ON THEORIES IN MATHEMATICS EDUCATION AND THEIR CONCEPTUAL DIFFERENCES

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Three “moments” in the evolution of mathematics education

• The *first moment* is the emergence of the research field around 1900.
• The *second moment* is the period from around 1950 to 2000.
• The *third moment* is today (ca. 2000 - today)
The first “moment”

Two major events:

• The foundation of the journal *L’Enseignement mathématique* in 1899, and

• The creation of the *International Commission on Mathematical Instruction (ICMI)* in 1908.
The “modern civilization”

What made this period modern was precisely its new form of production, namely industrial production.
“Without the principles of mechanics, analytical geometry and differential calculus, nothing of what constitutes modern civilization would exist” (1914)

Émile Borel
The goal of mathematical instruction
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- The socialist camp
- The humanist camp
“[W]e can now no longer afford to present mathematical science to our students in its purely speculative aspects ... we must endeavour, no matter what —more as a favour to society than as a favour to our students— to bend mathematical abstractions to the necessities of [social] reality.”

Carlo Bourlet
The humanist camp

“If industrialism or utilitarianism has had preponderant influences in the teaching of middle schools, mathematicians should fight them.”

Giuseppe Veronese, 1911
I alternate without distinction the theorems of plane geometry with those of solid geometry, since [this sharpens] the intellect and help[s] the development of that geometric imagination that is an essential quality to the engineer, so that he can think of the figures in space even without the aid of a design or a model.

Luigi Cremona, 1873
The merging of plane and solid geometry may be something preferable from a logical point of view.

But it seems to me that, pedagogically, we must first think of dividing difficulties. "Seeing in space" is a serious problem in itself, which I do not consider should be added first of all to the other problems [of plane geometry].

Jacques Hadamard, 1906.
Intuition versus Rigour

A rigorous proof in abstract geometry can never be based only on intuition; it must be founded on *logical deduction*.

Nevertheless *intuition* cannot be substituted by logical considerations. Intuition helps us to construct a proof and to gain an overview, it is, moreover, a source of inventions and new mental connections.

Felix Klein, 1901.
How to teach?

• Sensible intuition is followed by rational intuition, the preponderance of memory is followed by reason.

• Teaching processes must follow this progression.

Zoel G. de Saldeano, 1899

(1) an active role for the student, and

(2) attention to the needs and interests of the student.
The student is brought to the fore

- “Often the authors give no natural and non-trivial problems which lead the student to feel a need for vector concepts. A number of these presentations seem overly abstract”
  
  (Rosenbloom 1969; p. 405).

- “The pupil himself should re-invent mathematics”

- “The stress is shifted from teaching to learning, from the teacher's action to the pupil's . . . The pupil must perform the action.”

  (Freudenthal, 1973, p. 110)
“[This] method forces the student to work and discover the results himself instead of attending a boring class.”

(Roumanet, 1969, p. 222)
Constructivism

Two general goals of mathematics instruction that follow from constructivism [are]: the construction of increasingly powerful [mental] conceptual structures and the development of intellectual autonomy.

(Cobb, 1988).
Theoretical Principles

• $p_1$: knowledge is not passively received but built up by the cognizing subject; and

• $p_2$: the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality.

(von Glasersfeld, 1995)
How does constructivism translate into practice?

The teacher’s role is not merely to convey to students information about mathematics. One of the teacher’s primary responsibilities is to facilitate profound cognitive restructuring and conceptual reorganizations.  

(Cobb, 1988)
Theoretical Principles

• $p_3$: the cognizing subject not only constructs her own knowledge but she must do so in an *autonomous* way.
Introducing 9-10-year-old students to algebra

- $a_{10} = 2 \times 10 + 1$
- $a_{103} = 2 \times 103 + 1$
- $a_n = 2 \times n + 1$

- $a_{10} = 2 \times 11 - 1$
- $a_{103} = 10 \times a_{10} - 9 + a_3$
The autonomy project

“Betraying a secret that could be discovered by the child itself is bad pedagogics; it is even a crime.”


What if the child cannot discover it?
The theory of didactical situations (TDS)
\( \rho_1 \): In accordance with Piaget’s genetic epistemology learning is a form of cognitive adaptation to the environment.
For constructivism, knowledge is a subjective psychological construct (i.e., the ideas that the student creates or constructs).

For the TDS, knowledge is the mathematical knowledge recognized by the community of mathematicians.
\( p_2 \): knowledge appears as the optimal solution to a certain situation or problem.
The role of the teacher

The modern conception of teaching . . . requires the teacher to provoke the expected adaptation in her students by a judicious choice of “problems” that she puts before them. These problems, chosen in such a way that students can accept them, must make the students act, speak, think, and evolve by their own motivation. (Brousseau)
$p_3$: for every piece of mathematical knowledge there is a family of situations to give it an appropriate meaning. This family is called a \textit{fundamental situation}. 
The Milieu

• Is the “objects (physical, cultural, social, human) with which the subject interacts in a situation”

• “Example: the sheet of paper, the ruler and the compass generate the milieu of the Euclidean plane geometry” (p. 3).

• “The student learns by adapting herself to a milieu.”

• “But a milieu without didactical intentions is manifestly insufficient to induce in the student all the cultural knowledge that we wish her to acquire.”

(Brousseau, 2003)
Theoretical Principles

$p_4$: the student’s autonomy is a necessary condition for the genuine learning of mathematics.
Concerning knowledge, and in contrast to constructivism, some contemporary sociocultural theories do not conceive of knowledge as something that the learner constructs. Knowledge (at school, at the university) is a cultural-historical entity that is there, in our culture, at the moment of our birth. We need, hence, to resort to other metaphors different from the one of construction to provide accounts of learning.
Rethinking the role of teachers and students

We talk about teachers and students working together in processes through which the students encounter in a critical manner and through meaningful, aesthetic, collective experiences a culturally and historically constituted knowledge.
Rigour versus Intuition

“For the pure geometer, this faculty [of intuition] is necessary; it is by logic that one demonstrates, but by intuition that one invents.”

(Henri Poincaré, 1899)
“To teach mathematics is to teach logical precision. A mathematical teacher who has not taught that has taught nothing,” insofar as “the object of a mathematical education is to acquire the powers of analysis, of generalization, and of reasoning” (1913) 

Alfred North Whitehead