Nonlocal modeling, analysis and computation: an invitation

Qiang Du

Dept. of Appl. Phys. and Appl. Math., and Data Science Institute, Columbia University

Thanks to ICM committees/panels
Outline

Nonlocal models

► What are they?
► Why do we study them?
► What is the math behind?
► How to solve them?
Nonlocal model: example and comparison

- Differential equation \(-\frac{d^2 u}{dx^2}(x) = f(x), \ x \in \mathbb{R}\). 

- Difference equation \(-D_h^2 u(x) = -\frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = f(x)\).
Nonlocal model: example and comparison

- Differential equation \(- \frac{d^2 u}{dx^2}(x) = f(x), \ x \in \mathbb{R}\). 

- Difference equation \(-D_h^2 u(x) = - \frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = f(x)\). 

- Nonlocal/integral equation \(-\mathcal{L}_\delta u_\delta(x) = f(x), \ x \in \mathbb{R}\), which for a given kernel \(\omega_\delta\) and a nonlocal horizon \(\delta\), is associated with a nonlocal diffusion operator

\[
\mathcal{L}_\delta u(x) = \int_{-\delta}^{\delta} \frac{u(x + s) - 2u(x) + u(x - s)}{s^2} \omega_\delta(s) ds.
\]
Nonlocal model: example and comparison

- **Differential equation** \( -\frac{d^2 u}{dx^2}(x) = f(x), \ x \in \mathbb{R}. \)

- **Difference equation** \( -D_h^2 u(x) = -\frac{u(x + h) - 2u(x) + u(x - h)}{h^2} = f(x). \)

- **Nonlocal/integral equation** \( -\mathcal{L}_\delta u_\delta(x) = f(x), \ x \in \mathbb{R}, \) which for a given kernel \( \omega_\delta \) and a nonlocal horizon \( \delta, \) is associated with a nonlocal diffusion operator

  \[
  \mathcal{L}_\delta u(x) = \int_{-\delta}^{\delta} \frac{u(x + s) - 2u(x) + u(x - s)}{s^2} \omega_\delta(s) \, ds.
  \]

- **More general nonlocal equations can be alternatives/tools/bridges to existing local/discrete/fractional models, possibly allowing more singular solutions.**

  \[
  \omega_\delta(s) \to \text{Dirac-delta measure at } s = 0 \text{ or } h \quad \Rightarrow \quad \mathcal{L}_\delta \to \mathcal{L}_0 = \frac{d^2}{dx^2} \text{ or } D_h^2,
  \]

  \[
  \omega_\delta(s) \to c_{\alpha,1}|s|^{1-2\alpha}, \ 0 < \alpha < 1, \ \delta \to \infty \quad \Rightarrow \quad \mathcal{L}_\delta \to \mathcal{L}_\infty = \left(-\frac{d^2}{dx^2}\right)^\alpha.
  \]
Nonlocal models in action: an illustrative example

Bridging local/fractional models: nonlocal derivative \( G^\delta_t u(t) = \int_0^\delta \rho_\delta(s) \frac{u(t) - u(t - s)}{s} ds \)

Normal diffusion  ←  Nonlocal-in-time dynamics  →  Fractional subdiffusion

\( u_t = \Delta u \)  \( (\text{NLTD}) \quad G^\delta_t u = \Delta u \)  \( D^\alpha_t u = \Delta u \)
Nonlocal models in action: an illustrative example

Bridging local/fractional models: nonlocal derivative

\[ \mathcal{G}_t^\delta u(t) = \int_0^\delta \rho_\delta(s) \frac{u(t) - u(t - s)}{s} ds \]

Normal diffusion \quad \leftarrow \text{Nonlocal-in-time dynamics} \quad \rightarrow \text{Fractional subdiffusion}

\[ u_t = \Delta u \quad \text{(NLTD)} \quad \mathcal{G}_t^\delta u = \Delta u \quad D_t^\alpha u = \Delta u \]

Eg. MSD from NLTD simulation (\( \alpha = 0.2 \), Du-Yang-Zhou 16, D-Z 18) vs reported experimental data\(^1\)

\(^1\) Jeon-Monne-Javanainen-Metzler 12 PRL; He-Song-Su-Geng-Ackerson-Peng-Tong 16 NatureComm
Nonlocal models in action: an illustrative example

Bridging local/fractional models: nonlocal derivative \( G^\delta_t u(t) = \int_0^\delta \rho(\delta, s) \frac{u(t) - u(t - s)}{s} ds \)

Normal diffusion ← Nonlocal-in-time dynamics → Fractional subdiffusion

\[ u_t = \Delta u \quad \text{(NLTD)} \quad G^\delta_t u = \Delta u \quad D^\alpha_t u = \Delta u \]

Eg. MSD from NLTD simulation (\( \alpha = 0.2, \text{Du-Yang-Zhou 16, D-Z 18} \)) vs reported experimental data

NLTD captures crossover from sub- to normal diffusion effectively.

\[ \rho(s) \sim s^{-\alpha}, \text{NLTD} \]

\[ <x^2> = t, \quad <x^2> \propto t^{0.5} \]

\[ <x^2> \propto t^{0.2} \]

\[ \text{PRL 2012, NatComm 2016} \]

\[ \text{1 Jeon-Monne-Javanainen-Metzler 12 PRL; He-Song-Su-Geng-Ackerson-Peng-Tong 16 NatureComm} \]
Nonlocal models in action: an illustrative example

Nonlocal relaxation/smoothing: tools to understand/approximate local models.

E.g. Smoothed Particle Hydrodynamics (SPH, Gingold-Monaghan/Lucy 77): popular particle discretization (> 5000 citations) based on nonlocal/integral smoothing/regulation of local continuum PDEs. Its usual two steps are:
Nonlocal models in action: an illustrative example

Nonlocal relaxation/smoothing: tools to understand/approximate local models.

E.g. Smoothed Particle Hydrodynamics (SPH, Gingold-Monaghan/Lucy 77): popular particle discretization (> 5000 citations) based on nonlocal/integral smoothing/regulation of local continuum PDEs. Its usual two steps are:

for a smooth kernel $W(|x|, \delta)$ with a smoothing length $\delta$ (support of $W$ in $|x|$)

$$\nabla_x u(x) \approx \int u(y) W_r(|x - y|, \delta) \frac{x - y}{|x - y|} dy \quad \text{nonlocal smoothing on scale } \delta \quad ①$$

$$\approx \sum u_j W_r(|x_k - x_j|, \delta) \frac{x_k - x_j}{|x_k - x_j|} |V_j| \quad \text{particle/quadrature with spacing } h \quad ②$$
Nonlocal models in action: an illustrative example

Nonlocal relaxation/smoothing: tools to understand/approximate local models.

