Quantitative estimates for advective equations with compressible flows

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Our goal

Consider the advection equation on the density $\rho(t, x)$:

$$\partial_t \rho + \mathrm{div}\,(\rho\, u) = 0,$$

for a velocity field $u(t, x)$ in some Sobolev space $L^1_t W^{1,p}_x$.
The field u is either given or coupled with ρ.

$\longrightarrow$ Many examples of such equations, linear or non-linear in physics (Fluid mechanics...), the Bio-Sciences and other fields (transport models in biophysics, animal migration, pedestrian dynamics...).

But we are interested in the case where $\mathrm{div}\, u \notin L^\infty$ so that the flow may be compressible. Instead we only know a priori that $\rho \in L^q_{t,x}$ for some $q > 1$.

Question: Can one propagate some explicit regularity on ρ?

Renormalized solutions

- Well posedness for linear advection eqs (u given) was obtained by DiPerna and Lions for $u \in L^1_t W^{1,p}_x$ with $p \geq 1$, extended to BV by Ambrosio, through renormalized solutions.
- Renormalization: Prove that for any weak solution ρ then

$$\partial_t \chi(\rho) + \mathrm{div}\,(\chi(\rho)\,u) + \mathrm{div}\,u\,(\chi'(\rho)\,\rho - \chi(\rho)) = 0.$$

- Main idea: For some convolution kernel K_h, write an equation on $K_h \star \rho$ and show a commutator estimate

$$\int K_h(x-y)\,(u(x)-u(y))\,\rho(y)\,dy \longrightarrow 0, \quad \text{in } L^1.$$

- There is a need for more quantitative estimates but those are essentially only available for nearly incompressible flows (Crippa-DeLellis, J., Léger, Seeger-Smart-Street, Seis...).

A first application: Compressible Fluid dynamics

The compressible Navier-Stokes system reads

$$\partial_t \rho + \mathrm{div}\,(u\,\rho) = 0,$$
$$\partial_t(\rho\,u) + \mathrm{div}\,(\rho\,u \otimes u) - \mu\Delta u - (\lambda + \mu)\,\nabla \mathrm{div}\,u = \; - \nabla p(\rho),$$

written here in the barotropic case, in a bounded domain $\Omega \subset \mathbb{R}^d$ with for instance Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0.$$

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$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho\, u \otimes u) - \mathrm{div}\,(\mu(\theta)\, D\, u) + \nabla(\lambda(\theta)\,\mathrm{div}\, u) = -\nabla p(\rho, \theta),$$
$$\partial_t(\rho\, E(\rho, \theta)) + \mathrm{div}\,(\rho\, u\, E) + \mathrm{div}\,(p\, u) = \mathrm{div}\,(\mathcal{S}\, u) + \mathrm{div}\,(\kappa(\theta)\,\nabla\theta).$$

where $D\,u = (\nabla u + \nabla u^T)/2$.

Variants of course exist:

- With temperature for the Navier-Stokes-Fourier system

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where $D\,u = (\nabla u + \nabla u^T)/2$.

Variants of course exist:

- With temperature for the Navier-Stokes-Fourier system
- With density dependent viscosity

(see nevertheless Bresch-Desjardins, Mellet-Vasseur, Vasseur-Yu...)

The difficulty

One has the following a priori estimates
Energy estimate: For $P(\rho)$ s.t. $P'\,\rho - P = p(\rho)$,

$$\int \left(P(\rho(t,x)) + \frac{1}{2}\,\rho\,u^2\right)\,dx + \int_0^t \int |\nabla u|^2 = const.$$

Note that if $C^{-1}\rho^\gamma \leq p \leq C\,\rho^\gamma$ then $P(\rho) \sim \rho^\gamma$.

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Pressure estimates

$$\int_0^t \int \rho^a\, p(\rho)\, dx\, dt \le C, \qquad a < \frac{2}{d}\,\gamma - 1.$$

$\longrightarrow$ The issue is to obtain compactness on ρ, *i.e.*
Control possible oscillations in the density.

The Lions-Feireisl theory

By using the monotonicity of the pressure law, Lions obtained that

Theorem *P.L. Lions*
Assume $p'(\rho) \sim \rho^{\gamma-1}$ *with* $\gamma > 9/5$ *and p monotone.*
Then there exists a weak solution to the compressible Navier-Stokes system.

With refined techniques, this was improved to

Theorem *E. Feireisl*
Assume $p'(\rho) \sim \rho^{\gamma-1}$ *with* $\gamma > 3/2$ *and p monotone.*
Then there exists a weak solution to compressible Navier-Stokes.

