Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs

Maria J. Esteban

CEREMADE

CNRS & Université Paris-Dauphine & PSL Research University

In collaboration with J. Dolbeault and M. Loss

ICM 2018, Rio de Janeiro, August 3, 2018

Outline

- Symmetry and symmetry breaking in PDEs... and further.

- Functional inequalities

- Use of linear and nonlinear flows to prove functional inequalities, and describe optimizers and best constants. On the flat espace, but also on manifolds.

- Applications to spectral estimates on compact and non compact Riemaniann manifolds, with and without magnetic fields.

Symmetry and symmetry breaking I

The issue of symmetry and symmetry breaking is fundamental in all areas of science.

- Symmetry is often assimilated to order and beauty.

Symmetries are fundamental properties of the laws of Physics. They impose constraints on modeling phenomena and, at a more basic level, they serve as criteria of classification

Pierre Curie probably was the first to start seriously speaking about symmetry in his works on crystalography.

Symmetry and symmetry breaking I

The issue of symmetry and symmetry breaking is fundamental in all areas of science.

• Symmetry is often assimilated to order and beauty.

Symmetries are fundamental properties of the laws of Physics. They impose constraints on modeling phenomena and, at a more basic level, they serve as criteria of classification

Pierre Curie probably was the first to start seriously speaking about symmetry in his works on crystalography.

• Symmetry breaking is the source of many interesting phenomena such as :

- phase transitions and complex dynamics

- instabilities

- segregation and self-organization, ...

Symmetry and symmetry breaking II

Knowing *a priori* the symmetry properties of solutions of PDEs of variational problems which are invariant under some symmetry group is of fundamental importance.

Symmetry and symmetry breaking II

Knowing *a priori* the symmetry properties of solutions of PDEs of variational problems which are invariant under some symmetry group is of fundamental importance.

By itself, of course, but also if one wants to compute the solution numerically,

because symmetry $\Rightarrow$ simplifications, fewer degrees of freedom.

Attainability and value of best constants in functional inequalities

$$F(Dv, v, x) \leq C\ G(D^2 v, Dv, v, x) \qquad \forall v \in X\,.$$

Functional inequalities play an important role in obtaining a priori estimates for solutions of PDEs, in analyzing the long time behavior of solutions of evolution problems, in describing the blow-up profile for finite time blow-up phenomena, etc

Many examples : Hardy, Hölder, Poincaré, Jensen, Nash, Sobolev, Gagliardo-Nirenberg, log Sobolev, Caffarelli-Kohn-Nirenberg,... : a very important toolbox in analysis and geometry.

Attainability and value of best constants in functional inequalities

$$F(Dv, v, x) \leq C\ G(D^2v, Dv, v, x) \qquad \forall v \in X\,.$$

Functional inequalities play an important role in obtaining a priori estimates for solutions of PDEs, in analyzing the long time behavior of solutions of evolution problems, in describing the blow-up profile for finite time blow-up phenomena, etc

Many examples : Hardy, Hölder, Poincaré, Jensen, Nash, Sobolev, Gagliardo-Nirenberg, log Sobolev, Caffarelli-Kohn-Nirenberg,... : a very important toolbox in analysis and geometry.

Important questions :

- Is C attained in X ?   What is its value ? ?

- If yes, how do the optimal functions v look like ?

Flows and inequalities

In the last 20 years, flows and entropy / entropy-production methods have been used to prove many kinds of inequalities, like Sobolev, log-Sobolev, Gagliardo-Nirenberg, Hardy-Sobolev, etc, etc

In most cases, flow methods provide not only the inequalities with their respective best contants , but also some information about properties of the extremal solutions.

In some cases similar methods can be used to obtain improved inequalities.

Flows and inequalities

In the last 20 years, flows and entropy / entropy-production methods have been used to prove many kinds of inequalities, like Sobolev, log-Sobolev, Gagliardo-Nirenberg, Hardy-Sobolev, etc, etc

In most cases, flow methods provide not only the inequalities with their respective best contants , but also some information about properties of the extremal solutions.

In some cases similar methods can be used to obtain improved inequalities.

Flows have been used also to prove important results in Geometry : Yamabe flow, Ricci flow, etc.