E.g. Smoothed Particle Hydrodynamics (SPH, Gingold-Monaghan/Lucy 77): popular particle discretization (> 5000 citations) based on nonlocal/integral smoothing/regulation of local continuum PDEs. Its usual two steps are:

for a smooth kernel $W(|x|, \delta)$ with a smoothing length $\delta$ (support of $W$ in $|x|$)

$$\nabla_x u(x) \approx \int u(y) W_r(|x - y|, \delta) \frac{x - y}{|x - y|} dy$$

nonlocal smoothing on scale $\delta$ ①

$$\approx \sum u_j W_r(|x_k - x_j|, \delta) \frac{x_k - x_j}{|x_k - x_j|} |V_j|$$

particle/quadrature with spacing $h$ ②

Better nonlocal continuum models (e.g. Du-X.Tian 18) ⇒ more robust discretization.
Nonlocal models: alternatives to traditional continuum mechanics, in the form of local PDEs, particularly when the latter may be questionable near materials defects/cracks.
Nonlocal models in action: an illustrative example

Nonlocal models: *alternatives* to traditional continuum mechanics. in the form of local PDEs, particular when the latter may be questionable near materials defects/cracks.

E.g., *Peridynamics* (PD, Silling 00): nonlocal model of continuum mechanics with spatial derivatives in classical Newton’s law replaced by *nonlocal/integral operators*.

\[
\rho \ u_{tt}(t, x) = \int_{B_\delta(x)} \left\{ T(\mathbf{u}(t, x), \mathbf{u}(t, y), x, y) - T(\mathbf{u}(t, y), \mathbf{u}(t, x), y, x) \right\} dy
\]
Nonlocal models in action: an illustrative example

Nonlocal models: alternatives to traditional continuum mechanics. in the form of local PDEs, particular when the latter may be questionable near materials defects/cracks.

E.g., Peridynamics (PD, Silling 00): nonlocal model of continuum mechanics with spatial derivatives in classical Newton’s law replaced by nonlocal/integral operators.

$$\rho \frac{\partial^2 u(t, x)}{\partial t^2} = \int_{B_\delta(x)} \left\{ T(u(t, x), u(t, y), x, y) - T(u(t, y), u(t, x), y, x) \right\} dy$$


Nonlocal PD allows discontinuous displacement ⇒ material failure predictions.

Nonlocal models

- Effective descriptions of anomalous-process and singular-behavior.
- Potential alternatives/tools/bridges to local PDEs and other models.
- Generic outcome of model-reduction and coarse-graining.
- Discussions by Rayleigh, van der Walls, Korteweg, ...
Nonlocal models

- Effective descriptions of anomalous-process and singular-behavior.
- Potential alternatives/tools/bridges to local PDEs and other models.
- Generic outcome of model-reduction and coarse-graining.
- Discussions by Rayleigh, van der Walls, Korteweg, ...

As our perceived world gets smaller, more connected each day, together with increasing attention to studies of complexity, singularity and anomaly $\Rightarrow$ growth in nonlocal modeling.
New area of nonlocal modeling

Recent development: a focus on the study of systems for vector and tensor fields having a finite range (horizon $\delta$) of nonlocal interactions, which

- is driven by applications to mechanics (fracture/peridynamics, CFD/SPH, multiscale coupling), to the geometry of data, ...
Recent development: a focus on the study of systems for vector and tensor fields having a finite range (horizon $\delta$) of nonlocal interactions, which

- is driven by applications to mechanics (fracture/peridynamics, CFD/SPH, multiscale coupling), to the geometry of data, ...

- enjoys richer structures than nonlocal models with only scalar nonlocal operators (such as nonlocal means, fractional derivatives, $^3$)

---

$^3$E.g., earlier studies by Gilboa-Osher, Caffarelli-Silvestre, Rossi et al, Kassmann et al,...
New area of nonlocal modeling

**Recent development:** a focus on the study of systems for *vector and tensor fields* having a *finite range* (horizon $\delta$) of nonlocal interactions, which

- is driven by applications to *mechanics* (fracture/peridynamics, CFD/SPH, multiscale coupling), to the *geometry* of data, ...

- enjoys *richer structures* than nonlocal models with only scalar nonlocal operators (such as nonlocal means, fractional derivatives, ...$^3$)

- motivates systematic development of new mathematical framework (*nonlocal vector calculus, nonlocal calculus of variations, ...$^4$*)

---

$^3$Du, *Nonlocal modeling, analysis and computation*, NSF-CBMS Research Monograph, SIAM

$^4$Du, *Nonlocal modeling, analysis and computation*, NSF-CBMS Research Monograph, SIAM
A Simple illustration: linear PD with small deformation

E.g., force balance for a continuum of (linear/isotropic) Hookean springs:
E.g., force balance for a continuum of (linear/isotropic) Hookean springs:

\[ \mathcal{L}_\delta u_\delta(x) = \int_{\Omega \cup \Omega_\delta} \omega_\delta(|y - x|) \frac{y - x}{|y - x|} \left( \frac{y - x}{|y - x|} \cdot \frac{u_\delta(y) - u_\delta(x)}{|y - x|} \right) dy. \]

- \( u_\delta(x) \): displacement at \( x \);
- \( \delta \): nonlocal horizon;
- \( \omega_\delta(|r|) \): nonlocal kernel, with support in \( B_\delta(0) \).
A Simple illustration: linear PD with small deformation

E.g., force balance for a continuum of (linear/isotropic) Hookean springs:

\[
\mathcal{L}_\delta \mathbf{u}_\delta(x) = \int_{\Omega \cup \Omega_\delta} \frac{\omega_\delta(|y - x|)}{|y - x|} \frac{y - x}{|y - x|} \left( \frac{y - x}{|y - x|} \cdot \frac{\mathbf{u}_\delta(y) - \mathbf{u}_\delta(x)}{|y - x|} \right) dy.
\]

Spring constant direction (linear volumetric strain)

Spring Bond Projected relative deformation

\( \mathbf{u}_\delta(x) \): displacement at \( x \);
\( \delta \): nonlocal horizon;
\( \omega_\delta(|r|) \): nonlocal kernel, with support in \( B_\delta(0) \).

Nonlocal/volumetric constraint in \( \Omega_\delta = \{ x \in \Omega^c, d(x, \partial \Omega) < \delta \} \), analog of local BC.

\(-\mathcal{L}_\delta \mathbf{u}_\delta = \mathbf{b} \) in \( \Omega \)
\( \mathbf{u}_\delta = 0 \) on \( \Omega_\delta \)
A Simple illustration: linear PD with small deformation

E.g., force balance for a continuum of (linear/isotropic) Hookean springs:

\[
\mathcal{L}_\delta u_\delta(x) = \int_{\Omega \cup \Omega_\delta} \frac{\omega_\delta(|y - x|)}{|y - x|} \frac{y - x}{|y - x|} \left( \frac{y - x}{|y - x|} \cdot \frac{u_\delta(y) - u_\delta(x)}{|y - x|} \right) dy.
\]

**Spring constant** | **Bond direction** | **Projected relative deformation** (linear volumetric strain)
--|--|--
\(u_\delta(x):\) displacement at \(x\);
\(\delta: \) nonlocal horizon;
\(\omega_\delta(|r|):\) nonlocal kernel, with support in \(B_\delta(0)\).

