Mathematical and Numerical Analysis for Multiscale Kinetic Equations with Uncertainty

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Where do kinetic equations sit in physics

- **Quantum Mechanics (Schrödinger)**: $10^{-15}$ s
- **Molecular Dynamics (Newton's Equation)**: $10^{-10}$ s
- **Kinetic Theory (Boltzmann)**: $10^{-6}$ s
- **Continuum Theory (Navier-Stokes)**: 1 s
Kinetic equations with applications

- Rarefied gas–astronautics (Boltzmann equation)
- Plasma (Vlasov-Poisson, Landau, Fokker-Planck, ⋯)
- Semiconductor device modeling
- Microfluidics
- Nuclear reactor (neutron transport)
- Astrophysics, medical imaging (radiative transfer)
- Multiphase flows
- Environmental science, energy, social science, neuronal networks, biology, etc.
Challenges in kinetic computation

- High dimension (phase space, 6D for Boltzmann)
- Multiple scales
- Uncertainty
Multiscale phenomena

C. Demaziere
Uncertainty in kinetic equations

- Kinetic equations are usually derived from N-body Newton’s second law, by mean-field limit, BBGKY hierarchy, Grad-Boltzmann limit, etc.
- Collision kernels are often empirical
- Initial and boundary data contain uncertainties due to measurement errors or modeling errors; geometry, forcing
- While UQ has been popular in solid mechanics, CFD, elliptic equations, etc. (Abgrall, Babuska, Ghanem, Gunzburger, Hesthaven, Hou, Karniadakis, Knio, Madja, Mishra, Neim, Nobile, Quarteroni, Schwab, Stuart, Tempone, Webster, Xiu, etc.) there had almost no effort for kinetic equations until very recently.
Data for scattering cross-section

Figure 2: Example of uncertainty associated with a nuclear cross-section (from (Chadwick et al., 2006)). Figure contains values corresponding to several data libraries and measurements.
UQ for kinetic equations

For kinetic models, the only thing certain is their uncertainty

- Quantify the propagation of the uncertainty
- Efficient numerical methods to study the uncertainty
- Understand its statistical moments
- Sensitivity analysis, long-time behavior of the uncertainty
- Control of the uncertainty
- Dimensional reduction of high dimensional uncertainty, etc.
Impact of uncertainty—an example
(Boltzmann with small Knudsen number)

\[
\begin{align*}
\rho_l &= 1 + 0.2\left(\frac{z+1}{2}\right), \quad u_l = (0, 0), \quad T_l = 1, \quad x \leq 0.5, \\
\rho_r &= 0.125, \quad u_r = (0, 0), \quad T_r = 0.25, \quad x > 0.5.
\end{align*}
\]
Karhunen-Loève expansion

- Any stochastic process can be approximated by a linear combination of uncorrelated random variables
- We will model uncertainty by random inputs in coefficients, initial/boundary data, source/forcing terms
- Random PDEs
Polynomial Chaos (PC) approximation

The PC or generalized PC (gPC) approach first introduced by Wiener, followed by Cameron-Martin, and generalized by Ghanem and Spanos, Xiu and Karniadakis etc. It has been shown to be very efficient in many UQ applications when the solution has enough regularity in the random variable.

Let $z$ be a random variable with pdf $\omega(z) > 0$

Let $\phi_m(z)$ be the orthonormal polynomials of degree $m$ corresponding to the weight $\omega(z) > 0$, 

$$\int \phi_i(z)\phi_j(z)\omega(z)dz > 0.$$
The Wiener-Askey polynomial chaos for random variables (table from Xiu-Karniadakis SISC 2002)

