Algorithms for Motion of Networks by Weighted Mean Curvature

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Variational models with surface area penalty:
- Soap bubbles, foam.
- Materials science: Grain boundary motion.
- Computer vision: Image segmentation.

Typical cost function:

\[ E(\Sigma_1, \ldots, \Sigma_N) = \sum_{m \neq n} \text{Area}(\Gamma_{m,n}) \text{ with} \]

\[ \bigcup \Sigma_m = D, \text{ and} \]

\[ \Sigma_m \cap \Sigma_n = (\partial \Sigma_m) \cap (\partial \Sigma_n) = \Gamma_{m,n} \text{ for } m \neq n. \]

Gradient descent:
- Mean curvature motion: \( v_{m,n}^\perp = \kappa_1 + \kappa_2 \)
- Junction conditions: All angles 120°. Natural bdry. cond.
Two Phase Mean Curvature Flow

- Cost function $E(\Sigma) = \text{Length}(\partial \Sigma)$
- Parametrize $\partial \Sigma$ by arclength, $\gamma(s) : [0, L] \to \mathbb{R}^2$.
- Perturb $\Sigma$ to $\Sigma(t)$ in the normal direction:
  \[
  \gamma(s) \rightarrow \gamma(s) + t \phi(s)\nu(s)
  \]

- Directional derivative:
  \[
  \left. \frac{d}{dt} E(\Sigma(t)) \right|_{t=0} = \int_0^L -\kappa(s)\phi(s) \, ds
  \]

- We get
  \[
  \nabla E = -\kappa \implies \nu^\perp = \kappa
  \]

w.r.t. $L^2(\partial \Sigma)$ inner product on normal vector fields.

- On $\mathbb{R}^d$ with $d \geq 3$:
  \[
  \nabla E = -(\kappa_1 + \kappa_2 + \cdots + \kappa_{d-1})
  \]
Curvature Motion
Two Phase Anisotropic Mean Curvature Flow

- **Cost function:**
  \[ E(\Sigma) = \int_{\partial \Sigma} \sigma(\nu(x)) \, dS(x) \]

- **Surface tension** \( \sigma : \mathbb{S}^2 \to \mathbb{R}^+ \setminus \{0\} \):
  \[ \sigma(x) = |x| \sigma \left( \frac{x}{|x|} \right) \text{ is convex} \]

- **Wulff shape of} \sigma:\)
  \[ W_\sigma = \{ x : \sigma^*(x) \leq 1 \} = \arg \min_{|\Sigma|=\lambda} E(\Sigma) \]
  where
  \[ \sigma^*(x) = \sup_{y : \sigma(y) \leq 1} x \cdot y \]

- **Gradient descent**
  \[ \nu_\perp = \mu(\nu) \left\{ (\partial_1^2 \sigma(\nu) + \sigma(\nu)) \kappa_1 + (\partial_2^2 \sigma(\nu) + \sigma(\nu)) \kappa_2 \right\} \]
  where \( \mu : \mathbb{S}^2 \to \mathbb{R}^+ \) is the mobility.
Two-Phase Mean Curvature Flow

- An important property of two-phase curvature flow:
  
  **Comparison principle**

- If $\Sigma_1(0) \subset \Sigma_2(0)$, then $\Sigma_1(t) \subset \Sigma_2(t)$ for all $t \geq 0$.

- Monotone numerical schemes respect it.
Partition Energy:

\[ E = \sum_{m,n}^{N} \int_{\Gamma_{m,n}} \sigma_{m,n}(\nu(x)) \, dS(x) \]

where:

\[ \Gamma_{m,n} = (\partial \Sigma_{m}) \cap (\partial \Sigma_{n}) \]
\[ \sigma_{m,n} : \mathbb{S}^2 \to \mathbb{R}^{+} \]
\[ \Sigma_{m} \cap \Sigma_{n} = \Gamma_{m,n} \text{ if } m \neq n \]
\[ D = \bigcup_{n=1}^{N} \Sigma_{n}. \]