Flows and inequalities

In the last 20 years, flows and entropy / entropy-production methods have been used to prove many kinds of inequalities, like Sobolev, log-Sobolev, Gagliardo-Nirenberg, Hardy-Sobolev, etc, etc

In most cases, flow methods provide not only the inequalities with their respective best contants , but also some information about properties of the extremal solutions.

In some cases similar methods can be used to obtain improved inequalities.

Flows have been used also to prove important results in Geometry : Yamabe flow, Ricci flow, etc.

Let us next look at some inequalities which are important, appear in various contexts of applications and for which symmetry/rigidity issues have been solved recently with the help of flow methods.

Sobolev-like inequalities on the sphere

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\nu) \quad (1)$$

where the measure $d\nu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the exposant $p \geq 1$, $p \neq 2$, is such that $p \leq 2^* := \frac{2\,d}{d-2}$ if $d \geq 3$.

Sobolev-like inequalities on the sphere

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \ge \frac{d}{p-2} \left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\nu) \;\; (1)$$

where the measure $d\nu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the exposant $p \ge 1$, $p \ne 2$, is such that $p \le 2^* := \frac{2\,d}{d-2}$ if $d \ge 3$.

The case $p = \frac{2\,d}{d-2}$ corresponds to the Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \ge S \left(\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right)^{\frac{d-2}{d}} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

Sobolev-like inequalities on the sphere

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\nu) \quad (1)$$

where the measure $d\nu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ and the exposant $p \ge 1$, $p \neq 2$, is such that $p \le 2^* := \frac{2\,d}{d-2}$ if $d \ge 3$.

The case $p = \frac{2\,d}{d-2}$ corresponds to the Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \ge S \left(\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right)^{\frac{d-2}{d}} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

Proofs of (1) + minimizers are constants by : **Bidaut-Véron – Véron** (PDE, rigidity methods, 1991) ; **Beckner** (harmonic analysis methods, 1993) ; **Bakry-Ledoux (based on Bakry-Emery et al)** ("carré du champ" method, linked to a flow method, 1996 + ; only for $2 < p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^*$).

Linear flow method

Let us define $\rho = |u|^p$. The two inequalities below are equivalent

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \geq \frac{d}{p-2} \left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)} \right) .$$

$$\int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \geq \frac{d}{p-2} \left[\left(\int_{S^d} \rho \, d\omega \right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}} \, d\omega \right] .$$

So, we need to prove $\mathcal{I}_p[\rho] \geq d \, \mathcal{E}_p[\rho]$, with

$$\mathcal{I}_p[\rho] := \int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \ , \ \mathcal{E}_p[\rho] := \frac{1}{p-2} \left[\left(\int_{S^d} \rho \, d\omega \right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}} \, d\omega \right]$$

Linear flow method

Let us define $\rho = |u|^p$. The two inequalities below are equivalent

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(S^d)} - \|u\|^2_{\mathrm{L}^2(S^d)}\right) .$$

$$\int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \geq \frac{d}{p-2}\left[\left(\int_{S^d} \rho \, d\omega\right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}} \, d\omega\right] .$$

So, we need to prove $\mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$, with

$$\mathcal{I}_p[\rho] := \int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\omega \;,\; \mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\left(\int_{S^d} \rho \, d\omega\right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}} \, d\omega\right]$$

To establish such inequalities, one can use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho \,, \quad \text{with } \Delta = \text{ the Laplace-Beltrami operator on } \mathbb{S}^d$$

We have

$$\frac{d}{dt}\left(\int_{S^d} \rho \, d\omega\right) = 0$$

and if $p \leq 2^\#$, $\quad \frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho] \quad$ and $\quad \frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]\,.$

Nonlinear versus linear flow

The goal is to prove $\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho] \geq 0$. For $p \leq 2^{\#}$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq (-d + d)\,\mathcal{I}_p[\rho] = 0\,.$$

Not difficult to prove that ρ converges to a constant as $t \to +\infty$ and

$$\lim_{t\to+\infty}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) = 0\,.$$

Nonlinear versus linear flow

The goal is to prove $\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho] \geq 0$. For $p \leq 2^{\#}$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq (-d+d)\,\mathcal{I}_p[\rho] = 0\,.$$

Not difficult to prove that ρ converges to a constant as $t \to +\infty$ and

$$\lim_{t\to+\infty}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) = 0\,.$$

What if $2^{\#} < p < 2^{\sharp}$?