Which notion of solutions?

- Strong/classical solutions are of course the most convenient. They provide uniqueness and they preserve the most physical properties such as conservation of energy...
- However strong solutions only exist for short times, even in dimension 2 (vacuum problem), or for small initial perturbations of an equilibrium in some cases.
- Weak solutions can be global in time and also allow to work with non smooth initial data with only a bound on the energy.

What should the pressure law be?

- Ideal gas (Clapeyron 1834):

$$p = \rho\,\theta.$$

- Van der Waals law (1873):

$$(p + a\,\rho^2)\,(1 - b\,\rho) = c\,\rho\,\theta.$$

- Polynomial barotropic flows:

$$p = p(\rho), \quad \text{with often} \quad p = \rho^\gamma.$$

- Virial equation of state (H. Kamerlingh Onnes 1901):

$$p = \rho\,\theta\,\left(1 + B(\theta)\,\rho + C(\theta)\,\rho^2 + \ldots\right).$$

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$$p = p(\rho), \quad \text{with often} \quad p = \rho^\gamma.$$

- Variants of the Virial such as the Benedict-Webb-Rubin:

$$p = \rho\,R\,\theta + \left(B_0\,R\,\theta - A_0 - \frac{C_0}{\theta^2}\right)\rho^2 + (b\,R\,\theta - a)\,\rho^3 + \alpha\,a\,\rho^6 + \frac{c\,\rho^3}{\theta^2}(1 + \gamma\,\rho^2)\,e^{-\gamma\,\rho^2}.$$

Monotone vs non-monotone pressure laws

- Thermodynamically the stability of the equilibrium is directly connected to the monotonicity of p.
- Monotone laws are also required for hyperbolicity.
- However, many physical models have p non monotone.
- It is not clear why a thermodynamical assumption should control the stability of solutions over bounded times.
- The same type of questions may be asked about the stress tensors. For instance in some geophysical flows, one needs to take

$$\partial_t(\rho\, u) + \mathrm{div}\,(\rho\, u \otimes u) - \mu\, \Delta u - \mu_z\, \partial_{zz} u = -\nabla p(\rho),$$

with $\mu_z \neq 0$ (see for instance Temam-Ziane).

Our new result

Main improvements:
No monotonicity assumption on p, explicit regularity.

Theorem

(Bresch-Jabin 2018) Assume that for $\gamma > 9/5$

$$\frac{\rho^\gamma}{C} \leq p(\rho) \leq C\,\rho^\gamma, \quad |p'(\rho)| \leq C\,\rho^{\gamma-1}.$$

Then there exists a weak solution to compressible Navier-Stokes, with d the space dimension

$$\int \frac{|\rho(x)-\rho(y)|}{(|x-y|+h)^d}\,dx\,dy \leq C(E^0)\,\frac{|\log h|}{\log|\log h|}.$$

This is a $\log\log$ scale vs the usual log scale obtained by Crippa and DeLellis but for nearly incompressible flows instead of compressible.

Advection equations with anelastic constraints

Consider the following advective equation

$$a\,(\partial_t\phi + u\cdot\nabla\phi) = 0 \text{ in } (0,T)\times\Omega$$

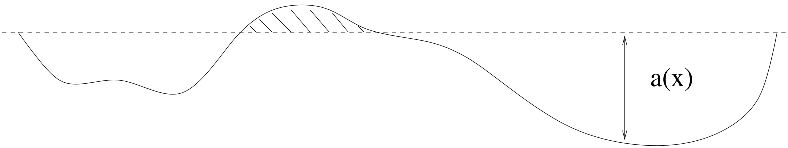
where $a = a(x)$ is given. and with a velocity field u s.t.

$$\operatorname{div}(au) = 0 \text{ in } (0,T)\times\Omega, \qquad a\,u\cdot n|_{(0,T)\times\Omega} = 0.$$

$\longrightarrow$ Many examples of such systems

- The so-called lake equation (see Bresch-Métivier, Lacave-Nguyen-Pausader, Levermore-Oliver-Titi, Masmoudi, Munteanu...) where a is the bathymetry and u is coupled to ϕ through $\operatorname{curl} u = a\,\phi$.
- Meteorology (Duran, Klein, Lipps-Hemler, Wilhelmon-Ogura), congestion (Perrin-Zatorska), floating objects (see Lannes)...

The difficulty



Main issue: The bathymetry a may be singular, and vanishes on a potentially complex set, where the equations degenerate.