Necessary conditions:

\[ \sigma_{m,n} : \mathbb{R}^3 \to \mathbb{R}^{+} \text{ convex and } \sigma_{i,k}(\nu) + \sigma_{k,j}(\nu) \geq \sigma_{i,j}(\nu) \text{ for all } \nu. \]
Junction Conditions: Isotropic Case

- Herring angle condition:

\[ \sigma_{1,2} T_{1,2} + \sigma_{1,3} T_{1,3} + \sigma_{2,3} T_{2,3} = 0 \]

\[ \iff \quad \frac{\sin \theta_1}{\sigma_{2,3}} = \frac{\sin \theta_2}{\sigma_{1,3}} = \frac{\sin \theta_3}{\sigma_{1,2}} \]

- Forces on junction point must vanish:

\[ \sigma_{1,2} T_{1,2} + \sigma_{1,3} T_{1,3} + \sigma_{2,3} T_{2,3} = 0 \]

- Example: \( \sigma_{i,j} = 1 \) for all \( i \& j \) \( \Rightarrow \) All angles 120°.

- In general, there may be stable multiple (≥ 4) junctions.
Multiphase, Isotropic, Equal Surface Tension Motion by Curvature
Herring angle condition:

There are additional torque terms:

\[
\sigma_{1,2}(T_{1,2}^\perp)T_{1,2} + \sigma_{1,3}(T_{1,3}^\perp)T_{1,3} + \sigma_{2,3}(T_{2,3}^\perp)T_{2,3} \\
\sigma'_{1,2}(T_{1,2}^\perp)T_{1,2}^\perp + \sigma'_{1,3}(T_{1,3}^\perp)T_{1,3}^\perp + \sigma'_{2,3}(T_{2,3}^\perp)T_{2,3}^\perp = 0.
\]

There may be stable multiple (\(\geq 4\)) junctions.
Minimizing Movements

- Gradient descent for $E$ w.r.t. $\langle \cdot, \cdot \rangle$:

$$\partial_t u(t) = -\nabla E(u(t))$$

- Discrete in time approximation:

$$u^{k+1} = \arg \min_v \left\{ E(v) + \frac{1}{2\delta t} \langle v - u^k, v - u^k \rangle \right\}$$

- Optimization algorithm $\implies$ Approximate gradient descent.

- Example: $E(u) = \int |\nabla u|^2 \, dx \implies u_t = \Delta u$ in $L^2$

$$u^{k+1} = \arg \min_v \int |\nabla v|^2 + \frac{1}{2\delta t} (v - u^k)^2 \, dx \implies \frac{u^{k+1} - u^k}{\delta t} = \Delta u^{k+1}$$

with natural (Neumann) boundary conditions.

- De Giorgi, Almgren-Taylor-Wang, Luckhaus-Sturzenhecker for geometric motions.
Fix a time step $\delta t > 0$ and generate a discrete in time approximation $\{\Sigma^k\}_{k=0}^\infty$ as follows:

**Merriman, Bence & Osher’89**

- **Convolution step:** Set
  \[ \phi_1 = G_{\delta t} * 1_{\Sigma^k} \text{ and } \phi_2 = G_{\delta t} * 1_{(\Sigma^k)^c}, \]

- **Thresholding step:** Set
  \[ \Sigma^{k+1} = \left\{ x : \phi_1(x) > \phi_2(x) \right\} \iff \Sigma^{k+1} = \left\{ x : \phi_1(x) \geq \frac{1}{2} \right\}. \]

**Benefits:**
- Unconditionally monotone (stable).
- Low per time step cost: $M \log(M)$, $M = \# \text{ of grid pts.}$

**Convergence:** Evans, Barles & Georgelin, Barles & Souganidis...

**Extension:** Replace $G$ by non-radially symmetric kernel for anisotropic motions.
Threshold Dynamics
Convolution step:

\[ \phi_n = G_{\delta t} * 1_{\Sigma_n^k} \text{ for } n = 1, 2, \ldots, N. \]