THM [Dolbeault, E., Loss, 2017]. When $2^{\#} < p < 2^{*}$, we can find a function ρ_0 such that ρ solution of $\frac{\partial\rho}{\partial t} = \Delta\rho\,,\ \ \rho(t=0) = \rho_0\,$, and

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big)\Big|_{t=0} > 0\,.$$

Then, we can get the same result by considering the flow $\frac{d\,\rho}{dt} = \Delta\rho^m\,$, for a well-chosen $m \neq 1$.

The computations are much more involved, but the idea is "more or less" the same. And we can also cover the case $p \in (1,2)$.

Caffarelli-Kohn-Nirenberg (CKN) inequalities (1984)

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a \neq \frac{d-2}{2}$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$b - a \to 0 \iff p \to \tfrac{2d}{d-2} \quad \text{(Sobolev)}$$

$$b - (a+1) \to 0 \iff p \to 2_+ \quad \text{(Hardy)}$$

Caffarelli-Kohn-Nirenberg (CKN) inequalities (1984)

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a \neq \frac{d-2}{2}$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$b-a \to 0 \iff p \to \tfrac{2d}{d-2} \quad \text{(Sobolev)}$$

$$b-(a+1) \to 0 \iff p \to 2_+ \quad \text{(Hardy)}$$

$$\frac{1}{C_{a,b}} = \inf_{\mathcal{D}_{a,b}} \quad \frac{\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx}{\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}}$$

The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$\mathsf{C}_{a,b}$ = best constant for general functions v
$\mathsf{C}^*_{a,b}$ = best constant for radially symmetric functions v

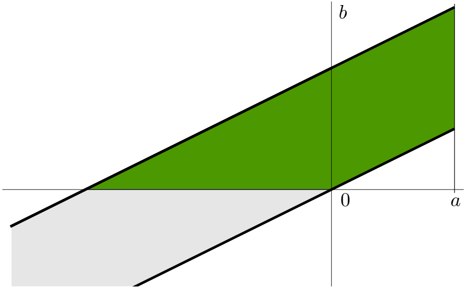
$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

$$v^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

Questions : is optimality (equality) achieved ? do we have $v_{a,b} = v^*_{a,b}$?

Symmetry $(d \geq 3)$



Case $a > 0$: Th. Aubin, G. Talenti, E. Lieb, Chou-Chu, P.L. Lions, Horiuchi,...

Case $a < 0$: Lin, Wang ; Dolbeault, E., Tarantello (d=2) ; Betta-Brock-Mercaldo-Posteraro ($b > 0$)

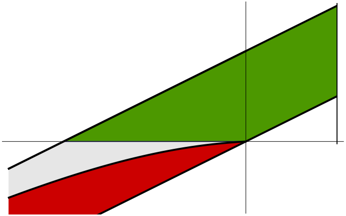
Linear instability of radial minimizers : the Felli-Schneider curve

Catrina, Wang (2001) looked for the set of pairs (a, b) such that the functional

$$\mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|w|^p}{|x|^{b\,p}}\,dx \right)^{2/p}$$

is linearly instable at $w = w^*_{a,b}$ (Catrina, Wang (2001) ; Felli, Schneider (2003)). This happens for

$$b < b^{FS}(a) := \frac{d(d-2-2a)}{2\sqrt{(d-2-2a)^2 + 4(d-1)}} - \frac{1}{2}(d-2-2a)$$



Generalized CKN inequalities

Let $d \geq 3$. For any $p \in [2, p(\theta, d) := \frac{2d}{d-2\theta}]$,

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta, p, a)$ and (see [Del Pino, Dolbeault, Filippas, Tertikas]) :

$$\mathsf{C}_{\rm CKN}(\theta,p,a) \geq \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\rm CKN}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}}\left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}}\left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta}\left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}}\left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

Generalized CKN inequalities

Let $d \geq 3$. For any $p \in [2, p(\theta,d) := \frac{2d}{d-2\theta}]$,

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and (see [Del Pino, Dolbeault, Filippas, Tertikas]) :

$$\mathsf{C}_{\rm CKN}(\theta,p,a) \geq \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

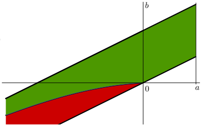
$$\mathsf{C}^*_{\rm CKN}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}}\left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}}\left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta}\left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}}\left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

For θ small, we have proved that there is symmetry breaking for certain values of (Λ, p) such that $u^*_{\Lambda,p}$ is stable ! In principle in all cases where we have observed this phenomenum, $\theta \leq 0.7$ approx.