Nonlocal/volumetric constraint in \(\Omega_\delta = \{x \in \Omega^c, d(x, \partial \Omega) < \delta\}\), analog of local BC.
A Simple illustration: linear PD with small deformation

E.g., force balance for a continuum of (linear/isotropic) Hookean springs:

\[ \mathcal{L}_\delta u_\delta(x) = \int_{\Omega \cup \Omega_\delta} \omega_\delta(|y - x|) \frac{y - x}{|y - x|} \left( \frac{y - x}{|y - x|} \cdot \frac{u_\delta(y) - u_\delta(x)}{|y - x|} \right) dy. \]

Spring constant direction Projected relative deformation (linear volumetric strain)

\( u_\delta(x) \): displacement at \( x \);
\( \delta \): nonlocal horizon;
\( \omega_\delta(|r|) \): nonlocal kernel, with support in \( B_\delta(0) \).

Nonlocal/volumetric constraint in \( \Omega_\delta = \{ x \in \Omega^c, d(x, \partial \Omega) < \delta \} \), analog of local BC.

1D scalar case:

\[ -\int_0^\delta \frac{u_\delta(x + s) - 2u_\delta(x) + u_\delta(x - s)}{s^2} \omega_\delta(|s|) ds = b(x) \rightarrow \ -u''(x) = b(x). \]
Application/Modeling  $\rightarrow$ Theory/Analysis
Reformulation via nonlocal vector calculus

In an attempt to rewrite \(-\mathcal{L}_\delta u_\delta = b\) as \(-\mathcal{D}(\omega_\delta \mathcal{D}^*)u_\delta = b\): we define

- \(\mathcal{D}^*(u)(x, y) = \frac{y - x}{|y - x|} \cdot \frac{u(y) - u(x)}{|y - x|}\) : linear nonlocal volumetric strain.

- \(\mathcal{D}\): dual/adjoint operator of \(\mathcal{D}^*\), \(\langle \mathcal{D}(\varphi), u \rangle = \langle \varphi, \mathcal{D}^*(u) \rangle\), \(\forall \varphi, u\).

\(\Rightarrow \mathcal{D}(\omega_\delta (\mathcal{D}^*(u)))(x) = \int \frac{\omega_\delta(|y - x|)}{|y - x|} \frac{y - x}{|y - x|} \mathcal{D}^*(u)(x, y) \, dy = \mathcal{L}_\delta u(x)\).
Reformulation via nonlocal vector calculus

In an attempt to rewrite \(-\mathcal{L}_\delta u_\delta = b\) as \(-\mathcal{D}(\omega_\delta \mathcal{D}^*)u_\delta = b\): we define

- \(\mathcal{D}^*(u)(x,y) = \frac{y-x}{|y-x|} \cdot \frac{u(y) - u(x)}{|y-x|}\) : linear nonlocal volumetric strain.
- \(\mathcal{D}\): dual/adjoint operator of \(\mathcal{D}^*, \langle \mathcal{D}(\phi), u \rangle = \langle \phi, \mathcal{D}^*(u) \rangle, \forall \phi, u.\)

\[\mathcal{D}(\omega_\delta (\mathcal{D}^*(u)))(x) = \int \omega_\delta \left(\frac{|y-x|}{|y-x|} \frac{y-x}{|y-x|}\right) \mathcal{D}^*(u)(x,y) \, dy = \mathcal{L}_\delta u(x).\]

Some examples of other basic nonlocal operators: for \(\tilde{\omega}_\delta(x) = \rho_\delta(|x|) \frac{x}{|x|},\)

- nonlocal gradient: \(\mathcal{G}_\delta p(x) = \int \tilde{\omega}_\delta (y-x) \frac{p(y) - p(x)}{|y-x|} \, dy,\)
- nonlocal divergence: \(\mathcal{D}_\delta u(x) = \int \tilde{\omega}_\delta (y-x) \cdot \frac{u(y) + u(x)}{|y-x|} \, dy.\)

\(\mathcal{D}^*, \mathcal{D}, \mathcal{G}_\delta, \mathcal{D}_\delta + \text{integral identities} \Rightarrow \text{nonlocal vector calculus}^5\)

---

^5 Du-Gunzburger-Lehoucq-Zhou 13, M3AS; 12, SIAM Rev; Mengesha-Du 13, 14, 15, 16
| Newton’s vector calculus                           | ⇔ | Nonlocal vector calculus\(^6\) |
| Local balance (PDE)                               | ⇔ | Nonlocal balance (PD) |
| Differential gradient/divergence                  | ⇔ | Nonlocal gradient/divergence |
| \(-\nabla \cdot (K \nabla u) = f\)                 | ⇔ | \(-\mathcal{D} \cdot (\omega_\delta \mathcal{D}^* u) = f\) |
| Boundary conditions                               | ⇔ | Volumetric constraints |
| Integral identities/integration by parts           | ⇔ | Nonlocal integration by parts |
| \(\int_\Omega u \Delta v - v \Delta u = \int_{\partial \Omega} u \partial_n v - v \partial_n u\) | ⇔ | \(\iint u \mathcal{D}(\mathcal{D}^* v) - v \mathcal{D}(\mathcal{D}^* u) = 0\) |

\(^6\)motivated by works on mechanics (Silling), image/topological-data (Gilboa-Osher, Smale et al), nonlocal space (Bourgain-Brezis-Mironescu, Ponce), fractional PDEs (Caffarelli-Silvestre), ...
Nonlocal vector calculus

|Newton’s vector calculus | \(\Leftrightarrow\) | Nonlocal vector calculus\(^6\) |
| Local balance (PDE) | \(\Leftrightarrow\) | Nonlocal balance (PD) |
| Differential gradient/divergence | \(\Leftrightarrow\) | Nonlocal gradient/divergence |
| \(-\nabla \cdot (K \nabla u) = f\) | \(\Leftrightarrow\) | \(-\mathcal{D} \cdot (\omega_\delta \mathcal{D}^* u) = f\) |
| Boundary conditions | \(\Leftrightarrow\) | Volumetric constraints |
| Integral identities/integration by parts | \(\Leftrightarrow\) | Nonlocal integration by parts |
| \(\int_\Omega u \Delta v - v \Delta u = \int_{\partial \Omega} u \partial_n v - v \partial_n u\) | \(\Leftrightarrow\) | \(\iint u \mathcal{D}(\mathcal{D}^* v) - v \mathcal{D}(\mathcal{D}^* u) = 0\) |

\(\Rightarrow\) characterization of nonlocal spaces/functionals/operators and their asymptotic limits;

\(\Rightarrow\) applications to nonlocal variational problems and nonlocal dynamics,…

**Systematic/axiomatic development of a new mathematical framework**

\(^6\)motivated by works on mechanics (Silling), image/topological-data (Gilboa-Osher, Smale et al), nonlocal space (Bourgain-Brezis-Mironescu, Ponce), fractional PDEs (Caffarelli-Silvestre), …
Well-posed nonlocal variational problems and their local limits

E.g. for $\omega_\delta \geq 0$: $\text{supp}(\omega_\delta) \subset (0, \delta)$, $\int \omega_\delta(|s|) \, ds = c_d$, $\lim_{\delta \to 0} \int_{|s| > \epsilon} \omega_\delta(|s|) \, ds = 0$, $\forall \epsilon > 0$.