The difficulty

Main issue: The bathymetry a may be singular, and vanishes on a potentially complex set, where the equations degenerate.

- First observe that a priori, $\phi \in L^\infty$ trivially and in fact $\phi \in L^\infty_t L^p_x(a\,dx)$ for any $p \geq 1$. But $\operatorname{div} u \notin L^\infty$ and it is not possible to reduce the problem to an incompressible equation.
- For given a and u, obtaining weak solutions is straightforward and require only minimal assumptions on a. This does not provide uniqueness, even in such a linear setting.
- Strong solutions require much more regularity on a and u.
- Obtaining renormalized solutions should still allow for a potentially singular bathymetry a but maintains the physical properties: Energy, uniqueness in the linear case...

$\longrightarrow$ Compactness and hence additional regularity on ϕ is required.

Our result

Theorem

Assume that for some $p > 1$ *and* $\frac{1}{q} + \frac{1}{p} < 1$, *there exists* $0 \le \alpha \le a$ *s.t.*

$$\int_\Omega \Big(a(x)\, |\log \alpha(x)| + |a(x)^{1/p^*} \nabla \log \alpha(x)|^q \Big)\, dx < \infty,$$

and that

$$\|u\|_{L^\infty_t L^p_a} + \int_0^T \int_\Omega a(x)\, |\nabla u(t,x)|\, \log\big(e + a(x) |\nabla (u(t,x))|\big)\, dx\, dt < \infty.$$

Then there exists a renormalized solution to the system.

$\longrightarrow$ Having $\alpha \neq a$ allows a wide range of possible a: For example if $a = 0$ on O^c for some Lipschitz domain O and on O

$$(d(x, \partial O))^k \lesssim a(x) \lesssim (d(x, \partial O))^l \implies \alpha(x) = (d(x, \partial O))^\theta, \; \theta \text{ large}.$$

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}\,dx\,dy,$$

for some $k \geq d$.

However this corresponds to a Sobolev like regularity on ρ which cannot work (see Alberti-Crippa-Mazzucato, J.). So instead...

The idea

Propagate some explicit regularity on ρ by computing

$$\int \frac{|\rho(t,x)-\rho(t,y)|}{(|x-y|+h)^k}\, W(t,x,y)\, dx\, dy,$$

for some $k \geq d$.
Where the weight W solves the same transport equation

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -D,$$

for a well chosen penalization D.

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for a well chosen penalization D.

Then explain that W cannot be too small, too often to bound

$$\int \frac{|\rho(x)-\rho(y)|}{(|x-y|+h)^k}\, dx\, dy,$$

in terms of h.

Because of the compressible flow, this will force $k=d$.

Sketch of the estimates: Compressible N-S

Define $w(t,x)$ solution to

$$\partial_t w + u \cdot \nabla w = -\lambda \, M \, |\nabla u| \, w,$$

with M the maximal operator.
Actually to obtain realistic conditions on γ, we take

$$\partial_t w + u \cdot \nabla w = -\lambda \left(\rho \, |\mathrm{div}\, u| + M \, |\nabla u| + \rho^\gamma\right) w.$$

But we will keep the simplest version here...

Sketch of the estimates: Compressible N-S

Define $w(t,x)$ solution to

$$\partial_t w + u \cdot \nabla w = -\lambda\, M\, |\nabla u|\, w,$$

with M the maximal operator.
One can simply show that

$$\int |\log w|\, \rho(t,x)\, dx \leq \lambda\, \|u\|_{L^2_t H^1_x}\, \|\rho\|_{L^2_{t,x}}.$$

This forces the choice

$$W(t,x,y) = w(t,x) + w(t,y),$$

so that $W = 0$ only if both $w(x) = 0$ and $w(y) = 0$, *i.e.* if $\rho(x) = 0$ and $\rho(y) = 0$.
This would not be needed for nearly incompressible flows where we could take $W = w(x)\, w(y)$ and it creates difficulties...