(Dolbeault, E., Tarantello, Tertikas (2011))

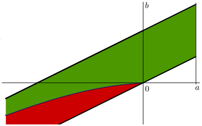
Solution of the conjecture : A Sobolev type inequality

THEOREM [2016].- In the stability region, if $d \geq 2$, the optimality is achieved by radial functions.



Solution of the conjecture : A Sobolev type inequality

THEOREM [2016].- In the stability region, if $d \geq 2$, the optimality is achieved by radial functions.



Idea of the proof : with the change of variables $r \mapsto r^\alpha$, $v(r,\omega) = w(r^\alpha,\omega)$, and with

$$n = \frac{d - b\,p}{\alpha} = \frac{d - 2\,a - 2}{\alpha} + 2 \quad \left(\text{equivalent to } \; p = \frac{2n}{n-2}\right)$$

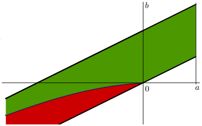
$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \leq \, \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \qquad \text{becomes}$$

$$\left(\int_{\mathbb{R}^d} |w|^{\frac{2n}{n-2}}\, r^{n-1}\,dr\,d\omega\right)^{\frac{n-2}{n}} \leq \, \mathsf{C}_{\alpha,n} \int_{\mathbb{R}^d} |\mathsf{D}_\alpha w|^2\, r^{n-1}\,dr\,d\omega$$

$$\mathsf{D}_\alpha w = \left(\alpha\,\frac{\partial w}{\partial r}, \frac{1}{r}\,\nabla_\omega w\right)$$

Solution of the conjecture : A Sobolev type inequality

THEOREM [2016].- In the stability region, if $d \geq 2$, the optimality is achieved by radial functions.



Idea of the proof : with the change of variables $r \mapsto r^\alpha$, $v(r,\omega) = w(r^\alpha, \omega)$, and with

$$n = \frac{d - b\,p}{\alpha} = \frac{d - 2\,a - 2}{\alpha} + 2 \quad \left(\text{equivalent to } \; p = \frac{2n}{n-2}\right)$$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \leq \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \qquad \text{becomes}$$

$$\left(\int_{\mathbb{R}^d} |w|^{\frac{2n}{n-2}}\,r^{n-1}\,dr\,d\omega\right)^{\frac{n-2}{n}} \leq \mathsf{C}_{\alpha,n} \int_{\mathbb{R}^d} |\mathsf{D}_\alpha w|^2\,r^{n-1}\,dr\,d\omega$$

$$\mathsf{D}_\alpha w = \left(\alpha\,\frac{\partial w}{\partial r}, \frac{1}{r}\,\nabla_\omega w\right)$$

The instability region is defined by $\alpha \leq \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$

Fisher information decay I

In the new set-up, if we define $u = |w|^p$, the CKN inequalities are

$$\int_{\mathbb{R}^d} u(r,\omega)\, r^{n-1}\, dr\, d\omega \le C_{\alpha,n}\, \mathcal{I}[u]$$

with

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\, |\mathsf{D}_\alpha \mathsf{p}|^2\, r^{n-1}\, dr\, d\omega\,, \quad \mathsf{p} = \frac{m}{1-m}\, u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

In terms of information theory, $\mathcal{I}$ is the Fisher information and p is the pressure function.

Fisher information decay I

In the new set-up, if we define $u = |w|^p$, the CKN inequalities are

$$\int_{\mathbb{R}^d} u(r,\omega)\, r^{n-1}\, dr\, d\omega \le C_{\alpha,n}\, \mathcal{I}[u]$$

with

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\, |\mathsf{D}_\alpha \mathsf{p}|^2\, r^{n-1}\, dr\, d\omega\,, \quad \mathsf{p} = \frac{m}{1-m}\, u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

In terms of information theory, $\mathcal{I}$ is the Fisher information and p is the pressure function.

STRATEGY : Find a flow such that for all $\alpha \le \alpha_{\rm FS}$,

1) for all $t \ge 0$, $\frac{d}{dt} \int_{\mathbb{R}^d} u(t)\, d\mu = 0$ and $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \le 0$,

2) prove $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ means, in particular, that u is radially symmetric.