Redistribution step:

\[ \Sigma_{n+1}^k = \left\{ x : \phi_n(x) = \max_{i \in \{1, 2, \ldots, N\}} \phi_i(x) \right\} \text{ for } n = 1, 2, \ldots, N. \]

Works only for \( \sigma_{m,n} = 1 \) and \( \mu_{m,n} = 1 \):

- \( E = \sum_{m \neq n} \text{Area}(\Gamma_{m,n}) \),
- \( v_{m,n}^\bot = H_i \),
- All junction angles 120°.
Challenges in Threshold Dynamics: Summary

1. Multiple phase, unequal surface tensions:

\[ E = \sum_{n \neq m} \sigma_{m,n} \text{Area}(\Gamma_{m,n}) \]

2. Two phase, anisotropic version:

\[ E = \int_{\partial \Sigma} \sigma(\nu(x)) \, dS(x) \quad \Rightarrow \quad \nu_\perp = \mu(\nu) \left\{ (\partial_1^2 \sigma + \sigma) \kappa_1 + (\partial_2^2 \sigma + \sigma) \kappa_2 \right\} \]

1. Kernel construction for arbitrary \( \sigma \) and \( \mu \).
2. Positivity of kernels (monotonicity of scheme).

3. Multiple phase, fully anisotropic version: \( \binom{N}{2} \) \( \sigma_{i,j} : S^2 \to \mathbb{R}^+ \)

\[ E = \sum_{m \neq n} \int_{\Gamma_{m,n}} \sigma_{m,n}(\nu(x)) \, dS(x) \]

along with \( \binom{N}{2} \) mobilities \( \mu_{i,j} : S^2 \to \mathbb{R}^+ \).
\[ \sum^c \] \[ \Sigma \]
\[ \int_{\Sigma^c} G_t * 1_\Sigma \, dx \]
The energy

$$E_t(\Sigma) = \frac{1}{\sqrt{t}} \int_{\Sigma^c} G_t \ast 1_{\Sigma} \, dx$$

approximates

$$E_t(\Sigma) \to c \cdot \text{Per}(\Sigma)$$

Intuitively: Amount of heat that escapes into $\Sigma^c$.

**Gamma convergence** by Alberti & Bellettini ’98.

Also: Miranda Jr., Pallara, Paronetto, Preunkert.
$E_t(\Sigma) = \frac{1}{\sqrt{t}} \int_{\Sigma^c} G_t \ast 1_\Sigma \, dx$

- **Question:** Does threshold dynamics dissipate $E_t$?

- **Extend to functions:** $u_1 \approx 1_\Sigma$ and $u_2 \approx 1_{\Sigma^c}$:

$$\min_{u_1, u_2 \geq 0 \atop u_1 + u_2 = 1} \int (G_t \ast u_1) \, u_2 \, dx \iff \min_{u_1 \in [0,1]} \hat{u}_1(0) - \int (\hat{G}_t(\xi) |\hat{u}_1(\xi)|^2 \, d\xi$$

- **Concave** on the constraint set $u_1 + u_2 = 1$.

- **Minimizer is binary:** $u_1(x), u_2(x) \in \{0, 1\}$ for all $x$.

- **$\hat{G} \geq 0 \implies E_t$ is dissipated.**
Threshold Dynamics as an Optimization Scheme (with F. Otto)

- **Linearization** of $E_t(u_1, u_2)$ at $(u_1^k, u_2^k)$:

$$\min_{u_1, u_2 \geq 0 \quad u_1 + u_2 = 1} \int u_1 \left( G_t * u_2^k \right) + u_2 \left( G_t * u_1^k \right) \, dx$$

- Minimization turns into **thresholding**:

$$u_1(x) = \begin{cases} 1 & \text{if } \left( G_t * u_2^k \right)(x) < \left( G_t * u_1^k \right)(x) \\ 0 & \text{otherwise.} \end{cases}$$

$$u_2(x) = \begin{cases} 1 & \text{if } \left( G_t * u_1^k \right)(x) < \left( G_t * u_2^k \right)(x) \\ 0 & \text{otherwise.} \end{cases}$$

- Identical to a step of **threshold dynamics**.