Sketch of the estimates: Compressible N-S, Part 2

Denote $\delta\rho = \rho(t,x) - \rho(t,y)$ and per Kruzkov' doubling of variables

$$\partial_t |\delta\rho|^2 + \mathrm{div}_x(u(t,x)\,|\delta\rho|^2) + \mathrm{div}_y(u(t,y)\,|\delta\rho|^2)$$
$$= -\frac{1}{2}\,(\mathrm{div}\,u(x) - \mathrm{div}\,u(y))\,\delta\rho\,(\rho(x)+\rho(y)).$$

Recall

$$\partial_t W + u(t,x)\cdot\nabla_x W + u(t,y)\cdot\nabla_y W = -D,$$

and calculate

$$\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,W$$
$$= -\frac{1}{2}\int \frac{\mathrm{div}\,u(x) - \mathrm{div}\,u(y)}{(|x-y|+h)^d}\,\delta\rho\,(\rho(x)+\rho(y))\,W$$
$$+ \int \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|\,(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W - \int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,D.$$

Sketch of the estimates: Compressible N-S, Part 2

$$\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,W$$
$$= -\frac{1}{2}\int \frac{\operatorname{div} u(x) - \operatorname{div} u(y)}{(|x-y|+h)^d}\,\delta\rho\,(\rho(x)+\rho(y))\,W$$
$$+\int \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|\,(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W - \int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,D.$$

The first term in the r.h.s. logically involves the regularity of $\operatorname{div} u$. For compressible Navier-Stokes, it is possible to couple this back to ρ through the pointwise relation

$$\operatorname{div} u = p(\rho(x)) + \underbrace{\Delta^{-1}\operatorname{div}\left(\partial_t(\rho\, u) + \operatorname{div}(\rho\, u\otimes u)\right)}_{OK}.$$

Sketch of the estimates: Compressible N-S, Part 2

$$
\begin{aligned}
&\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,W \\
&= -\frac{1}{2}\int \frac{\operatorname{div} u(x)-\operatorname{div} u(y)}{(|x-y|+h)^d}\,\delta\rho\,(\rho(x)+\rho(y))\,W \\
&+\int \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|\,(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W-\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,D.
\end{aligned}
$$

The third term will help bound the others

$$
\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,D=2\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,M\,|\nabla u|(x)\,w(t,x).
$$

Sketch of the estimates: Compressible N-S, Part 2

$$\frac{d}{dt}\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,W$$
$$= -\frac{1}{2}\int \frac{\mathrm{div}\,u(x)-\mathrm{div}\,u(y)}{(|x-y|+h)^d}\,\delta\rho\,(\rho(x)+\rho(y))\,W$$
$$+\int \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|\,(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W - \int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\,D.$$

The second term is a commutator estimate but cannot be simply controlled...

Commutator estimate: The problem

We need to control the commutator by the dissipation

$$\int \frac{|u(x)-u(y)|}{(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W \lesssim \int \frac{|\delta\rho|^2}{|(|x-y|+h)^d}\,M\,|\nabla u|(x)\,w(x)+sym.$$

Following Crippa-DeLellis, one could try to use

$$|u(x)-u(y)| \leq C\,(M\,|\nabla u|(x)+M\,|\nabla u|(y))\,|x-y|,$$

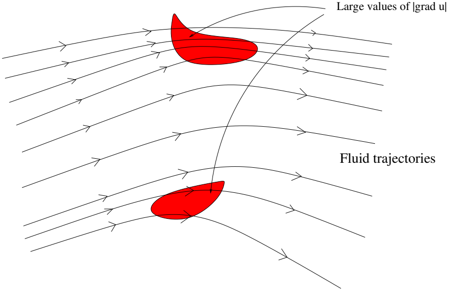
to get

$$\int \frac{|u(x)-u(y)|}{(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,W$$

$$\leq C \int \frac{|\delta\rho|^2}{|(|x-y|+h)^d}\,(\underbrace{M\,|\nabla u|(x)\,w(x)}_{OK}+\underbrace{M\,|\nabla u|(y)\,w(x)}_{???})+sym.$$

$\longrightarrow$ Due to choosing $W(x,y)=w(x)+w(y)$, not $w(x)\,w(y)$.

Commutator estimate: Lagrangian view



Good trajectory, $w(t, X(t,x)) > 0$: $\int_0^t M\,|\nabla u|(X(s,x))\,ds \sim 1$,

Bad trajectory, $w(t, X(t,x)) << 1$: $\int_0^t M\,|\nabla u|(X(s,x))\,ds >> 1$.

Nearly compressible case: Only compare good vs. good

Our compressible case: Also compare good vs. bad

$\longrightarrow$ Cannot work if good and bad trajectories intertwine...

