition for blow-ups**

Consider local scheme \( X \) (\( d = \dim(X) \)), \( Y \hookrightarrow X \),
\( q : \tilde{X} = \text{Bl}_Y X \to X \) and \( E = q^{-1}(Y) \rightsquigarrow \) exact descent sequence

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## Derived algebraic geometry

### Definition (Toën-Vezzosi, Lurie)

A **derived scheme** is a topological space $|\mathcal{X}|$ together with an $\infty$-sheaf $\mathcal{O}_\mathcal{X} : \text{Opn}(|\mathcal{X}|)^{\text{op}} \to \text{sRing}$, where $\text{sRing} = (\text{com. simplicial rings})/(\text{weak eq.})$ is an $(\infty, 1)$-category, s.t.

- $t\mathcal{X} := (|\mathcal{X}|, \pi_0 \mathcal{O}_\mathcal{X})$ is a scheme,
- $\pi_i \mathcal{O}_\mathcal{X}$ is a quasi-coherent sheaf.

Note that $t$ is a functor from derived schemes to schemes preserving finite limits.

### Example

Commutative simplicial ring $A \rightsquigarrow$ affine derived scheme $\mathcal{X} = \text{Spec}(A)$ with $t\mathcal{X} = \text{Spec}(\pi_0 A)$ and $\Gamma(|\mathcal{X}|, \pi_i \mathcal{O}_\mathcal{X}) = \pi_i A$. 
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A derived scheme is topological space $|\mathcal{X}|$ together with an ∞-sheaf $\mathcal{O}_X : \text{Opn}(|\mathcal{X}|)^{\text{op}} \to \text{sRing}$, where

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\( \mathcal{X} \) derived scheme \( \leadsto \) stable \( \infty \)-category \( \text{Perf}(\mathcal{X}) \) of perfect \( \mathcal{O}_{\mathcal{X}} \)-modules \( \leadsto K(\mathcal{X}) = K(\text{Perf}(\mathcal{X})) \) non-connective \( K \)-theory spectrum (via Waldhausen construction + delooping technique)

**Proposition (Clausen-Mathew-Naumann-Noel)**

\( K \)-theory of derived schemes satisfies Zariski Mayer-Vietoris.

**Example**

For \( \mathcal{X} \) affine and \( i \leq 1 \) we have \( K_i(\mathcal{X}) \xrightarrow{\sim} K_i(t\mathcal{X}) \).

**Corollary**

If \( \mathcal{X} \) is separated and covered by \( r + 1 \) affines then \( K_i(\mathcal{X}) \xrightarrow{\sim} K_i(t\mathcal{X}) \) for \( i \leq -r + 1 \).
$\mathcal{X}$ derived scheme $\leadsto$ stable $\infty$-category $\text{Perf}(\mathcal{X})$ of perfect $\mathcal{O}_\mathcal{X}$-modules $\leadsto K(\mathcal{X}) = K(\text{Perf}(\mathcal{X}))$ non-connective $K$-theory spectrum (via Waldhausen construction + delooping technique)

**Proposition (Clausen-Mathew-Naumann-Noel)**

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The derived scheme $\mathcal{X}$ is equivalent to a stable $\infty$-category $\text{Perf}(\mathcal{X})$ of perfect $\mathcal{O}_\mathcal{X}$-modules, which is equivalent to $K(\mathcal{X}) = K(\text{Perf}(\mathcal{X}))$ non-connective $K$-theory spectrum (via Waldhausen construction + delooping technique).

**Proposition (Clausen-Mathew-Naumann-Noel)**

The $K$-theory of derived schemes satisfies the Zariski Mayer-Vietoris property.

**Example**

For $\mathcal{X}$ affine and $i \leq 1$ we have $K_i(\mathcal{X}) \xrightarrow{\sim} K_i(t\mathcal{X})$.

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Proposition (Clausen-Mathew-Naumann-Noel)

\[ K \text{-theory of derived schemes satisfies Zariski Mayer-Vietoris.} \]

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Derived blow-ups

Given an (ordinary) affine scheme $X = \text{Spec}(A)$ and a sequence $a = (a_0, \ldots, a_r) \in A^{r+1}$, there is a commutative diagram of derived schemes (derived blow-up square)

\[
\begin{array}{ccc}
\tilde{X} & \xleftarrow{\varepsilon} & E \\
\downarrow & & \downarrow \\
X & \xleftarrow{\eta} & Y
\end{array}
\]

Properties:

- $Y = \text{Spec}(B)$ with $\pi_i B = H_i(A, a)$ (Koszul homology).
- For $a$ regular we have $\tilde{X} = \text{Bl}_{(a)}(X)$ and square is Cartesian in the category of schemes.
- Derived blow-up squares are compatible with (derived) pullback along morphism of affine schemes $X' = \text{Spec}(A') \to X$. 
Derived blow-ups

Given an (ordinary) affine scheme \( X = \text{Spec}(A) \) and a sequence \( \mathbf{a} = (a_0, \ldots, a_r) \in A^{r+1} \) 
\( \leadsto \) commutative diagram of derived schemes (\textit{derived blow-up square})

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\[ \downarrow \quad \downarrow \]

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Moritz Kerz  
Negative algebraic $K$-theory
Blow-up Theorem (Thomason, K-Strunk-Tamme)

\[ \cdots \to K_{i+1}(\mathcal{E}) \to K_i(X) \to K_i(Y) \oplus K_i(\tilde{X}) \to K_i(\mathcal{E}) \to \cdots \]

or equivalently \( K(X, Y) \cong K(\tilde{X}, \mathcal{E}) \).