<table>
<thead>
<tr>
<th>Random variables $\zeta$</th>
<th>Wiener-Askey chaos ${\Phi(\zeta)}$</th>
<th>Support</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gaussian</td>
<td>Hermite-Chaos</td>
<td>$(-\infty, \infty)$</td>
</tr>
<tr>
<td>Gamma</td>
<td>Laguerre-Chaos</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>Beta</td>
<td>Jacobi-Chaos</td>
<td>$[a, b]$</td>
</tr>
<tr>
<td>Uniform</td>
<td>Legendre-Chaos</td>
<td>$[a, b]$</td>
</tr>
<tr>
<td>Discrete</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Poisson</td>
<td>Charlier-Chaos</td>
<td>${0, 1, 2, \ldots}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>Krawtchouk-Chaos</td>
<td>${0, 1, \ldots, N}$</td>
</tr>
<tr>
<td>Negative Binomial</td>
<td>Meixner-Chaos</td>
<td>${0, 1, 2, \ldots}$</td>
</tr>
<tr>
<td>Hypergeometric</td>
<td>Hahn-Chaos</td>
<td>${0, 1, \ldots, N}$</td>
</tr>
</tbody>
</table>

**Table 4.1**

The correspondence of the type of Wiener-Askey polynomial chaos and their underlying random variables ($N \geq 0$ is a finite integer).
Generalized polynomial chaos stochastic Galerkin (gPC-SG) methods

- Take an orthonormal polynomial basis \( \{ \phi_j(z) \} \) in the random space
- Expand functions into Fourier series and truncate:

\[
f(z) = \sum_{j=0}^{\infty} f_j \phi_j(z) \approx \sum_{j=0}^{K} f_j \phi_j(z) := f^K(z).
\]

- Substitute into system and do Galerkin projection. Then one gets a deterministic system of the gPC coefficients \( (f_0, \ldots, f_K) \).

\[
\mathbb{E}[f^K] = f_0, \quad \text{Var}[f^K] = \sum_{m=0}^{M} f_m^2.
\]
Accuracy and efficiency

- We will consider the gPC-stochastic Galerkin (gPC-SG) method.
- Under suitable regularity assumptions, this method has a spectral accuracy.
- Much more efficient than Monte-Carlo samplings (halfth-order).
- Our regularity analysis is also important for stochastic collocation method.
- How to deal with multiple scales?
A multiscale paradigm: Asymptotic-preserving (AP) methods

Consider the dynamical system with multiscale and uncertainty:

$$\partial_t u^\varepsilon = \mathcal{L}^\varepsilon(t, x, z, u^\varepsilon; \varepsilon).$$  \hspace{1cm} (1)

It has the asymptotic (or macroscopic limit) for each \(z\):

$$\partial_t u = \mathcal{L}(t, x, z, u).$$  \hspace{1cm} (2)

s-AP: a SG scheme for (1), as \(\varepsilon \to 0\), becomes a SG for (2)
Stochastic AP

Uncertain Kinetic Model $\xrightarrow{K}$ Kinetic gPC-sG $\xrightarrow{\Delta x, \Delta v, \Delta t}$ Kinetic gPC-sG discretized

$\epsilon \rightarrow 0$

Uncertain Fluid Model $\xrightarrow{K}$ Fluid gPC-sG $\xrightarrow{\Delta x, \Delta v, \Delta t}$ Fluid gPC-sG discretized

$\epsilon \rightarrow 0, \epsilon \rightarrow 0, \epsilon \rightarrow 0$
Example: random linear neutron transport equation
(Jin-Xiu-Zhu JCP’ 14)

\[ \varepsilon \partial_t f(v) + v \partial_x f(v) = \frac{\sigma(x, z)}{\varepsilon} \left[ \frac{1}{2} \int_{-1}^{1} f(v') dv' - f(v) \right], \]

\( \sigma(x, z) \) the scattering cross-section, is random.