Linear bond-based peridynamics $\rightarrow$ Linear Navier system of elasticity

Nonlocal space $S_\delta$ ($S_0 \subset S_\delta \subset L^2$) $\rightarrow$ Local space $S_0 = H^1_0(\Omega)$ (more regular)

$|u|_{S_\delta}^2 = \iint \omega_\delta(|x - y|) |\mathcal{D}^* u(x, y)|^2$ $\rightarrow$ $|u|_{S_0}^2 = 2|\text{Sym} \nabla u|_{L^2}^2 + |\text{div} u|_{L^2}^2$
Well-posed nonlocal variational problems and their local limits

E.g. for $\omega_\delta \geq 0$: $\text{supp}(\omega_\delta) \subset (0, \delta)$, $\int \omega_\delta(|s|) \, ds = c_d$, $\lim_{\delta \to 0} \int_{|s| > \epsilon} \omega_\delta(|s|) \, ds = 0$, $\forall \epsilon > 0$.

Linear bond-based peridynamics $\rightarrow$ Linear Navier system of elasticity
Nonlocal space $S_\delta$ ($S_0 \subset S_\delta \subset L^2$) $\rightarrow$ Local space $S_0 = H^1_0(\Omega)$ (more regular)

$|u|^2_{S_\delta} = \int \omega_\delta(|x - y|)|D^*u(x, y)|^2$ $\rightarrow$ $|u|^2_{S_0} = 2|\text{Sym}\nabla u|^2_{L^2} + |\text{div} u|^2_{L^2}

Nonlocal problem $u_\delta \in S_\delta \xrightarrow{L^2} u_0 \in S_0$
Local PDE limit

$\Omega$

$-\mathcal{L}_\delta u_\delta = b$ $\Omega_\delta$

Volumetric constraint $\leftarrow$ Well-posed with a unique solution $\rightarrow$
Boundary condition $\leftarrow$ Key: nonlocal Kohn-Poincare\(^7\)

\(^7\)Extending works of Bougain-Brezis-Mironescu, Ponce, ... to vector/tensor-fields/nonlocal-systems.
Well-posed nonlocal variational problems and their local limits

E.g. for $\omega_\delta \geq 0$: \( \text{supp}(\omega_\delta) \subset (0, \delta) \), \( \int \omega_\delta(|s|) \, ds = c_d \), \( \lim_{\delta \to 0} \int_{|s| > \epsilon} \omega_\delta(|s|) \, ds = 0 \), \( \forall \epsilon > 0 \).

Linear bond-based peridynamics $\rightarrow$ Linear Navier system of elasticity

Nonlocal space $S_\delta$ ($S_0 \subset S_\delta \subset L^2$) $\rightarrow$ Local space $S_0 = H_0^1(\Omega)$ (more regular)

$$|u|_{S_\delta}^2 = \int \omega_\delta(|x - y|) |D^* u(x, y)|^2$$

$\rightarrow$

$$|u|_{S_0}^2 = 2|\text{Sym} \nabla u|_{L^2}^2 + |\text{div} u|_{L^2}^2$$

Nonlocal problem $u_\delta \in S_\delta \xrightarrow{L^2} u_0 \in S_0$ $\rightarrow$ Local PDE limit

$\Omega$

$-\mathcal{L}_\delta u_\delta = b$

$\Omega_\delta$

$\Omega$

$-\mathcal{L}_0 u_0 = b$ $\partial \Omega$

$u_\delta = 0$

$\leftarrow$ Volumetric constraint $\rightarrow$ Boundary condition

$\leftarrow$ Well-posed with a unique solution $\rightarrow$

Key: nonlocal Kohn-Poincare$^7$

Main distinctions: systems (vectors/tensors), $\delta$-dependence, minimal regularity$^8$.

$^8$D-G-L-Z 13; Mengesha-Du 13, 14, 15, 16; X.Tian-Du 15, 16; X.Tian-Du 16; ...
Mathematics of nonlocal models

Insight/guidance for well-posed models and robust simulations

- **Mathematical foundation for nonlocal models like peridynamics**\(^9\):
  - linear bond-based and state-based peridynamics;
  - general volumetric constraints (nonlocal BC);
  - nonlinear peridynamics for hyperelastic materials;
  - crack propagation in prototype micro-elastic brittle materials,...

---

\(^9\)Mengesha-Du 14 JoE; 15, PRSE-A; 15, Nonlinearity; 16 NA; Du-Tao-X.Tian 17, JoE.
Mathematics of nonlocal models

Insight/guidance for well-posed models and robust simulations

- **Mathematical foundation for nonlocal models like peridynamics**:
  - linear bond-based and state-based peridynamics;
  - general volumetric constraints (nonlocal BC);
  - nonlinear peridynamics for hyperelastic materials;
  - crack propagation in prototype micro-elastic brittle materials,…

- **Remedy for popular but controversial practices**:
  - stability of peridynamic correspondence modeling;
  - stability of SPH Laplacian, local and nonlocal incompressible fields;
  - incompatible PD based simulations with local elasticity,…

---

\(^{10}\) Du-X. Tian 18, SIAM Appl Math; 18 arXiv. X.Tian-Du 13, 14, 15, SINUM.
Theory/Analysis $\rightarrow$ Algorithms/Simulations
To simulate peridynamics, a nonlocal model with a horizon $\delta$,

- various discretizations have been studied: particle method (popular), conforming Galerkin FE (gaining share), DG/nonconforming FE (with potential), adaptive/fast algorithms (in need);
To simulate peridynamics, a nonlocal model with a horizon \( \delta \),

- various discretizations have been studied: particle method (popular), conforming Galerkin FE (gaining share), DG/nonconforming FE (with potential), adaptive/fast algorithms (in need);

- public code development (pdlammps/peridigm) has been ongoing: PD allows less-regular solutions (thus more challenging to perform predictive simulations).

It is crucial to check and validate codes on benchmark problems that have been experimentally tested and solved by traditional means (see Bobaru-Yang-Alves-Silling-Askari-Xu, Chen-Gunzburger, X.Tian-Du, ...);
Numerical solution of nonlocal models

To simulate peridynamics, a nonlocal model with a horizon $\delta$,

- various discretizations have been studied: particle method (popular), conforming Galerkin FE (gaining share), DG/nonconforming FE (with potential), adaptive/fast algorithms (in need);

- public code development (pdlammps/peridigm) has been ongoing: PD allows less-regular solutions (thus more challenging to perform predictive simulations). It is crucial to check and validate codes on benchmark problems that have been experimentally tested and solved by traditional means (see Bobaru-Yang-Alves-Silling-Askari-Xu, Chen-Gunzburger, X.Tian-Du, ...);

- puzzling claims of numerical discrepancy have been reported: even with proven consistency between nonlocal linear PD and local linear elasticity (i.e., $u_\delta \rightarrow u_0$ as $\delta \rightarrow 0$), due to the lack of robustness in some computational algorithms w.r.t. parameters $\delta$ and $h$.
Robust numerical solution of nonlocal models

Robustness is essential to predictive simulations: algorithms sensitive to parameter changes may be undesirable and/or hard to verify/validate.