This is done by performing lots of calculations, using some differential geometry tools and proving delicate regularity results

The flow

$$\frac{\partial u}{\partial t} = \mathrm{L}_\alpha u^m\,, \quad m = 1 - \frac{1}{n}$$

where we define the self-adjoint operator L_α by

$$\mathrm{L}_\alpha w := -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha\,w = \alpha^2\,w'' + \alpha^2\,\frac{n-1}{r}\,w' + \frac{\Delta_\omega\,w}{r^2}$$

The fundamental property of L_α is the fact that

$$\int_{\mathbb{R}^d} w_1\,\mathrm{L}_\alpha w_2\,d\mu = -\int_{\mathbb{R}^d} \mathsf{D}_\alpha w_1 \cdot \mathsf{D}_\alpha w_2\,d\mu\,, \quad d\mu = r^{n-1}\,dr\,d\omega$$

Fisher information decay II

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\int_{\mathbb{R}^d} \mathsf{k}[\mathsf{p}]\;\mathsf{p}^{1-n}\,d\mu$$

$$\mathsf{k}[\mathsf{p}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{p}''-\frac{\mathsf{p}'}{r}-\frac{\Delta_\omega\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla_\omega\mathsf{p}'-\frac{\nabla_\omega\mathsf{p}}{r}\right|^2+\frac{1}{r^4}\,\mathsf{k}_{\mathcal{M}}[\mathsf{p}]\,,$$

with

$$\mathsf{k}_{\mathcal{M}}[\mathsf{p}] := \frac{1}{2}\,\Delta_\omega\,|\nabla_\omega\mathsf{p}|^2-\nabla_\omega\mathsf{p}\cdot\nabla_\omega\Delta_\omega\,\mathsf{p}-\tfrac{1}{n-1}\,(\Delta_\omega\,\mathsf{p})^2-(n-2)\,\alpha^2\,|\nabla_\omega\mathsf{p}|^2$$
$$\ge (n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf{p}|^2+\zeta_\star\,(n-d)|\nabla_\omega\mathsf{p}|^4\,,\quad \zeta_\star>0$$

Fisher information decay II

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\int_{\mathbb{R}^d} \mathsf{k}[\mathsf{p}]\;\mathsf{p}^{1-n}\,d\mu$$

$$\mathsf{k}[\mathsf{p}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{p}''-\frac{\mathsf{p}'}{r}-\frac{\Delta_\omega\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla_\omega\mathsf{p}'-\frac{\nabla_\omega\mathsf{p}}{r}\right|^2+\frac{1}{r^4}\,\mathsf{k}_{\mathcal{M}}[\mathsf{p}]\,,$$

with

$$\mathsf{k}_{\mathcal{M}}[\mathsf{p}] := \frac{1}{2}\,\Delta_\omega\,|\nabla_\omega\mathsf{p}|^2-\nabla_\omega\mathsf{p}\cdot\nabla_\omega\Delta_\omega\,\mathsf{p}-\tfrac{1}{n-1}\,(\Delta_\omega\,\mathsf{p})^2-(n-2)\,\alpha^2\,|\nabla_\omega\mathsf{p}|^2$$
$$\geq (n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf{p}|^2+\zeta_\star\,(n-d)|\nabla_\omega\mathsf{p}|^4\,,\quad \zeta_\star>0$$

So that

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \leq -\left(\text{sum of squares} + (n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)\int_{\mathbb{R}^d}|\nabla_\omega\mathsf{p}|^2\,\mathsf{p}^{1-n}\,d\mu\right)$$

$$\leq 0 \quad \text{if } \alpha\leq\alpha_{\rm FS}, \quad \text{and } = 0 \to \nabla_\omega\mathsf{p}\equiv 0 \ \text{ a.e.}$$

