Limits of discrete structures

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Supported by the ERC grant: Limits of discrete structures

A far-reaching story in mathematics

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- In 1998 Gowers found an extension of Fourier analysis (called higher order Fourier analysis) that can be used to give explicit bounds in Szemerédi's theorem.

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- The hypergraph regularity method was developed around 2000 by Rödl, Skokan, Nagle, Schacht and Gowers (analytic approach to hypergraph regularity: Elek-Sz. 2007)

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- Higher order Fourier analysis led to the algebraic theory of nilspaces (Camarena-Sz.) rooted in works of Host and Kra. These objects generalize nilmanifolds and put them into a categorical setting. Nil-spaces can be used to formalize higher order Fourier analysis as an algebraic theory which studies morphisms between nilspaces. Nilspa-ces are also useful in ergodic theory (Candela-Sz.) and in topological dynamic (Gutman-Manners-Varju).

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- The limit language provides a clean and convenient way to formulate exact statements about asymptotic properties.

From discrete to continuous

Furstenberg's approach to Szemerédi's theorem is a structural limit theory. It can be used to study the asymptotic properties of subsets in integer intervals $\{1, 2, \ldots, N\}$ as N goes to infinity. Similarity can be defined through pattern frequency. Limit objects are expressed through dynamical systems (Furstenberg's correspondence principle). Asymptotic behavior becomes precise behavior in the limit. Algebraic and geometric structures such as compact abelian groups and nilmanifolds appear in this limit.

Ergodic theory is one of the first examples for an analytic theory that gives a powerful new point of view on discrete structures by expressing asymptotic properties in the frame of a limit theory.

From discrete to continuous

For a bounded measurable function $W : [0,1]^2 \to \mathbb{R}$ let

$$\|W\|_\square := \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} W \right|.$$

Let X be the space of two variable symmetric functions of the form $W : [0,1]^2 \to [0,1]$ and for two functions U, W let

$$\delta_\square(U,W) := \inf_{\psi:[0,1]\to[0,1]} \|U^\psi - W\|_\square$$

where ψ is measure preserving and $U^\psi(x,y) := U(\psi(x),\psi(y))$. Note that $\delta_\square$ is a pseudo metric. Let $U \sim W$ if $\delta_\square(U,W) = 0$.

Let G be a graph on the vertex set $V(G) = \{0,1,\ldots,n\}$. We can represent G by the graphon W_G defined by

$$W_G(x,y) = 1 \text{ if } (\lceil xn\rceil, \lceil yn\rceil) \in E(G) \text{ and } W_G(x,y) = 0 \text{ otherwise.}$$

Distance of graphs:

$$d(G,H) := \delta_\square(W_G, W_H)$$

From discrete to continuous

Theorem (Lovász, Sz. 2006): $(X/\sim, \delta_\square)$ is a compact space.

Compactness of $(X/\sim, \delta_\square)$ implies two famous results:

- Aldous-Hoover theorem on exchangeability (a result in probability theory)
- Szemerédi's regularity lemma even in the stronger form by Alon, Fischer, Krivelevich, Szegedy.

The space $X/\sim$ is the completion of the set of finite graphs in $\delta_\square$.

Graph limit theory

A "movement" started in the early 2000's:

Bounded degree case: Benjamini, Schramm

Reflection positivity: Freedman, Lovász, Schrijver

Dense graph limits: Chayes, Borgs, Lovász, Sós, Sz., Vesztergombi

Various extensions: Bollobás, Riordan

One of the main challenges in graph limit theory is that, depending on the number of edges, there are different useful limit notions. A graph is dense if the number of edges is "quadratic" in the number of vertices. Bounded degree graphs are at the other end of the spectrum. The number of edges is "linear" in the number of vertices. The hardest case is the sub quadratic and super linear case.

Some more limit theories

- Local-global limits for bounded degree graphs (Bollobás, Riordan, Hatami, Lovász, Sz.). This is a refinement of the Benjamini-Schramm limit. Local global limits together with information theory were successfully applied in the study of random regular graph (eigenvector Gaussianity: Backhausz, Sz.).
- Limits of hypergraphs, and the ultralimit method (Elek, Sz..)
- Limits of Permutations (Hoppen, Kohayakawa, Moreira, Ráth, Sampaio)
- Limits of subsets in abelian groups (Sz.)

Limit theory for additive (arithmetic) combinatorics?

The famous correspondence principle by Furstenberg allows us to view ergodic theory of $\mathbb{Z}$-actions as a limit theory for ordered 0-1 sequences.