- **UPSHOT**: Threshold dynamics is an optimization scheme that minimizes the linearization of the energy over the constraint set.

- **Concavity of** $E_t \implies$ Unconditional gradient stability.
Minimizing Movements Interpretation (with F. Otto)

- Important:

\[ L(u_1, u_2) = E_t(u_1, u_2) - E_t(u_1 - u^k_1, u_2 - u^k_2) \]
\[ \approx \text{Per} \quad \text{movement limiter} \]

\[-E_t(u_1 - u^k_1, u_2 - u^k_2) \approx \int_{\partial \Sigma^k} \phi^2(x) dS(x)\]

- Analogous to minimizing movements procedure of Almgren, Taylor & Wang, and Luckhaus & Sturzenhecker.

- **UPSHOT**: Threshold dynamics generates gradient descent for approximately the right energy w.r.t approximately the right metric.

- In fact: Laux & Otto prove conditional convergence of algorithm to right dynamics, including the multiphase version below.
Strategy:

- Avoid guessing the extension of threshold dynamics.
- Instead, extend the variational formulation:
  - Extend the non-local energy.
  - Turn the crank: Apply optimization procedure:
    - Relax, linearize, minimize over constraint set.
- Systematic procedure for deriving MBO-type fast algorithms.
- Will look at:
  - Isotropic, multiphase model, with \( \binom{n}{2} \) distinct \( \sigma_{m,n} \).
  - Anisotropic, two-phase model. Kernel construction.
  - Multiphase, anisotropic model, with \( \binom{n}{2} \) distinct \( (\sigma_{m,n}, \mu_{m,n}) \).
Extension to Multiple Phases (with F. Otto)
Extension to Multiple Phases (with F. Otto)

\[ \int_{\Sigma_j} G_t \ast 1_{\Sigma_i} \, dx \]

\[ (\partial \Sigma_i) \cap (\partial \Sigma_j) \]
Recall Mullins’ Energy:

\[ E(\Sigma_1, \ldots, \Sigma_N) = \sum_{m,n=1}^{N} \sigma_{m,n} \text{Area}(\partial \Sigma_m \cap \partial \Sigma_n) \]

Our Approximation

\[ E_t(u_1, \ldots, u_N) = \frac{1}{\sqrt{t}} \sum_{m,n=1}^{N} \sigma_{m,n} \int u_n G_t * u_m \, dx \]

Theorem

As \( t \to 0^+ \), approximate energies \( E_t \) converge to Mullins’ sharp interface model \( E \) in the sense of Gamma convergence.
The New Algorithm

1. **Convolution step:**

\[ \phi^k_m = G\delta t * \left( \sum_{n=1}^{N} \sigma_{m,n} 1_{\Sigma_n^k} \right) \text{ for } m = 1, 2, \ldots, N. \]

2. **Redistribution step:**

\[ \Sigma_{m}^{k+1} = \left\{ x : \phi^k_m(x) = \min_{n \in \{1,2,\ldots,N\}} \phi^k_n(x) \right\}. \]

- Reduces to original MBO’89 scheme if \( \sigma_{m,n} = \delta_{m,n} \).
- **Caveat:** Mobilities \( \mu_{m,n} = \frac{1}{\sigma_{m,n}} \).
Algorithm is unconditionally stable whenever $E_t$ is concave.

Concavity depends on the surface tensions $\sigma_{m,n}$:

Need $\sigma_{m,n}$ to be C.N.D.: $\sum \sigma_{m,n} \xi_m \xi_n \leq 0$ for all $\xi \in 1^\perp$.

Schoenberg 1930s: $\sigma$ is C.N.D. iff

$$\sigma_{m,n} = |p_n - p_m|^2_2$$

for some $\{p_1, \ldots, p_N\} \subset \mathbb{R}^M$.

$\sigma_{m,n}$ given by the Read-Shockley model (1950s):

- Orientations $\theta_n \in SO(3)$.
- $\sigma_{m,n} = d(\theta_m, \theta_n)$ for some “distance” $d$ on $SO(3)$.

