On Grothendieck–Serre conjecture concerning principal bundles

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Let $R$ be a regular local ring. Let $G$ be a reductive group scheme over $R$. A well-known conjecture due to Grothendieck and Serre asserts that a principal $G$-bundle over $R$ is trivial, if it is trivial over the fraction field of $R$.

The conjecture was stated by J.-P. Serre in 1958 in so-called constant case and by A. Grothendieck in 1968 in the general case.

The conjecture is solved in positive if $R$ contains a field.

In the first part of the talk we will discuss smooth complex algebraic varieties and some examples to the conjecture in which as the group $G$, so the principal $G$-bundle are involved only tacitely (non-explicitly).
In this introduction we give couple results motivating the conjecture in the constant case. To do that recall some notation.

Let $X$ be an affine complex algebraic variety, smooth and irreducible. Let $\mathbb{C}[X]$ be the ring of regular functions on $X$ and $f \in \mathbb{C}[X]$ be a non-zero function. Let

$$X_f := \{ x \in X : f(x) \neq 0 \}.$$
This open subset is called the principal open subset of $X$ corresponding to the function $f$.

This open subset $X_f$ is itself an affine algebraic variety and its ring of regular functions $\mathbb{C}[X_f]$ is the localization $\mathbb{C}[X]_f$ of the ring $\mathbb{C}[X]$ with respect to the element $f$.

If $A$ is a $\mathbb{C}[X]$-algebra, then we write $A_f$ for the localization of $A$ with respect to $f \in \mathbb{C}[X]$. 
Serre’s theorem (1958)

Let $A$ be a $\mathbb{C}[X]$-algebra, which is a free finitely generated $\mathbb{C}[X]$-module of rank $n$. Suppose that $A$ is isomorphic to the matrix algebra $M_r(\mathbb{C}[X])$ locally for the complex topology on $X$. Suppose further that for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras

$$A_f \quad \text{and} \quad M_r(\mathbb{C}[X_f])$$

are isomorphic.

Then for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and

$$A_g \cong M_r(\mathbb{C}[X_g])$$

as the $\mathbb{C}[X_g]$-algebras. In the other words, the $\mathbb{C}[X]$-algebras

$$A \quad \text{and} \quad M_r(\mathbb{C}[X])$$

are isomorphic locally for the Zariski topology on $X$. 
Ojanguren’s theorem (1982)

Let $X$ and $\mathbb{C}[X]$ be as above and let $a_i, b_i \in \mathbb{C}[X]$ be invertible functions on $X$, where $i \in \{1, \ldots, r\}$. Consider two quadratic spaces

$$P := \sum_{i=1}^{r} a_i T_i^2 \quad \text{and} \quad Q := \sum_{i=1}^{r} b_i T_i^2$$

over $\mathbb{C}[X]$. Suppose for a non-zero function $f \in \mathbb{C}[X]$ these quadratic spaces are isomorphic over the ring $\mathbb{C}[X_f]$.

Then the quadratic spaces $P$ and $Q$

are isomorphic locally for the Zariski topology on $X$.

In other words, for any point $x \in X$ there is a regular function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and quadratic spaces $P$ and $Q$ are isomorphic as quadratic spaces over $\mathbb{C}[X]_g$. 
A comment

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It is pretty clear now that one can try to state a rather general theorem in terms of principal $G$-bundles. To do that recall the notion of a

**PRINCIPAL $G$-bundle**
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Let $G$ be a linear complex algebraic group. Let $X$ be as above. Let $(E, \nu : G \times E \to E)$ be a pair such that $E$ is a complex algebraic variety together with a regular map $p : E \to X$ and $\nu$ is a $G$-action on $E$ respecting the map $p$. Write $g \cdot e$ for $\nu(g, e)$.

A principal $G$-bundle over $X$ is a pair $(E, \nu : G \times E \to E)$ above such that the map $p : E \to X$ is smooth surjective and

- the regular map $G \times E \to E \times_X E$ taking $(g, e)$ to $(g \cdot e, e)$ is an isomorphism of algebraic varieties;