Idea of proof: The perfect modules on \( \mathcal{E} = \mathbb{P}^r_Y \) have a filtration

\[ \text{Perf}_l(\mathcal{E}) = \langle \mathcal{O}_\mathcal{E}(0), \ldots, \mathcal{O}_\mathcal{E}(-l) \rangle, \quad l \in \{0, \ldots, r\} \]

with graded pieces \( \text{gr}_l(\mathcal{E}) \cong \text{Perf}(Y) \).

Similar filtration for perfect modules on \( \tilde{X} \) with \( \text{gr}_0(\tilde{X}) \cong \text{Perf}(X) \) and \( \text{gr}_l(\tilde{X}) \cong \text{Perf}(Y) \) for \( l > 0 \).

□
Blow-up Theorem (Thomason, K-Strunk-Tamme)

$K$-theory satisfies descent for derived blow-up squares, i.e. we get a long exact sequence

$$\cdots \to K_{i+1}(\mathcal{E}) \to K_i(X) \to K_i(Y) \oplus K_i(\tilde{X}) \to K_i(\mathcal{E}) \to \cdots$$

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Similar filtration for perfect modules on $\tilde{X}$ with $\text{gr}_0(\tilde{X}) \cong \text{Perf}(X)$ and $\text{gr}_l(\tilde{X}) \cong \text{Perf}(Y)$ for $l > 0$. \qed
Proof of the Descent Proposition

Let $X = \text{Spec}(A)$, $Y = \text{Spec}(A/I)$. Choose reduction $(a_0, \ldots, a_r) \subset I$ with $r < d = \dim(X) \Rightarrow$ derived blow-up square $\sim K(X, Y) \simeq K(\tilde{X}, \mathcal{E})$. We get isomorphisms

$$K_i(X, Y = tY) \overset{\sim}{=} K_i(X, Y) \overset{\text{b-u thm.}}{=} K_i(\tilde{X}, \mathcal{E}) \overset{\text{cor., } i \leq -r}{=} K_i(t\tilde{X}, t\mathcal{E}) \overset{\text{excision, } i < -d}{=} K_i(\tilde{X}, E)$$

The isomorphism $K_i(X, Y) \simeq K_i(\tilde{X}, E)$ for $(i < -d)$ induces the long exact sequence of the Descent Proposition. \qed
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cdh-cohomology

The cdh-topology of a scheme \( X \) is generated Zariski covers + covers \( \tilde{X} \amalg Y \to X \) with \( Y \to X \) closed immersion and with \( \tilde{X} \to X \) proper, isomorphism away from \( Y \) (Voevodsky).

Theorem (K-Strunk-Tamme, Cortinas-Haesemeyer-Schlichting-Weibel)

For \( d = \text{dim}(X) < \infty \) there is an isomorphism \( K_{-d}(X) \cong H^d(X_{cdh}, \mathbb{Z}) \).

Example

If \( X \) has only an isolated singularity \( x \) with desingularization \( q : \tilde{X} \to X, E = q^{-1}(x) \) then \( K_{-d}(X) \cong H^{d-1}(\text{Com}(E), \mathbb{Z}) \).

Here \( \text{Com}(E) \) is the configuration complex (points of \( \text{Com}(E) \cong \text{irred. comp.} \ (E) \)).
The cdh-topology of a scheme $X$ is generated Zariski covers $\tilde{X} \amalg Y \to X$ with $Y \hookrightarrow X$ closed immersion and with $\tilde{X} \to X$ proper, isomorphism away from $Y$ (Voevodsky).

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Continuous $K$-theory of affinoid algebras I

$k$ non-archimedean valued field, $A/k$ affinoid algebra

**Definition (Karoubi-Villamayor, Calvo)**

- $K_{0}^{\text{cont}}(A) = K_{0}(A)$,
- $K_{i}^{\text{cont}}(A) = \text{coker}(K_{i+1}^{\text{cont}}(A[t])) \times K_{i+1}^{\text{cont}}(A[t^{-1}]) \to K_{i+1}^{\text{cont}}(A[t, t^{-1}]))$ for $i < 0$.

Here $A[t] \subset A[[t]]$ consists of power series $a_0 + a_1 t + \cdots$ with $\lim_i |a_i| = 0$ etc.

**Remark**

In order to extend $K^{\text{cont}}$ in a sensible way to positive degrees one has to work with pro-abelian groups (Morrow, K-Saito-Tamme).
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As an analog of Weibel’s conjecture one proves:

**Theorem (K)**

If \( k \) is discretely valued and \( d = \text{dim}(X) \) we have

(i) \( K^\text{cont}_i(A) = 0 \) for \( i < -d \),

(ii) \( K^\text{cont}_i(A) \xrightarrow{\cong} K^\text{cont}_i(A\langle t_1, \ldots, t_r \rangle) \) is an isomorphism for \( i \leq -d \).

An analog of the cdh-topology calculation would be:

**Conjecture**

For \( d = \text{dim}(A) \) there is an isomorphism

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K_{-d}^\text{cont}(A) \cong H^d(\mathcal{M}(A), \mathbb{Z}).
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Here \( \mathcal{M}(A) \) is the Berkovich space of mult. seminorms of \( A \).
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