Theorem

The new algorithms is unconditionally stable if $\sigma_{m,n}$ are obtained from the Read-Shockley model.
Restricted to two-phase setting:

- Ruuth & Merriman (2000 & 2001). 2D. $\sigma = ? \mu = ?$
- Bonnetier, Bretin, Chambolle (2011): $\mu = \sigma$, $K \not\geq 0$, slow decay.

Questions:

- Given $\sigma(\nu)$ & $\mu(\nu)$, find the kernel $K(x)$.
- $K \geq 0$? $\hat{K} \geq 0$?
- $K$ smooth, with rapid decay?
To evaluate $E_t$ on smooth $\partial \Sigma$:

$$\lim_{t \to 0} E_t(1_{\Sigma}) = \int_{\partial \Sigma} \sigma_K(\nu(x)) \, dS(x)$$

where

$$\sigma_K(\nu) = \frac{1}{2} \int_{\mathbb{R}^d} |\nu \cdot x| K(x) \, dx$$

To evaluate movement limiter on smooth $\partial \Sigma$:

$$-E_t(1_{\Sigma} - 1_{\Sigma^k}) \approx \int_{\partial \Sigma^k} \frac{1}{\mu_K(\nu(x))} \phi^2(x) \, dS(x)$$

where

$$\mu_K^{-1}(\nu) = \int_{\nu} K(x) \, dS(x)$$

**UPSHOT:** Each kernel comes with its surface tension and mobility.
Variational Formulation

- **Surface tension:**

\[
\sigma(\nu) = - \text{F. P.} \int_{\mathbb{R}} \frac{\hat{K}(\xi\nu)}{\xi^2} d\xi \\
= - \int_{\mathbb{R}} \frac{\hat{K}(\xi\nu) - \hat{K}(0)}{\xi^2} d\xi.
\]

- **Mobility:**

\[
\frac{1}{\mu(\nu)} = \int_{\mathbb{R}} \hat{K}(\xi\nu) d\xi
\]

- **UPSHOT:** Each kernel comes with its preferred energy and mobility.
Kernel Construction

- Given desired $\sigma, \mu : \mathbb{S}^d \to \mathbb{R}^+$, need to solve

$$\int_{\mathbb{R}^d} K(x)|x \cdot \nu| \, dx = \sigma(\nu), \text{ and } \int_{\nu \perp} K(x) \, dS(x) = \frac{1}{\mu(\nu)}.$$ 

- In polar coordinates:

$$\int_{0}^{\infty} K(r\nu) r^d \, dr = \mathcal{T}^{-1} \sigma(\nu), \text{ and } \int_{0}^{\infty} K(r\nu) r^{d-2} \, dr = \mathcal{J}_s^{-1} \left[ \frac{1}{\mu} \right] (\nu)$$

where $\mathcal{T}, \mathcal{J}_s : C_e^\infty(\mathbb{S}^{d-1}) \to C_e^\infty(\mathbb{S}^{d-1})$ are defined as

$$\mathcal{T} f(\nu) = \int_{\mathbb{S}^{d-1}} f(x)|x \cdot \nu| \, dS(x) \text{ and } \mathcal{J}_s f(\nu) = \int_{\mathbb{S}^{d-1} \cap \nu \perp} f(x) \, ds(x)$$
Kernel Construction

\[ \mathcal{T} = \text{Cosine transform} \]
\[ \mathcal{J}_s = \text{Spherical Radon transform}. \]

- Letting \( \Box = \Delta_{S^{d-1}} + (d - 1)I \):
  \[ \Box \mathcal{T} = \mathcal{T} \Box = \mathcal{J}_s. \]

- Our equations in polar coordinates:
  \[ \int_0^\infty K(r\nu)r^d \, dr = \mathcal{T}^{-1}\sigma(\nu), \text{ and} \]
  \[ \int_0^\infty K(r\nu)r^{d-2} \, dr = \mathcal{J}_s^{-1}\left[\frac{1}{\mu}\right](\nu) \]