Asymptotically compatible (AC) discretization schemes (X.Tian-Du 13, 14, 15; Ph.D thesis of X.Tian 17): robust for PD w.r.t. changing $\delta$ and $h$.\(^\text{11}\)

\(^\text{11}\)Similar in spirit to AP schemes, Locking free schemes (Jin, Filbet, Degond, Carrillo, Bao, Lafitte, Guermond-Kanschat, Arnold-Brezzi, Girault-Raviart, ...), schemes for other parameterized problems, ...
Robust numerical solution of nonlocal models

Robustness is essential to predictive simulations: algorithms sensitive to parameter changes may be undesirable and/or hard to verify/validate.

**Asymptotically compatible (AC) discretization schemes (X.Tian-Du 13, 14, 15; Ph.D thesis of X.Tian 17):** robust for PD w.r.t. changing $\delta$ and $h$.\(^{11}\)

$$-\mathcal{L}_\delta u_\delta = f$$  
Continuum Nonlocal $h = 0$  

$$-\mathcal{L}^h_\delta u^h_\delta = f^h$$  
Discrete Nonlocal

$$-\mathcal{L}^0_\delta u^0_\delta = f^0$$  
Discrete Local $\delta = 0$

$$-\mathcal{L}^h_0 u^h_0 = f$$  
Continuum Local $h = 0$
Asymptotically compatible schemes

Surprisingly, some of the most popular schemes are not AC, e.g.,

- particle discretization with Riemann sum quadrature;
- finite element with piecewise constant functions.

Using \( h = 3 h_0 \) to simulate a 1-d benchmark (X. Tian-Du 2013).
Asymptotically compatible schemes

Surprisingly, some of the most popular schemes are not AC, e.g.,
- particle discretization with Riemann sum quadrature;
- finite element with piecewise constant functions.

Keeping the ratio $\frac{\delta}{h}$ unchanged (popular practice for sparsity/bandedness), such approximations may converge,

Using $\delta = 3h$ to simulate a 1-d benchmark (X. Tian-Du 13)
Asymptotically compatible schemes

Surprisingly, some of the most popular schemes are not AC, e.g.,

- particle discretization with Riemann sum quadrature;
- finite element with piecewise constant functions.

Keeping the ratio $\frac{\delta}{h}$ unchanged (popular practice for sparsity/bandedness), such approximations may converge, but to a wrong local limit! They over-estimate elastic constants by constant factors as $h \to 0$, thus incompatible to the correct local limit ($\delta = 0$).

Using $\delta = 3h$ to simulate a 1-d benchmark (X. Tian-Du 13)
Surprisingly, some of the **most popular** schemes are not AC, e.g.,

- particle discretization with Riemann sum quadrature;
- finite element with piecewise constant functions.

Keeping the ratio $\frac{\delta}{h}$ unchanged (popular practice for sparsity/bandedness), such approximations may **converge**, but to a **wrong** local limit!

They **over-estimate** elastic constants by constant factors as $h \to 0$, thus incompatible to the correct local limit ($\delta = 0$).
Consider a scalar linear nonlocal equation \(-\mathcal{L}_\delta u_\delta = b\) and its local limit \(-u'' = b\) with

\[-\mathcal{L}_\delta u_\delta (x) = -\frac{3}{\delta^3} \int_0^\delta (u(x + s) - 2u(x) + u(x - s)) \, ds \quad \text{in } (0,1) \quad \text{and} \quad u_\delta = 0 \quad \text{in } (-\delta,0) \cup (1,1+\delta).

Extension to linear high dimensional nonlocal scalar model: AC+DMP (discrete-maximum-principle) on a uniform mesh (Du-Tao-X.Tian-Yang 18, IMA Num Ana,)
Consider a scalar linear nonlocal equation $-\mathcal{L}_\delta \mathcal{u}_\delta = b$ and its local limit $-u'' = b$ with 

$$-\mathcal{L}_\delta \mathcal{u}_\delta (x) = \frac{3}{\delta^3} \int_0^\delta (u(x + s) - 2u(x) + u(x - s)) \, ds \quad \text{in} \ (0, 1) \quad \text{and} \quad u_\delta = 0 \quad \text{in} \ (\delta, 0) \cup (1, 1 + \delta).$$

Given $h=1/N=\delta/r$, $\{x_j = j \, h\}$. As $h \to 0$, the Riemann sum quadrature discretization

$$-\mathcal{L}_h^\delta \mathcal{u}_j = \frac{3h}{\delta^3} \sum_{m=1}^r (u_{j-m} - 2u_j + u_{j+m}) = b(x_j)$$

is convergent to the nonlocal equation with a fixed $\delta > 0$,
Consider a scalar linear nonlocal equation \(-\mathcal{L}_\delta u_\delta = b\) and its local limit \(-u'' = b\) with
\[
-\mathcal{L}_\delta u_\delta (x) = -\frac{3}{\delta^3} \int_0^\delta (u(x + s) - 2u(x) + u(x - s)) \, ds \quad \text{in (0, 1)} \quad \text{and} \quad u_\delta = 0 \quad \text{in (-\delta, 0) \cup (1, 1 + \delta)}.
\]
Given \(h = 1/N = \delta/r\), \(\{x_j = j \, h\}\). As \(h \to 0\), the Riemann sum quadrature discretization
\[
-\mathcal{L}_\delta^h u_j = -\frac{3h}{\delta^3} \sum_{m=1}^r (u_{j-m} - 2u_j + u_{j+m}) = b(x_j) \quad \text{is convergent to the nonlocal equation with a fixed} \ \delta > 0, \quad \text{but with} \ r \ \text{unchanged, it converges to} \ -(1 + \frac{3}{2r^2} + \frac{1}{2r^2})u'' = b \ \text{instead}!
\]
Consider a scalar linear nonlocal equation \(-L_\delta u_\delta = b\) and its local limit \(-u''=b\) with
\[-L_\delta u_\delta (x) = -\frac{3}{\delta^3} \int_0^\delta (u(x + s) - 2u(x) + u(x - s)) \, ds \quad \text{in } (0, 1) \quad \text{and } u_\delta = 0 \text{ in } (-\delta, 0) \cup (1, 1 + \delta).

Given \(h=1/N=\delta/r, \{x_j = j \, h\}\). As \(h \to 0\), the Riemann sum quadrature discretization
\[-L_\delta^h u_j = -\frac{3h}{\delta^3} \sum_{m=1}^r (u_{j-m} - 2u_j + u_{j+m}) = b(x_j) \quad \text{is convergent to the nonlocal equation with a fixed } \delta > 0, \quad \text{but with } r \text{ unchanged, it converges to } -(1 + \frac{3}{2r} + \frac{1}{2r^2})u''=b \text{ instead}!

As one possible remedy, a quadrature with modified weights leads to AC:
\[-L_\delta^{2,h} u_j = \frac{3h}{\delta^3} \sum_{m=1}^r \frac{m^3 - (m-1)^3}{3m^2} (u_{j-m} - 2u_j + u_{j+m}) = b(x_j).