Elliptic proof

If $\alpha \leq \alpha_{\rm FS}$ and if u_0 , $\mathsf{p}_0 = \frac{m}{1-m}\, u_0^{m-1}$ are critical points of the Euler-Lagrange equations for CKN, written in the good variables, then

$$\frac{\partial}{\partial t}\,\mathcal{I}[u(t)]_{|_{t=0}} = \mathcal{I}'[\mathsf{u}(t)]\cdot\frac{\partial}{\partial t}\,u(t)_{|_{t=0}} = \mathcal{I}'[\mathsf{u}_0]\cdot \mathrm{L}_\alpha\, u_0^m = 0 = -C\,K[\mathsf{p}_0]$$

$$0 = \mathcal{K}[\mathsf{p}_0] \geq \int_{\mathbb{R}^d} \alpha^4 \left(1-\frac{1}{n}\right)\left[\mathsf{p}_0'' - \frac{\mathsf{p}_0'}{r} - \frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2}\right]^2 \mathsf{p}_0^{1-n}\,d\mu$$

$$+\int_{\mathbb{R}^d}(n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega \mathsf{p}_0|^2\,\mathsf{p}_0^{1-n}\,d\mu + \int_{\mathbb{R}^d}\zeta_\star\,(n-d)\,|\nabla_\omega \mathsf{p}_0|^4\,\mathsf{p}_0^{1-n}\,d\mu$$

where $\zeta_\star > 0$ and $n > d$.

So, $\nabla_\omega \mathsf{p}_0 \equiv 0$, that is, p_0 does not depend on ω, which means radial symmetry.

Moreover, $\mathsf{p}_0'' - \frac{\mathsf{p}_0'}{r} - \frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2} \equiv 0$, which implies that for some $a, b > 0$, $\mathsf{p}_0 = a + b\,r^2$.

Consequences on cylinders

By using a well-adapted Endem-Fowler transformation, the above theorem implies, in particular, that, up to translations, the unique positive solution of the equation

$$-\Delta v + \Lambda v = |v|^{p-1} v \,, \quad \text{in} \quad \mathbb{R} \times S^{d-1} \,, \quad \Delta = \frac{\partial^2}{\partial s^2} + \Delta_\omega \,,$$

depends only on the variable s and is equal to

$$v_\star(s) := \left(\frac{p}{2}\,\Lambda\right)^{\frac{1}{p-2}} \left(\cosh\left(\frac{p-2}{2}\,\sqrt{\Lambda}\,s\right)\right)^{-\frac{2}{p-2}}$$

for all $\Lambda \le \Lambda_{FS} = 4\,\frac{d-1}{p^2-4}$. And the result is optimal.

Spectral consequences

The above theorem provides optimal estimates for the first eigenvalue of Schrödinger operators $-\Delta - V$ on cylinders $\mathcal{C} = \mathbb{R} \times S^d$ of the type

$$\lambda_1(-\Delta - V) \geq f(\|V_+\|_{L^q})$$

The optimality for these estimates, known as Keller–Lieb-Thirring estimates, were only known for the whole space $\mathbb{R}^d$, for balls, and in a not optimal way, for compact manifolds only.

Other inequalities

A similar method has been applied to subcritical CKN inequalities, and the same kind of optimal results proved.

The method seems quite robust and extendable to deal with very different situations.

The problem is always to find a “good” flow that does the business needed.

Other inequalities

A similar method has been applied to subcritical CKN inequalities, and the same kind of optimal results proved.

The method seems quite robust and extendable to deal with very different situations.

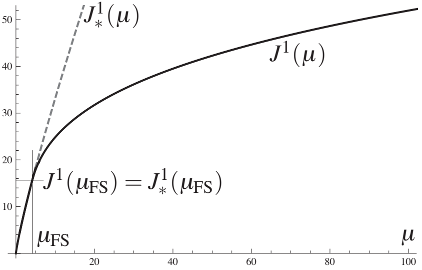
The problem is always to find a “good” flow that does the business needed.

BUT, beware ! this is not a method to prove symmetry... but uniqueness/rigidity....

Why the conjecture is not true for θ small ?

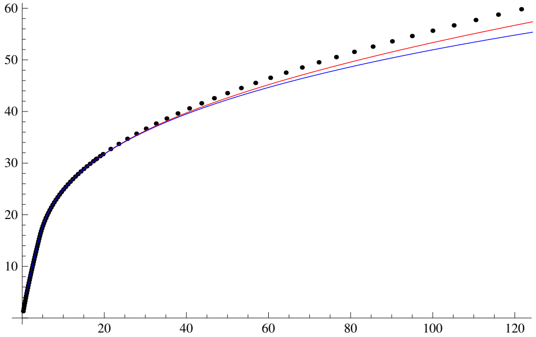
Branches for p fixed and $\theta = 1$

$$J^1(\mu) := \inf_{v \neq 0} \frac{\|\nabla v\|^2_{L^2(\mathcal{C})} + \mu \|v\|^2_{L^2(\mathcal{C})}}{\|v\|^2_{L^p(\mathcal{C})}}$$