- Immediate observation:
  \[ K \geq 0 \implies \mathcal{T}^{-1}\sigma \geq 0 \text{ and } \mathcal{J}_s^{-1}\left[\frac{1}{\mu}\right] \geq 0. \]

- Necessary condition for monotonicity:
  \[ \mathcal{T}^{-1}\sigma \geq 0 \text{ and } \mathcal{J}_s^{-1}\left[\frac{1}{\mu}\right] \geq 0. \]
Kernel Construction

- $T^{-1}\sigma \geq 0? \ J_s^{-1}\left[\frac{1}{\mu}\right] \geq 0$?

- **Facts:**
  1. $T^{-1}\sigma \geq 0$ for all $\sigma$ in 2D.
  2. In 3D, $T^{-1}\sigma \geq 0$ iff $W_\sigma$ is a zonoid.

- **Zonotopes** are convex polytopes all faces of which are centrally symmetric polygons.

- **Zonoids** are the closure of zonotopes in the Hausdorff topology.

A Non-Zonoid

- Anisotropy: \( \sigma(x) = \max_{j \in \{1,2,3\}} |x_j| \)
- Unit ball: \( B_\sigma = \text{Cube} \).
- Wulff shape: \( W_\sigma = \text{Octahedron} \).

- No zonoids in a nhd. of octahedron!
How about $J_s^{-1} \left[ \frac{1}{\mu} \right] \geq 0$?

**Busemann-Petty’56** problem of convex geometry: For centered, convex bodies $K_1, K_2 \subseteq \mathbb{R}^d$, does

$$\mathcal{H}^{d-1}(K_1 \cap n^\perp) \geq \mathcal{H}^{d-1}(K_2 \cap n^\perp)$$

for all $n \in \mathbb{S}^{d-1}$ imply

$$\mathcal{H}^d(K_1) \geq \mathcal{H}^d(K_2)$$

Many contributors: E. Lutwak’88; V. D. Milman & A. Pajor’89; J. Bourgain & J. Lindenstrauss’89; J. Bourgain’91; G. Zhang’94; A. J. Gardner’94; Koldobsky’98; etc.

Answer they reached: Yes for $d \leq 4$, No for $d \geq 5$.

We need to use case $d = 3$ with minor modifications.

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Barrier Theorem

A threshold dynamics scheme that is consistent with an anisotropic surface tension $\sigma$ and mobility given by a norm $\mu$ cannot possibly be monotone unless $W_\sigma$ is the Minkowski sum of a zonoid and a sphere.

- **Converse** is also essentially true. Assume:
  1. $B_\sigma$ is strongly convex and $\partial B_\sigma$ is smooth,
  2. $W_\sigma$ is the dilation of a zonoid by a sphere,
  3. $\mu : \mathbb{S}^2 \rightarrow \mathbb{R}^+ \setminus \{0\}$ is smooth, and
  4. $\mu(x) = |x| \mu\left(\frac{x}{|x|}\right)$ is convex.

Then: There exists a positive, smooth, compactly supported convolution kernel that yields a monotone, consistent threshold dynamics scheme for the pair $\sigma$ & $\mu$.

- **Simple explicit construction:**

  $$K(\theta, \phi) = \alpha(\theta, \phi)\eta\left(\beta(\theta, \phi)r\right)$$

where

$$m_3 \frac{\alpha(\theta, \phi)}{\beta^4(\theta, \phi)} = T^{-1}\sigma(\theta, \phi)$$
and

$$m_1 \frac{\alpha(\theta, \phi)}{\beta^2(\theta, \phi)} = J_\sigma^{-1}\left[\frac{1}{\mu}\right](\theta, \phi).$$
Theorem

If $K$ has a large positive core and sufficient decay:

$$|K(x)| \leq \frac{C}{1 + |x|^{d+2}}$$

then the non-local energies

$$E_t(\Sigma) = \frac{1}{t} \int_{\Sigma_c} K_t \ast 1_{\Sigma} \, dx$$

$\Gamma$-converge to

$$E(\Sigma) = \int_{\partial \Sigma} \sigma_K(\nu(x)) \, dS(x) \text{ where } \sigma_K(\nu) = \frac{1}{2} \int |x \cdot \nu|K(x) \, dx.$$
Kernel Construction in the Fourier Domain