Extension to linear high dimensional nonlocal scalar model: AC+DMP (discrete-maximum-principle) on a uniform mesh (Du-Tao-X.Tian-Yang 18, IMA Num Ana,)
For nonlocal systems possibly without DMP, AC conforming Galerkin schemes have been developed for general parametrized variational problems (X. Tian-Du 14, 15).

\[ \mathcal{T}_\sigma, \sigma \in [0, \infty] \}: \text{a family of Hilbert spaces}^{12}. \]

\[ \mathcal{T}_{\sigma_2} \text{ is a dense subspace of } \mathcal{T}_{\sigma_1}, 0 \leq \sigma_1 \leq \sigma_2 \leq \infty. \]

Symmetric bilinear form \( a_\sigma : \mathcal{T}_\sigma \times \mathcal{T}_\sigma \to \mathbb{R}. \)

Associated operator \( (A_\sigma u_\sigma, v) = a_\sigma(u_\sigma, v), \forall v \in \mathcal{T}_\sigma \)

---

\(^{12}\)For nonlocal model \( (\delta = 1/\sigma), \mathcal{T}_0 = L^2_0, \mathcal{T}_\infty = H^1_0, \mathcal{T}_* = C^\infty_0, \mathcal{T}_\sigma \): nonlocal space.
For nonlocal systems possibly without DMP, AC conforming Galerkin schemes have been developed for general parametrized variational problems (X. Tian-Du 14, 15).

\[ \{ T_\sigma, \sigma \in [0, \infty] \} : \text{a family of Hilbert spaces}^{12}. \]

\( T_{\sigma_2} \) is a dense subspace of \( T_{\sigma_1}, 0 \leq \sigma_1 \leq \sigma_2 \leq \infty. \)

Symmetric bilinear form \( a_\sigma : T_\sigma \times T_\sigma \rightarrow \mathbb{R}. \)

Associated operator \( (A_\sigma u_\sigma, v) = a_\sigma(u_\sigma, v), \forall v \in T_\sigma \)

**Parametrized linear variational problems:** find \( u_\sigma \in T_\sigma \) s.t.

\[ a_\sigma(u_\sigma, v) = (f, v), \forall v \in T_\sigma. \]

**Galerkin approximations:** find \( u^h \in W_{\sigma,h} \subset T_\sigma \) s.t.

\[ a_\sigma(u^h, v^h) = (A_h u^h, v^h) = (f, v^h), \forall v^h \in W_{\sigma,h}. \]

---

\(^{12}\) For nonlocal model \((\delta = 1/\sigma), T_0 = L_0^2, T_\infty = H_0^1, T_* = C_0^\infty, T_\sigma: \text{nonlocal space.}\)
Asymptotically compatible scheme

Properties: with positive constants $M_1$, $M_2$, $C_1$ and $C_2$ independent of $\sigma$.

About the spaces:

A-i) Uniform embedding: $M_1 \|u\|_{T_0} \leq \|u\|_{T_\sigma} \leq M_2 \|u\|_{T_\infty}, \forall u \in T_\sigma$.

A-ii) Asymptotically compact embedding: for any $\{\|u_n\|_{T_n}\}$ uniformly bounded $\Rightarrow \{u_n\}$ relatively compact in $T_0$ with limit points in $T_\infty$.

About the bilinear forms:

B-i) $a_\sigma$ is bounded: $a_\sigma(u, v) \leq C_2 \|u\|_{T_\sigma} \|v\|_{T_\sigma}, \forall u, v \in T_\sigma$.

B-ii) $a_\sigma$ is coercive: $a_\sigma(u, u) \geq C_1 \|u\|_{T_\sigma}^2, \forall u \in T_\sigma$.

About the operators:

C-i) $\exists$ a dense subspace $T_*$ of $T_\infty$ s.t. $A_\sigma u \in T_0, \forall u \in T_*$

C-ii) $A_\infty$ is the limit of $A_\sigma$ in $T_*$, i.e., $\lim_{\sigma \to \infty} \|A_\sigma u - A_\infty u\|_{T_{-\sigma}} = 0, \forall u \in T_*$.

For nonlocal problems, A-ii) is based on ideas of Bourgain-Brezis-Mironescu, Ponce, Mengesha-Du on nonlocal space characterization. B-ii) are uniform nonlocal Korn's/Poincare's inequalities based on A-i)-A-ii).
Asymptotically compatible scheme

Properties: with positive constants $M_1$, $M_2$, $C_1$ and $C_2$ independent of $\sigma$.

About the spaces:

A-i) Uniform embedding: $M_1\|u\|_{T_0} \leq \|u\|_{T\sigma} \leq M_2\|u\|_{T_\infty}$, $\forall u \in T\sigma$.

A-ii) Asymptotically compact embedding: for any $\{\|u_n\|_{T_n}\}$ uniformly bounded $\Rightarrow \{u_n\}$ relatively compact in $T_0$ with limit points in $T_\infty$.

About the bilinear forms:

B-i) $a_\sigma$ is bounded: $a_\sigma(u, v) \leq C_2\|u\|_{T\sigma}\|v\|_{T\sigma}$, $\forall u, v \in T\sigma$.

B-ii) $a_\sigma$ is coercive: $a_\sigma(u, u) \geq C_1\|u\|_{T\sigma}^2$, $\forall u \in T\sigma$.

About the operators:

C-i) $\exists$ a dense subspace $T_*$ of $T_\infty$ s.t. $A_\sigma u \in T_0$, $\forall u \in T_*$

C-ii) $A_\infty$ is the limit of $A_\sigma$ in $T_*$, i.e., $\lim_{\sigma \to \infty} \|A_\sigma u - A_\infty u\|_{T_{-\sigma}} = 0$, $\forall u \in T_*$.

* For nonlocal problems, A-ii) is based on ideas of Bourgain-Brezis-Mironescu, Ponce, Mengesha-Du on nonlocal space characterization. B-ii) are uniform nonlocal Korn’s/Poincare’s inequalities based on A-i)-A-ii).
Asymptotically compatible scheme

**More properties:** on approximation,

D-i) Given $\sigma \in (0, \infty)$, $\forall \nu \in \mathcal{T}_\sigma$, $\inf \{ \| \nu - \nu^h \|_{\mathcal{T}_\sigma} \lvert \nu^h \in \mathcal{W}_{\sigma,h} \} \to 0$ as $h \to 0$

D-ii) $\{ \mathcal{W}_{\sigma,h}, \sigma \in (0, \infty), h \in (0, h_0) \}$ is asymptotically dense in $\mathcal{T}_\infty$, i.e., $\forall \nu \in \mathcal{T}_\infty$,

$$\exists \{ \nu_n \in \mathcal{W}_{\sigma_n,h_n} \}_{\sigma_n \to \infty} \text{ as } n \to \infty, \text{ s.t. } \| \nu - \nu_n \|_{\mathcal{T}_\infty} \to 0.$$
Asymptotically compatible scheme

More properties: on approximation,

D-i) Given $\sigma \in (0, \infty)$, $\forall v \in \mathcal{T}_\sigma$, $\inf \{ \| v - v^h \|_{\mathcal{T}_\sigma} | v^h \in W_{\sigma, h} \} \to 0$ as $h \to 0$

D-ii) $\{ W_{\sigma, h}, \sigma \in (0, \infty), h \in (0, h_0) \}$ is asymptotically dense in $\mathcal{T}_\infty$, i.e., $\forall v \in \mathcal{T}_\infty$,

$$\exists \{ v_n \in W_{\sigma_n, h_n} \}_{n \to \infty}^{\sigma_n \to \infty} \text{ as } n \to \infty, \text{ s.t. } \| v - v_n \|_{\mathcal{T}_\infty} \to 0.$$