**Diffusion limit:** Larsen-Keller, Bardos-Santos-Sentis, Bensoussan-Lions-Papanicolaou (for each \( z \))

As \( \varepsilon \to 0^+ \), \( f \to \rho(t, x) = \frac{1}{2} \int_{-1}^{1} f(v') dv' \), and

\[ \rho_t = \partial_x \left[ \frac{1}{3 \sigma(x, z)} \partial_x \rho \right]. \]
Linear transport equation with random coefficients

To understand its diffusion limit, we first split this equation into two equations for $v > 0$:

$$
\varepsilon \partial_t f(v) + v \partial_x f(v) = \frac{\sigma(x, z)}{\varepsilon} \left[ \frac{1}{2} \int_{-1}^{1} f(v') dv' - f(v) \right],
$$

$$
\varepsilon \partial_t f(-v) - v \partial_x f(-v) = \frac{\sigma(x, z)}{\varepsilon} \left[ \frac{1}{2} \int_{-1}^{1} f(v') dv' - f(-v) \right],
$$

and then consider its even and odd parities

$$
r(t, x, v) = \frac{1}{2} \left[ f(t, x, v) + f(t, x, -v) \right],
$$

$$
\quad j(t, x, v) = \frac{1}{2\varepsilon} \left[ f(t, x, v) - f(t, x, -v) \right].
$$
The above even-odd parity system can be rewritten as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} r + v \frac{\partial}{\partial x} j &= \frac{\sigma(x, z)}{\varepsilon^2} (\bar{r} - r), \\
\frac{\partial}{\partial t} j + \frac{\nu}{\varepsilon^2} \frac{\partial}{\partial x} r &= -\frac{\sigma(x, z)}{\varepsilon^2} j,
\end{align*}
\]

where \( \bar{r}(t, x) = \int_0^1 r d\nu \).

As \( \varepsilon \to 0^+ \), (3) yields

\[
\begin{align*}
r &= \bar{r}, \\
j &= -\frac{\nu}{\sigma(x, \nu)} \frac{\partial}{\partial x} \bar{r}.
\end{align*}
\]

Substituting it into system (3) and integrating over \( \nu \), one gets the limiting diffusion equation

\[
\frac{\partial}{\partial t} \bar{r} = \frac{1}{3\sigma(x, z)} \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \bar{r} \right].
\]
gPC approximations

Let

\[ r_N(x, z, t) = \sum_{m=1}^{M} \hat{r}_m(t, x) \Phi_m(z), \quad j_N(x, z, t) = \sum_{m=0}^{M} \hat{j}_m(t, x) \Phi_m(z) \]

be the Nth-order gPC expansion for the solutions and

\[ \hat{r} = (\hat{r}_1, \ldots, \hat{r}_M)^T, \quad \hat{j} = (\hat{j}_1, \ldots, \hat{j}_M)^T, \]

After Galerkin projection one gets

\[
\begin{aligned}
\partial_t \hat{r} + v \partial_x \hat{j} &= \frac{1}{\varepsilon^2} S(x)(\bar{r} - \hat{r}), \\
\partial_t \hat{j} + \frac{v}{\varepsilon^2} \partial_x \hat{r} &= -\frac{1}{\varepsilon^2} S(x)\hat{j},
\end{aligned}
\]

where

\[ \bar{r}(x, t) = \int_0^1 \hat{r} dv, \]

and \( S(x) = (s_{ij}(x))_{1 \leq i,j \leq M} \) is a \( M \times M \) matrix with entries

\[ s_{ij}(x) = \int \sigma(x, z) \Phi_i(z) \Phi_j(z) \rho(z) dz. \]
Vectorized version of the deterministic problem

- One can now use deterministic AP schemes to solve this system.

- Why s-AP?