- **Recall:**

  \[
  \sigma(\nu) = - \text{F. P.} \int_{\mathbb{R}} \frac{\hat{K}(\xi \nu)}{\xi^2} d\xi \quad \text{and} \quad \frac{1}{\mu(\nu)} = \int_{\mathbb{R}} \hat{K}(\xi \nu) d\xi
  \]

- **Kernel of the form:***

  \[
  \hat{K}(\xi) = \frac{1}{2} \exp \left( - \eta(\alpha(\xi)) \right) + \frac{1}{2} \exp \left( - \eta(\beta(\xi)) \right)
  \]

  where \( \eta(x) \) looks like:

- **Explicit expressions for** \( \alpha(\xi) \) **and** \( \beta(\xi) \) **in terms of** \( \sigma(\xi) \) **and** \( \mu(\xi) \).  
- **Compare with Bonnetier, Bretin, Chambolle kernels.**
Multi-Phase, Anisotropic Threshold Dynamics

- Example:

\[
\begin{align*}
\sigma_{1,2}(x_1, x_2) &= \sqrt{x_1^2 + x_2^2} \\
\sigma_{1,3}(x_1, x_2) &= \sqrt{\frac{1}{4}x_1^2 + x_2^2} + \sqrt{x_1^2 + \frac{1}{4}x_2^2} \\
\sigma_{2,3}(x_1, x_2) &= \sqrt{x_1^2 + \frac{25}{16}x_2^2}
\end{align*}
\]

\[
\begin{align*}
\mu_{1,2}(x_1, x_2) &= 1, \\
\mu_{1,3}(x_1, x_2) &= \frac{2x_1^2 + 3x_2^2}{4\sqrt{x_1^2 + x_2^2}} \\
\mu_{2,3}(x_1, x_2) &= 1.
\end{align*}
\]

- Convolution kernels \(K_{m,n}\) with \((\sigma_{m,n}, \mu_{m,n})\) baked in:
Multi-Phase, Anisotropic Threshold Dynamics (with M. Elsey)

**Approximate Energy**

\[ E_t(u_1, \ldots, u_N) = \frac{1}{\sqrt{t}} \sum_{m,n=1}^{N} \int u_m(K_{m,n} * u_n) \, dx \]

\[ \min_{u_j \geq 0, \sum u_1 + \ldots + u_N = 1} E_t(u_1, \ldots, u_N) \]

**Resulting Scheme**

\[ u^{k+1}_m(x) = 1 \text{ if } \sum_{n \neq m} (K_{m,n} * u^k_n)(x) = \min_{j \in \{1,2,\ldots,N\}} \sum_{n \neq j} (K_{j,n} * u^k_n)(x) \]

- When is \( E_t \) concave? Convergence of \( E_t \)?
Comparison with **front tracking** (parametrized curves) before topological changes:
Conclusions

1. Threshold dynamics has a **variational** formulation:
   - As an optimization scheme,
   - As **minimizing movements**

2. Dissipates approx. the **right energy** w.r.t. approx. the **right metric**.

3. Gamma convergence of energies.

4. Systematic procedure to derive MBO-type algorithms.

5. **New algorithm.** for model with \( \binom{N}{2} \) distinct \((\sigma, \mu)\) pairs.

6. **Explicit formulas** for anisotropic \(\sigma\) and \(\mu\) in terms of \(K\).

7. How to **construct** \(K\) for given anisotropic \((\sigma, \mu)\) pair.

8. **New algorithm** for model with \( \binom{N}{2} \) distinct anisotropic \((\sigma, \mu)\) pairs.

   - “Barrier” statement.

10. Connections to isometric embedding of finite metric spaces, convex geometry (Busemann-Petty problem, zonoids, etc.)