**Theorem:** X. Tian-Du 14 Properties A-B-C-D imply

1) $\lim_{\sigma \to \infty} \| u_\sigma - u_\infty \|_{\mathcal{T}_0} = 0$. (convergence of parametrized problems)

2) $\lim_{h \to 0} \| u_{\sigma, h} - u_\sigma \|_{\mathcal{T}_\sigma} = 0$. (convergence of approximations for a given $\sigma$)

3) $\lim_{\sigma \to \infty, h \to 0} \| u_{\sigma, h} - u_\infty \|_{\mathcal{T}_0} = 0$. (asymptotically compatible approximations)

4) $\lim_{\sigma \to \infty} \| u_{\sigma, h} - u_\infty, h \|_{\mathcal{T}_0} = 0$ for $W_{\infty, h} = \mathcal{T}_\infty \cap (\bigcap_{\sigma > 0} W_{\sigma, h})$

(discrete asymptotic convergence)

- For nonlocal problems, D-ii) requires good approximation of $\mathcal{T}_\infty = H^1_0$ by $W_{\sigma, h}$ as $\sigma \to \infty$, $h \to 0$. 


AC schemes for peridynamics

**Theorem (X.Tian-Du 14):** conforming Galerkin approximation schemes for nonlocal PD are AC, if the finite dimensional subspaces contain all $C^0$ piecewise-linear elements.

Discrete Nonlocal $\mathbf{u}_h$\(\rightarrow\mathbf{u}_0\)

Discrete Local $h = 0$

Continuum Nonlocal $h = 0$

Continuum PDE $\delta = h = 0$
AC schemes for peridynamics

**Theorem (X. Tian-Du 14):** conforming Galerkin approximation schemes for nonlocal PD are AC, if the finite dimensional subspaces contain all $C^0$ piecewise-linear elements.

Theory is applicable to linear bond/state PD systems in any space dimension, with uniform or unstructured mesh, in strong or weak forms.

This finding helps us design robust discretization of nonlocal models w.r.t changing parameters ($\delta$ and $h$), showing the **practical impact** of mathematical development.
Nonlocal Modeling, Analysis and Computation

Algorithms/Simulations $\rightarrow$ Modeling/Analysis
Nonlocal models allow singular solutions by retaining more physics and working with nonlocal interactions. Coupling with local models can make simulations more effective.
Effective nonlocal modeling and simulations

Nonlocal models allow singular solutions by retaining more physics and working with nonlocal interactions. Coupling with local models can make simulations more effective.

**Coupling strategies**\(^\text{13}\): force or energy based blending, coupling by optimization, quasinonlocal coupling (free from ghost forces), heterogeneous localization, ...
Effective nonlocal modeling and simulations

Nonlocal models allow singular solutions by retaining more physics and working with nonlocal interactions. Coupling with local models can make simulations more effective.

**Coupling strategies**: force or energy based blending, coupling by optimization, quasinonlocal coupling (free from ghost forces), heterogeneous localization, ...

Away from singularity, nonlocal models can in principle be localized \( \Rightarrow \) heterogeneous localization (X.Tian-Du 17, Du-Tao-X.Tian 18).

\[
-\Delta u = f \\
\delta = 0
\]

\[
\begin{align*}

\Omega_- & : u \in H^1(\Omega_-) \\
\Omega_+ & : u \in S(\Omega_+) \\
\Gamma & : \delta(x) = \text{dist}(x, \Gamma) \rightarrow 0 \text{ as } x \rightarrow \Gamma
\end{align*}
\]

A PDE (in \( \Omega_- \)) coupled with a nonlocal model (in \( \Omega_+ \)) via heterogeneous localization.
Coupling via heterogeneous localization

Coupled model: minimizing, s.t. a proper interface condition on $u$, the energy

$$\int_{\Omega^-} |\nabla u(x)|^2 dx + \int_{\Omega^+} \int_{\Omega^+} \gamma_\delta(x) \frac{|u(y) - u(x)|^2}{|y - x|^2} dy dx - (f, u)_{\Omega^- \cup \Omega^+}.$$ 

**Theorem** [X. Tian-Du 16, strengthening the classical trace inequality]

$\Omega$ bounded, simply connected, Lipschitz boundary $\Rightarrow$

$$\|u\|_{H^{\frac{1}{2}}(\partial \Omega)} \leq C_1 \|u\|_{S(\Omega)}, \forall u \in S(\Omega) \text{ and } \|u\|_{S(\Omega)} \leq C_2 \|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega).$$
Coupling via heterogeneous localization

Coupled model: minimizing, s.t. a proper interface condition on $u$, the energy

$$
\int_{\Omega_-} |\nabla u(x)|^2 dx + \int_{\Omega_+} \int_{\Omega_+} \gamma_\delta(x)(|y - x|) \frac{|u(y) - u(x)|^2}{|y - x|^2} dy dx - (f, u)_{\Omega_- \cup \Omega_+}.
$$

**Theorem** [X. Tian-Du 16, strengthening the classical trace inequality]

$\Omega$ bounded, simply connected, Lipschitz boundary $\Rightarrow$

$$
\|u\|_{H^\frac{1}{2}(\partial\Omega)} \leq C_1 \|u\|_{S(\Omega)}, \ \forall u \in S(\Omega) \ \text{and} \ \|u\|_{S(\Omega)} \leq C_2 \|u\|_{H^1(\Omega)}, \ \forall u \in H^1(\Omega).
$$

▶ More general/natural (imposing regularity only at $\partial\Omega$, not away)$^{14}$.

---

$^{14}$A follow-up question (Caffarelli): characterizing the largest space to afford such a trace inequality?
Coupling via heterogeneous localization

Coupled model: minimizing, s.t. a proper interface condition on $u$, the energy
\[
\int_{\Omega^-} |\nabla u(x)|^2 \, dx + \int_{\Omega^+} \int_{\Omega^+} \gamma_\delta(x)(|y - x|) \frac{|u(y) - u(x)|^2}{|y - x|^2} \, dy \, dx - (f, u)_{\Omega^- \cup \Omega^+}.
\]

**Theorem** [X. Tian-Du 16, strengthening the classical trace inequality]

$\Omega$ bounded, simply connected, Lipschitz boundary $\Rightarrow$

\[
\|u\|_{H^\frac{1}{2}(\partial \Omega)} \leq C_1 \|u\|_{S(\Omega)}, \, \forall u \in S(\Omega) \text{ and } \|u\|_{S(\Omega)} \leq C_2 \|u\|_{H^1(\Omega)}, \, \forall u \in H^1(\Omega).
\]

- More general/natural (imposing regularity only at $\partial \Omega$, not away)$^{14}$.
- Implication: heterogeneous localization $\Rightarrow$ well-posed BVPs with local Dirichlet data and coupled local-nonlocal models.
- AC schemes are robust w.r.t. heterogeneous localization $\Rightarrow$ a full loop: nonlocal modeling, application, analysis, computation!
Nonlocal modeling/analysis/computation

- Nonlocal phase field (Ginzburg-Landau/Allen-Cahn), PFC, DFT.