In this case there exists a cover $X = \bigcup V_i$ in the complex topology on $X$ and holomorphic isomorphisms $\varphi_i : G \times V_i \to E|_{V_i} := p^{-1}(V_i)$ respecting as the projections onto $V_i$ so the $G$-actions on both sides.
An isomorphism between principal $G$-bundles $(E_1, \nu_1)$ and $(E_2, \nu_2)$ is a morphism $\psi : E_1 \to E_2$ respecting the projections on $X$, and the $G$-actions.
A trivial $G$-bundle is a $G$-bundle isomorphic to $G$-bundle of the form $(G \times X, \mu)$, where $g' \cdot (g, x) = ((g' \cdot g), x)$. A trivial bundle has a section. If a bundle $E$ has a section $s$, then it is trivial. Indeed, the map $(g, x) \mapsto g \cdot s(x)$ identifies $G \times X$ with $E$. 
Many examples of principal $G$-bundles are obtained by the following simple construction. Consider a closed embedding of algebraic groups $G \subset H$ and set $X = G \backslash H$ (the orbit variety of right cosets with respect to $G$). Then the pair

$$(H, \nu : G \times H \to H),$$

where $\nu$ takes $(g, h)$ to $g \cdot h$ is a principal $G$-bundle over $X$. The fibres of the projection $p : H \to X$ are right cosets of $H$ with respect to the subgroup $G$. 

[Diagram showing the construction and the orbit variety $X = G \backslash H$.]

[Diagram highlighting the principal $G$-bundle structure and the fibers.]
A principal $G$-bundle $E$ over $X$ is not necessarily trivial locally for the Zariski topology on $X$. However it is always trivial locally for the étale topology on $X$. In a picture the latter means the following: here $X'$ is smooth, $\pi : X' \to X$ is surjective and any point $x' \in X'$ one has $T_{X',x'} \cong T_{X,\pi(x')}$. 

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\[\begin{tikzcd}
X' \arrow{d}[swap]{\pi} \arrow{r}{\sim} & E \arrow{d}[swap]{\pi(x')} \\
X' \times G \arrow{r}{x'} & X
\end{tikzcd}\]
Examples of simple, semi-simple, reductive complex algebraic groups.

A reductive group is connected by an agreement due to Demazure and Grothendieck. 
$SL_n$, $PGL_n$, $SO_n$, $Spin_n$, $PGO_n^+$, $Sp_{2n}$, $PSp_{2n}$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$,  
$SL_3 \times E_6$, $Sp_{2n} \times Spin_m$

$GL_n$ and $GO_n$, $GSp_{2n}$ (the groups of similitudes).

We are ready now to state a very general result concerning principal $G$-bundles and extending the results from the introduction.
Theorem (R. Fedorov, I. Panin; 2013)

Let $G$ be a simple (or a semi-simple, or even a reductive) complex algebraic group. Let $X$ be an affine complex algebraic variety, smooth and irreducible and let $E_1$, $E_2$ be two principal $G$-bundles over $X$. Suppose there is a non-zero regular function $f \in \mathbb{C}[X]$ such that the principal $G$-bundles $E_1|_{X_f}$ and $E_2|_{X_f}$ are isomorphic over $X_f$.

Then the principal $G$-bundles $E_1$ and $E_2$ are isomorphic locally for the Zariski topology on $X$.

Remark. Particularly, if $E_1$ is trivial over a non-empty Zariski open subset of $X$, then $E_1$ is trivial locally for the Zariski topology on $X$. 
Examples illustrating the Theorem.

- Let $A_1$ and $A_2$ be two algebras as in the Serre’s theorem above. They are called Azumaya $\mathbb{C}[X]$-algebras. Suppose for a non-zero function $f \in \mathbb{C}[X]$ the $\mathbb{C}[X_f]$-algebras $(A_1)_f$ and $(A_2)_f$ are isomorphic. Then the $\mathbb{C}[X]$-algebras $A_1$ and $A_2$ are isomorphic locally for the Zariski topology on $X$.

- Let $P$ and $Q$ be the quadratic spaces over $\mathbb{C}[X]$ as in Ojanguren’s theorem. Suppose they are in the same similarity class over the field $\mathbb{C}(X)$, then they are in the same similarity class locally for the Zariski topology on $X$. 
The Conjecture

Non-constant case of the conjecture for complex algebraic varieties.

• Example 1. Let $a, b \in \mathbb{C}[X]^\times$. Consider an equation

$$T_1^2 - aT_2^2 = b$$

If this equation has a solution over the field $\mathbb{C}(X)$ then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the equation (1) has a solution in $\mathbb{C}[X_g]$.

• Example 2. Let $a, b, c \in \mathbb{C}[X]^\times$. Consider an equation

$$T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = c$$

Suppose this equation has a solution over the field $\mathbb{C}(X)$. Then for any point $x \in X$ there is a function $g \in \mathbb{C}[X]$ such that $g(x) \neq 0$ and the equation (2) has a solution in $\mathbb{C}[X_g]$. 
Reformulate these statements in terms of principal $G$-bundles for reductive group schemes over our complex algebraic variety $X$.

Recall for that notion of a reductive group $X$-scheme and a principal $G$-bundle.
Let $X$ be as above. A smooth $X$-group scheme consists of the data $p : G \to X$, $\mu : G \times_X G \to G$, $i : G \to G$, $e : X \to G$, where $p$, $\mu$, $i$, $e$ are regular maps. The requirements are the obvious ones.