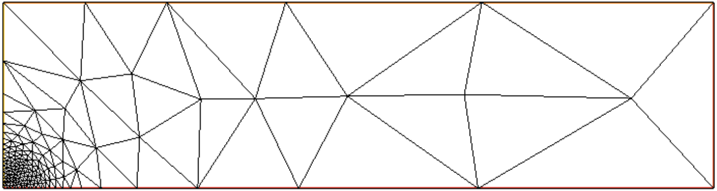
Branches computed for $d = 5$ and $p = 2.1, 2.2, \ldots, 3.3$ (Freefem++)

Non symmetric optimal functions : grid issues

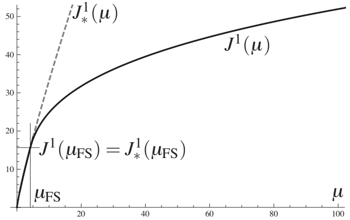


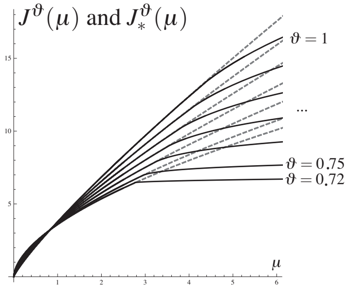
Coarse / refined / self-adaptive grids

A self-adaptive grid



Numerical computation of branches with Freefem++

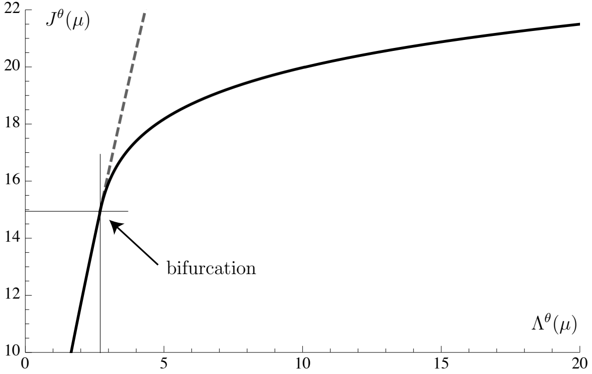




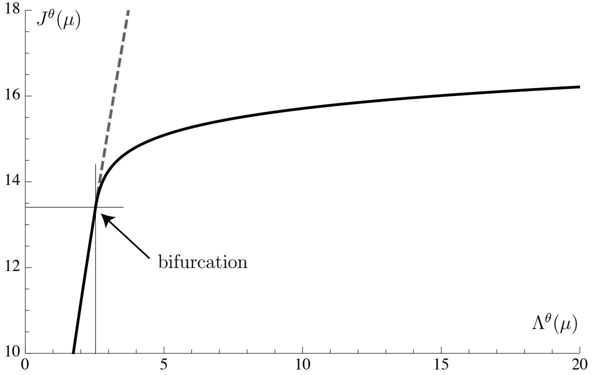
Branch for $d = 5$, $p = 2.8$, $\theta = 1$
branches for $\theta < 1$

Reparametrized

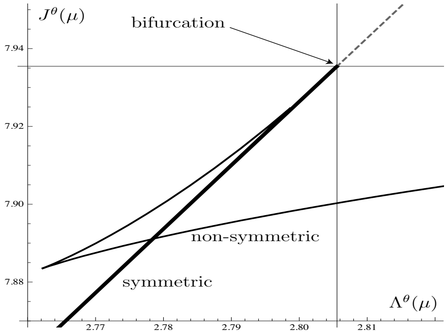
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 1$

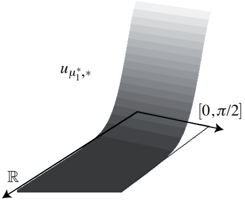


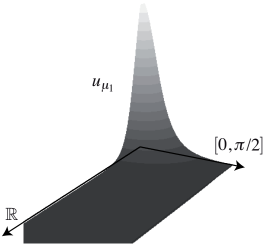
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 0.95$



Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.72$







THANK YOU FOR YOUR ATTENTION !