- Nonlocal conservation laws (Du-Kamm-Lehoucq-Parks 11).

- Nonlocal Stokes system (X.Tian-Du 18): aim at improving SPH.
Nonlocal modeling/analysis/computation

- Nonlocal phase field (Ginzburg-Landau/Allen-Cahn), PFC, DFT.

  Interplay between $\delta$ and $\epsilon$ (Yang-Du 16)

  \[ E(u) = \frac{\epsilon}{2} \iint \omega_\delta |D^* u|^2 + \frac{1}{4\epsilon} \int (u^2 - 1)^2. \]

  Reduction to local model: Landau expansion.

- Nonlocal conservation laws (Du-Kamm-Lehoucq-Parks 11).

  Retaining unwinding in nonlocal balance laws with entropy conditions encoded automatically (see Huang-Du-Lefloch 17)

  \[ \frac{\partial u}{\partial t} + \int_0^\delta \frac{g(u, \tau_h u) - g(\tau_{-h} u, u)}{h} \, d\omega_\delta(h) = 0 \overset{\delta \to 0}{\implies} \text{Entropy solution of } u_t + (f(u))_x = 0. \]

- Nonlocal Stokes system (X.Tian-Du 18): aim at improving SPH.
An invitation to nonlocal modeling

- Multiscale, multiphysics problems, rigorous derivation of nonlocal models
- Nonlocal inverse problems, design and control, data-driven modeling
An invitation to nonlocal modeling

- Multiscale, multiphysics problems, rigorous derivation of nonlocal models
- Nonlocal inverse problems, design and control, data-driven modeling
- Nonlocal exterior calculus and geometry
- Nonlocal models and stochastic processes and their localization
An invitation to nonlocal modeling

- Multiscale, multiphysics problems, rigorous derivation of nonlocal models
- Nonlocal inverse problems, design and control, data-driven modeling

- Nonlocal exterior calculus and geometry
- Nonlocal models and stochastic processes and their localization

- Nonlinear, non-convex problems, inhomogeneities and anisotropies
- Nonlocal space-time dynamics (non-Lipshitz forces, phase transformations)
- Nonlocal boundary conditions (systems, local approximations, regularity)
An invitation to nonlocal modeling

- Multiscale, multiphysics problems, rigorous derivation of nonlocal models
- **Nonlocal inverse problems**, design and control, data-driven modeling

- Nonlocal exterior calculus and geometry
- Nonlocal models and stochastic processes and their localization

- Nonlinear, non-convex problems, inhomogeneities and anisotropies
- Nonlocal space-time dynamics (non-Lipschitz forces, phase transformations)
- Nonlocal boundary conditions (systems, local approximations, regularity)

- Robust error estimates/control, adaptive mesh, time-steps and quadrature
- Domain decomposition, multilevel, sub-structuring, effective coupling
- Property preserving discretization (conservation, maximum principle, AC)
An invitation to nonlocal modeling

- Multiscale, multiphysics problems, rigorous derivation of nonlocal models
- **Nonlocal inverse problems**, design and control, data-driven modeling

- Nonlocal exterior calculus and geometry
- Nonlocal models and stochastic processes and their localization

- Nonlinear, non-convex problems, inhomogeneities and anisotropies
- Nonlocal space-time dynamics (non-Lipschitz forces, phase transformations)
- Nonlocal boundary conditions (systems, local approximations, regularity)

- Robust error estimates/control, adaptive mesh, time-steps and quadrature
- Domain decomposition, multilevel, sub-structuring, effective coupling
- Property preserving discretization (conservation, maximum principle, AC)

- Fracture mechanics, fluids, neural and network sciences, social dynamics
- Data/image modeling, kernel methods, deep learning
- ...

...
Nonlocal models are \textbf{generic} and effective near defects and singularities; they motivate new mathematical thinking and may lead to broad applications.
Nonlocal models are generic and effective near defects and singularities; they motivate new mathematical thinking and may lead to broad applications.

With increasing computing power, nonlocal models with finite range of interactions serve to bridge local/discrete/fractional models; much more are to be explored.

More discussions, "Nonlocal modeling, analysis and computation", CBMS monograph to be published by SIAM 2018, which contains contributions from many collaborators.
<table>
<thead>
<tr>
<th>Collaborators</th>
<th>Institutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Xiaochuan Tian</td>
<td>UT Austin</td>
</tr>
<tr>
<td>Max Gunzburger</td>
<td>FSU</td>
</tr>
<tr>
<td>Kun Zhou</td>
<td>PSU/WallSt</td>
</tr>
<tr>
<td>Jiang Yang</td>
<td>SUST</td>
</tr>
<tr>
<td>Yunzhe Tao</td>
<td>Columbia</td>
</tr>
<tr>
<td>Li Tian</td>
<td>PSU/Epic</td>
</tr>
<tr>
<td>Jianfang Lu</td>
<td>CSRC</td>
</tr>
<tr>
<td>Wei Zhang</td>
<td>HKBU-Zhuhai</td>
</tr>
<tr>
<td>Hao Tian</td>
<td>OUCHina</td>
</tr>
<tr>
<td>Tadele Mengesha</td>
<td>Tennessee</td>
</tr>
<tr>
<td>Richard Lehoucq</td>
<td>Sandia</td>
</tr>
<tr>
<td>Lili Ju</td>
<td>S.Carolina</td>
</tr>
<tr>
<td>Zhi Zhou</td>
<td>HKPoly</td>
</tr>
<tr>
<td>Hwi Lee</td>
<td>Columbia</td>
</tr>
<tr>
<td>Zhan Huang</td>
<td>PSU/Mellon</td>
</tr>
<tr>
<td>Xiao Li</td>
<td>CSRC</td>
</tr>
<tr>
<td>Xuying Zhao</td>
<td>AMSS,CAS</td>
</tr>
<tr>
<td>Hadrien Montanelli</td>
<td>Columbia</td>
</tr>
</tbody>
</table>
Collaborators:

- Michael Parks (Sandia)
- Pablo Seleson (Oak Ridge)
- Alex Tartakovsky (PNNL)
- Mark Meerschaert (MSU)
- Jiwei Zhang (CSRC)
- John Burkardt (FSU)
- Chunxiong Zheng (Tsinghua)
- Philippe LeFloch (Paris 6)
- An Chen (GuiLin)
- Helen Li (UNCC)
- Lorenzo Toniazzi (Warwick)
- Robert Lipton (LSU)
- Marta D'Elia (Sandia)
- Jack Kamm (Sandia)
- O. Defterili (MSU)
- Zhonghua Qiao (HKPU)
- Feifei Xu (UNC)
- Jiashun Hu (Tsinghua)
- Houde Han (Tsinghua)
- Changpin Li (ShanghaiU)
- Jianfeng Lu (Duke)
- Richard Slevinsky (Manitoba)

Thanks to all my collaborators, colleagues and team members, and support from NSF, Sandia, DOE, AFOSR, ARO, DARPA, CSRC and Columbia
Thanks to ICM, to Rio and Brazil

Thank you!  谢谢，  Obrigado