- In the example (1) consider an $X$-group scheme defined by the equation $T_1^2 - aT_2^2 = 1$. Call it $T$.

- In the example (2) consider an $X$-group scheme defined by the equation $T_1^2 - aT_2^2 - bT_3^2 + abT_4^2 = 1$. Call it $SL_{1,A}$, where $A$ is the generalized quaternion $\mathbb{C}[X]$-algebra for the pair $a, b$.

One has $T \cong \mathbb{C}^\times \times X$, $SL_{1,A} \cong SL_2(\mathbb{C}) \times X$, locally for the complex topology on $X$. 
The following well-known definition shows that the two $X$-group schemes $T$ and $SL_{1,A}$ are REDUCTIVE $X$-GROUP SCHEMES.

Being a bit non-precise, an $X$-group scheme $G$ is called a reductive if for a complex algebraic reductive group $G_0$

$$G \cong G_0 \times X$$

holomorphically isomorphic locally for the complex topology on $X$. Recall that $G_0$ is required to be connected. The class of reductive group schemes contains the class of semi-simple group schemes which in turn contains the class of simple group schemes.

Examples: $T$, $SL_{1,A}$, $PGL_n$, $Spin_Q$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$. 
Let $G$ be a reductive $X$-group scheme. A principal $G$-bundle over $X$ consists of data $(p : E \to X, \nu : G \times_X E \to E)$ such that $p$ is a smooth surjective regular map, $\nu$ is a $G$-action respecting the projections on $X$ and 1) the regular map $G \times_X E \to E \times_X E$ taking $(g, e)$ to $(ge, e)$ is an isomorphism of algebraic varieties;

A principal $G$-bundle $E$ is called trivial if there is an isomorphism $E \to G$ over $X$, which respects the obvious left $G$-action on both sides. $E$ is trivial if and only if there is a section $s : X \to E$ of the projection $p : E \to X$.

The equation (1) above defines a principal $T$-bundle. The equation (2) above defines a principal $SL_1,A$-bundle.
Theorem non-constant case: R.Fedorov, I.Panin; 2013

Let $G$ be a complex algebraic reductive $X$-group scheme and $E$ be a principal $G$-bundle. Suppose for a non-zero function $f$ the principal $G$-bundle $E|_{X_f}$ is trivial over $X_f$. Then $E$ is trivial locally for the Zariski topology on $X$.

Corollary

Let $H$, $\mu : H \to G_{m,X}$ and $\lambda \in \mathbb{C}[X]^{\times}$. Suppose the kernel $\ker(\mu)$ is a reductive $X$-group scheme. If the equation $\mu(h) = \lambda$ has a solution over $\mathbb{C}(X)$, then it has a solution locally for the Zariski topology on $X$. 
The Conjecture

Let $U = \text{Spec}(R)$ be an irreducible regular scheme and $G$ be a reductive $U$-group scheme. Recall that a $U$-scheme $E$ with an action of $G$ is called a principal $G$-bundle over $U$, if $E$ is smooth and surjective over $U$ and the morphism $G \times_U E \to E \times_U E$ taking $(g, e)$ to $(ge, e)$ is an isomorphism (see [Gro5, Section 6]).

Conjecture[Serre (1958), Grothendieck (1968)]. Let $K$ be the fraction field of a regular local ring $R$. If $E(K) \neq \emptyset$, then $E(R) \neq \emptyset$.

Theorem. If $R$ is a regular local ring containing a field, then the above conjecture holds. That is $[E(K) \neq \emptyset \Rightarrow E(R) \neq \emptyset]$. 
This theorem is proved by R. Fedorov and the author in [FP, 2013] in the case, when $R$ contains an infinite field. It is proved by the author in [Pan, 2015], when $R$ contains a finite field.

Corollary. Let $R$ be a regular local ring, $K$ be its field of fractions, $U = \text{Spec}(R)$. Let $\mu : H \to \mathbb{G}_{m,U}$ be a smooth $U$-group morphism, where $H$ is a reductive $U$-group scheme. Suppose the kernel $\ker(\mu)$ is a reductive $U$-group scheme. Then the inclusion of $R$ into $K$ induces an injection

$$R^\times / \mu(H(R)) \hookrightarrow K^\times / \mu(H(K)).$$
History of the topic

History of the topic. — In his 1958 paper Jean-Pierre Serre asked whether a principal bundle is Zariski locally trivial, once it has a rational section (see [Ser, Remarque, p. 31]). In his setup the group is any algebraic group over an algebraically closed field. He gave an affirmative answer to the question when the group is PGL(n) (see [Ser, Prop. 18]) and when the group is an abelian variety (see [Ser, Lemme 4]). In the same year, Alexander Grothendieck asked a similar question (see [Gro1, Remarque 3, pp. 26–27]). A few years later, Grothendieck conjectured that the statement is true for any semi-simple group scheme over any regular local scheme (see [Gro 4, Remarque 1.11.a]). Now by the Grothendieck–Serre conjecture we mean Conjecture 1 though this may be slightly inaccurate from historical perspective. Many results corroborating the conjecture are known.
Here is a list of known results in the same vein, corroborating the Grothendieck–Serre conjecture.