When $\varepsilon \rightarrow 0$, the gPC-SG for transport equation becomes the gPC-SG for the limiting diffusion equation.
Uniform spectral accuracy

- via **hypocoercivity**: uniform exponential decay to the local equilibrium: Jin-J.-G. Liu-Ma (RMS ‘17)

- Define the following norms

\[ \langle f, g \rangle_\omega = \int_{I_z} fg \omega(z)dz, \quad \| f \|^2_\omega = \langle f, f \rangle_\omega. \]

\[ \| f(t, x, v, \cdot) \|^2_{H^k} := \sum_{\alpha \leq k} \| D^\alpha f(t, x, v, \cdot) \|^2_\omega. \]

\[ \| f(t, \cdot, \cdot, \cdot, \cdot) \|^2_\Gamma := \int_Q \| f(t, x, v, \cdot) \|^2_\omega \, dx dv. \]
Uniform regularity

- The regularity in the random space is preserved in time, uniformly in $\varepsilon$.

$$D^k f(t, x, v, z) := \partial_z^k f(t, x, v, z).$$

**Theorem**

Assume

$$\sigma(z) \geq \sigma_{\text{min}} > 0.$$ 

If for some integer $m \geq 0$,

$$\|D^k \sigma(z)\|_{L^\infty} \leq C_{\sigma}, \quad \|D^k f_0\|_\Gamma \leq C_0, \quad k = 0, \ldots, m,$$

then

$$\|D^k f\|_\Gamma \leq C, \quad k = 0, \ldots, m, \quad \forall t > 0,$$

where $C_{\sigma}, C_0$ and $C$ are constants independent of $\varepsilon$.

- A good problem to use the gPC-SG for UQ.
Key estimates

- **Energy estimate:** For any \( k \geq 0 \), there exist \( k \) constants \( c_{kj} > 0 \), \( j = 0, \ldots, k - 1 \) such that

\[
\varepsilon^2 \partial_t \left( \| D^k f \|_\Gamma^2 + \sum_{j=0}^{k-1} c_{kj} \| D^j f \|_\Gamma^2 \right) \leq \begin{cases} 
-2\sigma_{\min} \|[f] - f\|_\Gamma^2, & k = 0, \\
-\sigma_{\min} \| D^k ([f] - f) \|_\Gamma^2, & k \geq 1.
\end{cases}
\]

**Theorem (\( \varepsilon^2 \) estimate on \([f] - f\))**

*With all the assumptions in Theorem 4.1 and Lemma 4.2, for a given time \( T > 0 \), the following regularity result of \([f] - f\) holds:*

\[
\| D^k ([f] - f) \|_\Gamma^2 \leq e^{-\sigma_{\min} t / 2\varepsilon^2} \| D^k ([f_0] - f_0) \|_\Gamma^2 + C' \varepsilon^2,
\]

*for any \( t \in (0, T) \) and \( 0 \leq k \leq m \), where \( C' \) and \( C \) are constants independent of \( \varepsilon \).*
Uniform spectral convergence (s-AP)

**Theorem**

(Uniform convergence in $\varepsilon$) Assume

$$\sigma(z) \geq \sigma_{\text{min}} > 0.$$ 

If for some integer $m \geq 0$,

- $\|\sigma(z)\|_{H^k} \leq C_\sigma$,
- $\|D^k f_0\|_\Gamma \leq C_0$,
- $\|D^k (\partial_x f_0)\|_\Gamma \leq C_x$, $k = 0, \ldots, m$,

then the error of the sG method is

$$\|f - f^K\|_\Gamma \leq \frac{C(T)}{K^k},$$

where $C(T)$ is a constant independent of $\varepsilon$. 
For a fully discrete scheme based on the deterministic micro-macro decomposition \((f = M + g)\) based approach (Klar-Schmeiser, Lemou-Mieussens) approach, we can also prove the following uniform stability:

\[
\Delta t \leq \frac{\sigma_{\text{min}}}{3} \Delta x^2 + \frac{2\varepsilon}{3} \Delta x.
\]
Numerical tests

Fig. 1  Example 1. Errors of the mean (solid line) and standard deviation (dash line) of $\rho$ with respect to the gPC order at $\varepsilon = 10^{-8}$: $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta x = 0.01$ (stars)
Numerical tests