Commutator estimate: The solution

Instead one uses the more precise estimate

$$|u(x)-u(y)| \le \underbrace{C\int_{|z-x|\le 2\,|x-y|} \frac{|\nabla u(z)|}{|z-x|^{d-1}}\,dz}_{=L_{|x-y|}\star|\nabla u|(x)} + sym.$$

The control is achieved through square function

$$\int \frac{|u(x)-u(y)|}{(|x-y|+h)^{d+1}}\,|\delta\rho|^2\,(w(x)+w(y))$$

$$\le C\int \frac{|\delta\rho|^2}{(|x-y|+h)^{d}}\,M\,|\nabla u|(x)\,w(x)$$

$$+\underbrace{C\int \frac{|\delta\rho|^2}{(|x-y|+h)^{d}}\,|L_{|x-y|}\star|\nabla u|(x)-L_{|x-y|}\star|\nabla u|(y)|\,w(x)}_{term\ \ to\ \ bound}.$$

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$$\int \frac{|\delta\rho|^2}{(|x-y|+h)^d}\, \big|L_{|x-y|}\star|\nabla u|(x) - L_{|x-y|}\star|\nabla u|(y)\big|\, w(x)\, dx\, dy$$
$$= \int_{S^{d-1}} \int_h^1 \int |\delta\rho|^2\, |L_r\star|\nabla u|(x) - L_r\star|\nabla u|(x+r\,\omega)|\, w(x)\, dx\, \frac{dr}{r}\, d\omega,$$

with the property that for a normalized convolution kernel L_r

$$\int_h^1 \frac{dr}{r}\, \|L_r\star f - L_r\star f(.+r\,\omega)\|_{L^p} \leq C\,|\log h|^{1/2}\, \|f\|_{L^p}.$$

Compressible N-S: Conclusion

Summing up, one finds

$$\frac{d}{dt}\int \frac{|\delta\rho|}{(|x-y|+h)^d}\,(w(x)+w(y)) \le C\,|\log h|^\theta\,\|\nabla u\|_{L^p}\,\|\rho\|^2_{L^{2\,p^*}},$$

for some $\theta < 1$.
Coupled with

$$\int |\log w|\,\rho(t,x)\,dx \le \lambda\,\|u\|_{L^2_t H^1_x}\,\|\rho\|_{L^2_{t,x}},$$

this leads to

$$\int \frac{|\delta\rho|^2}{(|x-y|+h)^d} \le C\,\frac{|\log h|}{\log|\log h|}.$$

Sketch of the estimates: Anelastic constraints

Lagrangian approach: Requires a to belong to Muckenhoupt space to bound $\int a\,|M\,f|^2$ by $\int a\,f^2$.

$\longrightarrow$ Introduce a three-level weight procedure with first a weight controlling large velocities

$$\partial_t w_u + u\cdot\nabla w_u = -w_u\,|u(t,x)|\,\frac{\alpha(x)+\int_0^t |\nabla(\alpha(x)u(s,x))|\,ds}{1+\int_0^t |\alpha(x)u(s,x)|\,ds},$$

which is used on the weight controlling shallow and fast-varying bathymetry

$$\partial_t w_a + u\cdot\nabla w_a = -\gamma\,\frac{|u\cdot\nabla\alpha|}{\alpha}\,w_a,\qquad w_a|_{t=0} = (\alpha(x))^\gamma,$$

through an estimate on

$$\frac{d}{dt}\int a(x)\,w_u(t,x)\,|\log w_a(t,x)|\,dx.$$

Sketch of the estimates: Anelastic constraints, Part 2

The weight w_a is used in turn to bound the final weight penalizing oscillations in u

$$\partial_t w + u \cdot \nabla w = \lambda \Big[\frac{M\,|\nabla(\alpha\, u)|}{\alpha} + (M\,|\nabla \alpha|(x))^\theta\,|u(x)|^\theta + |\alpha(x)|^{-\theta^*} \Big],$$

again through an estimate on

$$\frac{d}{dt}\int a(x)\, w_a(t,x)\, |\log w(t,x)|\, dx.$$

Finally all three weights are used to obtain the regularity of $a\, f(\phi)$ for any f by estimating

$$\int a(x)\, a(y) \frac{|\phi(x)-\phi(y)|}{(|x-y|+h)^d}\, w_u(x)\, w_u(y)\, w_a(x)\, w_a(y)\, w(x)\, w(y).$$

$$\Longrightarrow \quad \int_{\Omega^2} \frac{|\phi(x)-\phi(y)|}{(h+|x-y|)^2}\, a(x)\, a(y)\, dx\, dy \le C\, \frac{|\log h|}{\log|\log h|}.$$

Conclusion

We have developed a method for advection equations where compressibility is an issue. The method is flexible, allowing for straightforward penalization of appropriately defined bad regions in the fluid.
This allows to handle for the first time physical phenomena that are a priori unstable.

Thank you!