- The case when the group is $\text{PGL}_n$ and the base field is algebraically closed is done by J.-P. Serre in 1958.
- The case when the group scheme is $\text{PGL}_n$ and the ring $R$ is an arbitrary regular local ring is done by A. Grothendieck in 1968.
- The case when the local ring $R$ contains a field of characteristic not 2 the group is $\text{SO}_n$ over the ground field is done by M. Ojanguren in 1982.
- The case of an arbitrary reductive group scheme over a discrete valuation ring or over a henselian ring is solved by Y. Nisnevich in 1984.
- The case, where $G$ is an arbitrary torus over a regular local ring, was settled by J.-L. Colliot-Thélène and J.-J. Sansuc in 1987.
The case, when $G$ is quasi-split reductive group scheme over arbitrary two-dimensional local rings, is solved by Y. Nisnevich in 1989.

The case, where the group scheme $G$ comes from an infinite perfect ground field, solved by J.-L. Colliot-Thélène, M. Ojanguren in 1992. As far as we know this work was inspired by the one [Oj1, 1982].

The case, where the group scheme $G$ comes from an arbitrary infinite ground field, solved by M. S. Raghunathan 1994.

O. Gabber announced in 1994 a proof for group schemes coming from arbitrary ground fields (including finite fields).
• For the group scheme $\text{SL}_{1,A}$, where $A$ is an Azumaya $R$-algebra and $R$ contains a field the conjecture is solved by A. Suslin and the author in 1998

• For the unitary group scheme $U^\varepsilon_{A,\sigma}$, where $(A, \sigma)$ is an Azumaya $R$-algebra with involution $R$ contains a field of characteristic not 2 the conjecture is solved by M. Ojanguren and the author in 2001

• For the special unitary group scheme $SU_{A,\sigma}$, where $(A, \sigma)$ is an Azumaya $R$-algebra with a unitary involution and $R$ contains a field of characteristic not 2 the conjecture is solved by K. Zainoulline in 2001

• For the spinor group scheme $\text{Spin}_Q$ of a quadratic space $Q$ over $R$ containing a field of characteristic not 2 the conjecture is solved M. Ojanguren, K. Zainoulline and the author in 2004
Under an isotropy condition on $G$ the conjecture is proved by A. Stavrova, N. Vavilov and the author in a series of preprints in 2009, published as papers in 2015 and in 2016.

The case of strongly inner simple adjoint group schemes of the types $E_6$ and $E_7$ is done by the second author, V. Petrov, A. Stavrova and the second author in 2009. No isotropy condition is imposed there.

The case, when $G$ is of the type $F_4$ with trivial $f_3$-invariant and the field is infinite and perfect, is settled by V. Petrov and A. Stavrova in 2009.

The case, when $G$ is of the type $F_4$ with trivial $g_3$-invariant and the field is of characteristic zero, is settled by V. Chernousov in 2010.
The conjecture is solved when $R$ contains an infinite field, by R.Fedorov and the author in a preprint in 2013 and published in 2015.

The conjecture is solved by the author in the case, when $R$ contains a finite field in 2015 (for a better structured proof see [Pan3,2017]).

So, the conjecture is solved in the case, when $R$ contains a field.

The case of mixed characteristic is widely open. Let us indicate two recent interesting preprints [F1] and [PS3]. In [F1] the conjecture is solved for a large class of regular local rings of mixed characteristic assuming that $G$ splits. In [PS3] the conjecture is solved for any semi-local Dedekind domain providing that $G$ is simple simply-connected and $G$ contains a torus $\mathbb{G}_{m,R}$. 
A sketch of the proof in the constant simply connected case assuming the base field is \( \mathbb{C} \). Suppose a principal \( G \)-bundle \( E \) over \( X \) is trivial over \( X_f \). Then there exists a principal \( G \)-bundle over \( \mathbb{C} \times U \) as on the picture. One has isomorphisms
\[
G \times U \cong E \big|_{1 \times U} \cong E \big|_{0 \times U} = E \big|_U.
\]

Show that (1)–(3) yield the constant simply connected case once the base field is \( \mathbb{C} \).