Fig. 4  Example 1. Differences in the mean (solid line) and standard deviation (dash line) of $\rho$ with respect to $\varepsilon^2$, between limiting analytical solution (103) and the 4th-order gPC solution with $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles) and $\Delta x = 0.01$ (stars)
**Fig. 11** The mean (left) and standard deviation (right) of $\rho$ at $\varepsilon = 10^{-8}$, obtained by 5th-order gPC Galerkin (circles) and the stochastic collocation method (crosses). The random input has dimension $d = 2$. 
One can extend hypocoercivity theory developed by Villani, Desvillettes, Guo, Mouhut, Briant, etc. in velocity space for deterministic problems to study the following properties in random space:

regularity, sensitivity in random parameter, long-time behavior (exponential decay to global equilibrium), spectral convergence and long-time exponential decay of numerical error for gPC-SG

also relevant contributions (Y. Zhu, R. Shu)

linear kinetic equation: Qin Li, Li Wang
Entropy decay in inhomogeneous Boltzmann equation

- Simulation of 1 + 2 D Boltzmann equation: wavy entropy decay
- $H_g(t)$: relative entropy w.r.t. the global Maxwellian

Nonlinear collisional kinetic equations

- Following Mouhot-Neumann (‘06), Briant (‘15), consider the initial value problem for kinetic equations of the form

\[
\begin{align*}
\partial_t f + \frac{1}{\epsilon^\alpha} v \cdot \nabla_x f &= \frac{1}{\epsilon^{1+\alpha}} Q(f), \\
\end{align*}
\]

\[
\begin{align*}
f(0, x, v, z) &= f_{\text{in}}(x, v, z), \quad x \in \Omega \subset \mathbb{T}^d, \; v \in \mathbb{R}^d, \; z \in I_z \subset \mathbb{R},
\end{align*}
\]

- $\epsilon$ is the Knudsen number. $z$ is the random variable. Periodic B.C. in space.
- $\alpha = 1$: the incompressible Navier-Stokes (INS) scaling;
- $\alpha = 0$: the Euler (acoustic) scaling.
Perturbative Setting with Small Scales

\[ f = \mathcal{M} + \epsilon M h, \]

\(\mathcal{M}\): global Maxwellian, \(M = \sqrt{\mathcal{M}}\); avoid compressible Euler limit, thus shocks

- Plug the above ansatz into the model equation,

\[
\begin{aligned}
\partial_t h + \frac{1}{\epsilon^\alpha} \nu \cdot \nabla_x h &= \frac{1}{\epsilon^{1+\alpha}} \mathcal{L}(h) + \frac{1}{\epsilon^\alpha} \mathcal{F}(h, h), \\
\end{aligned}
\]

\(h(0, x, \nu, z) = h_{\text{in}}(x, \nu, z), \quad x \in \Omega \subset \mathbb{T}^d, \ \nu \in \mathbb{R}^d, \ z \in I_z, \)

\(\mathcal{L}\) is the linearized operator, \(\mathcal{F}\) is the nonlinear operator.

- Why it works: hypocoercivity decay of the linear part dominates the bounded (weaker) nonlinear part; need to construct a new Lyapunov functional which involves mixed \(x, \nu\) derivatives
Hypocoercivity Assumptions

List two among the several hypocoercivity assumptions (Mouhot-Neumann-09'):

Denote $N(L) = \text{Span}\{\varphi_1, \cdots, \varphi_n\}$, and $\Pi_L(h)$ is the orthogonal projection in $L^2_\nu$ on $N(L)$. $L$ has the local coercivity property; $\exists \lambda > 0$ such that $\forall h \in L^2_\nu$,

$$\langle L(h), h \rangle_{L^2_\nu} \leq -\lambda ||h^\perp||^2_{\Lambda_\nu},$$

where $h^\perp = h - \Pi_L(h)$ is the microscopic part of $h$.

For example, for the Boltzmann equation, the coercivity norm is defined by

$$||h||_{\Lambda_\nu} = ||h(1 + |\nu|)^{\gamma/2}||_{L^2_\nu}.$$
With random dependence

\[
\left| \langle \partial^m \partial^j \mathcal{F}(h, h), f \rangle_{L^2_{x,v}} \right| \leq \begin{cases} 
G^{s,m}_{x,v,z}(h, h) \| f \|_\Lambda, & \text{if } j \neq 0, \\
G^{s,m}_{x,z}(h, h) \| f \|_\Lambda, & \text{if } j = 0.
\end{cases}
\]

There exists a z-independent \( C_{\mathcal{F}} > 0 \) such that for all \( z \),

\[
\sum_{|m| \leq r} (G^{s,m}_{x,v,z}(h, h))^2 \leq C_{\mathcal{F}} \| h \|_{H^s_{x,v}}^2 \| h \|_{H^s_\Lambda}^2,
\]

\[
\sum_{|m| \leq r} (G^{s,m}_{x,z}(h, h))^2 \leq C_{\mathcal{F}} \| h \|_{H^s_{x,v} L^2_{x,v}}^2 \| h \|_{H^s_\Lambda}^2.
\]

- Define the sum of Sobolev norms of the \( z \) derivatives by

\[
\| h \|_{H^s_{x,v}}^2 = \sum_{|m| \leq r} \| \partial^m h \|_{H^s_{x,v}}^2,
\]

\[
\| h \|_{H^s_\Lambda}^2 = \sum_{|j| + |l| \leq s} \| \partial^j_h \|_\Lambda^2.
\]

\[
\| h \|_{H^s_{x,v} L^2_{x,v}}^2 = \sum_{|m| \leq r} \| \partial^m h \|_{H^s_{x,v} L^2_{x,v}}^2.
\]

\[
\| h \|_{H^s_{x,v}}^2 \leq C_{\mathcal{F}} \| h \|_{H^s_{x,v} L^2_{x,v}}^2 \| h \|_{H^s_\Lambda}^2.
\]
A Lyapunov Functional $|| \cdot ||_{\mathcal{H}^s_{\epsilon \perp}}$ (following Briant ‘15)

- Define $|| \cdot ||_{\mathcal{H}^s_{\epsilon \perp}}$ by

$$|| \cdot ||^2_{\mathcal{H}^s_{\epsilon \perp}} = \sum_{|j| + |l| \leq s, |j| \geq 1} b^{(s)}_{j,l} ||\partial_j (\mathbb{I} - \Pi_L) \cdot ||^2_{L^2_{x,v}} + \sum_{|l| \leq s} \alpha^{(s)}_l ||\partial^0_l \cdot ||^2_{L^2_{x,v}}$$

$$+ \sum_{|l| \leq s, i, c_i(l) > 0} \epsilon \ a^{(s)}_{i,l} \langle \partial^{\delta_i}_l \cdot, \partial^0_l \cdot \rangle_{L^2_{x,v}},$$

We have

$$|| \cdot ||_{\mathcal{H}^s_{\epsilon \perp}} \equiv || \cdot ||_{\mathcal{H}^s_{x,v}}.$$

- The case $s = 1$:

$$||h||^2_{\mathcal{H}^1_{\epsilon \perp}} = A ||h||^2_{L^2_{x,v}} + \alpha ||\nabla_x h||^2_{L^2_{x,v}} + b ||\nabla_v h^\perp||^2_{L^2_{x,v}} + a \epsilon \langle \nabla_x h, \nabla_v h \rangle_{L^2_{x,v}}.$$
Sensitivity and regularity results

In the case of random initial data, we have the following results on the convergence to global equilibrium:

\textbf{Theorem}

Assume $\|h_{in}\|_{H_{x,v}^{s,r} L^\infty_z} \leq C_I$, then

(i) For incompressible Navier-Stokes scaling:

$$
\|h\|_{H_{x,v}^{s,r} L^\infty_z} \leq C_I e^{-\tau_s t}, \quad \|h\|_{H_{x,v}^s H_r^r} \leq C_I e^{-\tau_s t}.
$$

(ii) For Euler (acoustic) scaling:

$$
\|h\|_{H_{x,v}^{s,r} L^\infty_z} \leq C_I e^{-\epsilon \tau_s t}, \quad \|h\|_{H_{x,v}^s H_r^r} \leq C_I e^{-\epsilon \tau_s t},
$$

where $C_I$, $\tau_s$ are positive constants independent of $\epsilon$. 


If the collision kernel is also random, we need the following assumptions (for Boltzmann)

\[ B(|v - v_*|, \cos \theta, z) = \phi(|v - v_*|) b(\cos \theta, z), \phi(\xi) = C_\phi \xi^\gamma, \text{ with } \gamma \in [0, 1], \]

\forall \eta \in [-1, 1], |b(\eta, z)| \leq C_b, |\partial_\eta b(\eta, z)| \leq C_b, \text{ and } |\partial_z^k b(\eta, z)| \leq C_b^*, \forall 0 \leq k \leq r.

One needs to use a weighted Sobolev norm in random space as in Jin-Ma-J.G.Liu

\[ ||g||_{L^2_{x,v}} : = \sum_{m=0}^{r} \tilde{C}_{m+1} ||\partial^m g||_{L^2_{x,v}}. \]

Similar decay rates can be obtained.
The gPC Galerkin System of the Boltzmann

- Insert the gPC ansatz and conduct a standard Galerkin projection, then obtain the gPC-Galerkin system for the gPC coefficients $h_k$ (INS scaling):

$$
\begin{aligned}
\partial_t h_k + \frac{1}{\epsilon} \nu \cdot \nabla_x h_k &= \frac{1}{\epsilon^2} L_k(h^K) + \frac{1}{\epsilon} F_k(h^K, h^K), \\
h_k(0, x, \nu) &= h_k^0(x, \nu), \quad x \in \Omega \subset \mathbb{T}^d, \ \nu \in \mathbb{R}^d,
\end{aligned}
$$

for each $1 \leq |k| \leq K$, with a periodic B.C. and the initial data

$$
h_k^0 := \int_{I_z} h^0(x, \nu, z) \psi_k(z) \pi(z) dz.
$$
Assumptions

- Assume that $B$ satisfies

$$B(|v - v_*|, \cos \theta, z) = \phi(|v - v_*|) b(\cos \theta, z),$$

and is linear in $z$, in the form of

$$b(\cos \theta, z) = b_0(\cos \theta) + b_1(\cos \theta) z,$$

with $|\partial_z b| \leq O(\epsilon)$.

- We also need the technical condition (following Shu-Jin ‘17):

$$||\psi_k||_{L^\infty} \leq C k^p, \quad \forall k,$$

with a parameter $p > 0$ (so the initial data are independent of $K$). Examples satisfying this condition include normalized Legendre polynomials, normalized Chebyshev polynomials, etc.
Hypocoercivity Estimates of the gPC Solution

Let $q > p + 2$, define the energy $E^K$ by $E^K(t) = E_{s,q}^K(t) = \sum_{k=1}^{K} \| k^q h_k \|_{H^s_{x,v}}^2$, with the initial data satisfying $E^K(0) \leq \eta$. If $h^K$ is the gPC solution, then

**Theorem**

(i) Under the INS scaling,

$$E^K(t) \leq \eta e^{-\tau t}, \quad \| h^K \|_{H^s_{x,v} L^\infty_z} \leq \eta e^{-\tau t}.$$ 

(ii) Under the acoustic scaling,

$$E^K(t) \leq \eta e^{-\epsilon \tau t}, \quad \| h^K \|_{H^s_{x,v} L^\infty_z} \leq \eta e^{-\epsilon \tau t}.$$ 

where $\eta, \tau$ are all positive constants that only depend on $s$ and $q$, independent of $K$ and $z$.

- Solution in the long time **insensitive** to random perturbation for initial data or collision kernel under the aforementioned assumptions
- Sensitivity if aforementioned assumptions are not satisfied? open
Hypocoercivity Estimates of the gPC error

- Define the norm \( \|g\|_{H^{s}_{x,v} L^{2}_{z}}^{2} := \int_{I_{z}} \|g\|_{H^{s}_{x,v}}^{2} \pi(z) \, dz \).

**Theorem**

Assume \( \|h_{in}\|_{H^{s} \cdot r L^{\infty}_{z}} \leq C_{I} \), then

(i) Under the INS scaling,

\[
\|h - h^{K}\|_{H^{s}_{x,v} L^{2}_{z}} \leq C_{e} \frac{e^{-\lambda t}}{K r},
\]

(ii) Under the Euler scaling,

\[
\|h - h^{K}\|_{H^{s}_{x,v} L^{2}_{z}} \leq C_{e} \frac{e^{-\epsilon \lambda t}}{K r},
\]

with the constants \( C_{e}, \lambda > 0 \) independent of \( K \) and \( \epsilon \).

The gPC-SG method for the Boltzmann equation with random inputs and both scalings is of **spectral accuracy**, and the total gPC error **decays exponentially** in time.
What about variance?

\[ f(t, v, z) = \sum_{k=0}^{\infty} \hat{f}_k(t, v) \phi_k(z). \]

\[ \mathbb{E}[f] = \hat{f}_0, \quad \text{Var}[f] = \sum_{k=0}^{\infty} \hat{f}_k^2. \]

By Parseval’s identity, our results directly imply the same accuracy and decay rate for the variance, since the global equilibrium is deterministic and our estimates are established in \( L^2 \) and Sobolev norms.
A general framework

- This framework works for general linear and nonlinear collisional kinetic equations.

- Also works for non-collision kinetic equation: Vlasov-Poisson-Fokker-Planck system (Jin-Y. Zhu (SIAM ‘18)).
gPC-SG for many different kinetic equations

- **Boltzmann**: a fast algorithm for collision operator (J. Hu-Jin, JCP ‘16), sparse grid for high dimensional random space (J. Hu-Jin-R. Shu ‘16). *initial regularity in the random space is preserved in time*; but not clear whether it is uniformly stable in the compressible Euler limit (s-AP?): gPC-SG for nonlinear hyperbolic system is not globally hyperbolic! (APUQ is open)

- **Landau equation** (J. Hu-Jin-R. Shu, ‘16)

- High dimensional random space: Best-$N$ approximations + greedy algorithms for linear kinetic equations (Jin-Y. Zhu-Zuazua)

- Kinetic-incompressible fluid couple models for disperse two phase flow: (efficient algorithm in multi-D: Jin-R. Shu. Theory open)

- Vlasov-Poisson system (**Landau Damping**): R. Shu-J
Conclusion

- gPC-SG allows us to treat kinetic equations with random inputs in the deterministic AP framework
- Many different kinetic equations can be solved this way
- Hypocoercivity based regularity and sensitivity analysis can be done for general linear and nonlinear collision kinetic equations and Vlasov-Poisson-Fokker-Planck system, which imply (uniform) spectral convergence of gPC methods
- Kinetic equations have the good regularity in the random space, even for the nonlinear kinetic equation: good problem for UQ!
- Can develop efficient algorithms and in the meantime develop deep, physically important, and beautiful mathematical theory and numerical analysis
- Many kinetic ideas useful for UQ problems: mean-field approximations; moment closure; (s-AP is one example); and vice versa
- Many open questions, very few existing works (e.g. boundary conditions)
- Kinetic equations are good problems for UQ!