Two such 0-1 sequences are considered to be similar if the frequencies of small consecutive patterns such as 001, 010, 111, etc... are similar

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Note that the Benjamini-Schramm limit theory of colored bounded degree graphs leads to a generalization of Furstenberg's framework. But there is a very different analytic way of thinking about additive combinatorics which is more similar to dense graph limits!

In 1998 Gowers gave a new proof of Szemerédi's theorem using the so-called Gowers uniformity norms for functions on abelian groups, denoted by $\|.\|_{U_k}$. For his method he coined the name **Higher order Fourier analysis**. A beautiful observation by Gowers is that if a function $f : A \to \mathbb{C}$ has the property that $\|f\|_{U_2}$ is separated enough form 0 then f correlates with a wave (a linear character) on A.

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What about $\|f\|_{U_k}$? What are higher order waves?

Gowers's work started a line of research to understand higher order waves. There are long and deep works by Green, Tao, Ziegler and an alternative approach by me which is also not simple. In my approach higher order waves are basically morphisms from abelian groups to nilmanifolds and more generally to nilspaces. Note that linear characters are homomorphisms to the circle which is the simplest nilmanifold.

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Tha Aldous-Hoover theorem is closely related to graph theory and graph limit theory. Is there an exchangeability problem which is related to additive and arithmetic combinatorics? YES! There are quite a few

Additive (arithmetic) exchangeability

Arithmetic exchangeability is related to the arithmetic groups!

Problem 1: Describe systems of random variables $\{X_v\}_{v \in \mathbb{Z}^\infty}$ that are symmetric under the induced action of $GL(\infty, \mathbb{Z})$.

Problem 2 (affine version): Describe systems of random variables $\{X_v\}_{v \in \mathbb{Z}^\infty}$ that are symmetric under the induced action of $\mathbb{Z}^\infty \rtimes GL(\infty, \mathbb{Z})$.

Problem 3 (affine cubic version): Describe systems of random variables $\{X_v\}_{v \in \{0,1\}^\infty_{\mathbf{0}}}$ such that for every $k \in \mathbb{N}$ the marginal distributions on affine subcubes are all the same.

Some explanation: relationship to additive combinatorics

Let A be a finite Abelian group. Let $S \subset A$ be a subset in A. The set $\hom(\mathbb{Z}^\infty, A)$ is a compact Abelian group so there is a Haar measure. Consequently we can talk about random homomorphisms from $\mathbb{Z}^\infty$ to A.

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For $v \in \mathbb{Z}^\infty$ let X_v be the random variable $(1_S \circ \psi)(v)$. Since $\hom(\mathbb{Z}^\infty, A)$ is symmetric under the action of $GL(\infty, \mathbb{Z})$ the system $\{X_v\}_{v \in \mathbb{Z}^\infty}$ is exchangeable. Similarly one can construct exchangeble systems satisfying problem 2) and problem 3).

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Method: Encode functions on Abelian groups into exchangeable systems. In the space of exchangeable systems use weak limits. The limit object is again an exchangeable system. However for proper limit theory a characterization of exchangeable systems is missing.

Good news!

We have a solution for problem 3 and it can be used for many applications.

Theorem (Candela-Sz. 2018) Every ergodic affine cubic exchangeable system comes from a so-called nilspace in a "similar way" as the above Abelian construction.

Remarks:

1) Nilspaces are slight generalizations of nilmanifolds. Interesting fact for probabilists: Nilmanifolds come up naturally in probability theory!

2) Ergodic measures form a closed set! (Bauer simplex... recall Bauer-Poulsen duality)

3) The proof is long (paper is 98 pages) but about 30 pages deals with some ground work in probability theory, it studies an abstract theory of couplings which can be of independent interest.

4) Difficulty: extract geometric structures (such as nilmanifolds) from a purely probabilistic situation. For this we use a certain topologization method which is inspired by one of my paper with Lovász "Regularity partitions and the topology of graphons"

Some background in ergodic theory: The Host-Kra seminorms and factors

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- For each Host-Kra seminorm $U(k)$ there is a σ-algebra $\mathcal{B}_k$ such that $\mathcal{B}_k$ is $\mathbb{Z}$-invariant (it gives a factor system) and furthermore $\mathbb{E}(f|\mathbb{B}_k) = 0$ if and only if $\|f\|_{U(k+1)} = 0$. We have that $U(k+1)$ is a norm on $L^\infty(\Omega, \mathcal{B}_k, \mu)$.

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- If $k = 1$ then $\mathcal{B}_1$ is the so-called Kronecker factor known from classical ergodic theory. These factors are known to have the structure of a compact abelian group with a given shift.
- The factor $\mathcal{B}_k$ turns out to be crucial for many problems in additive combinatorics. In particular it is responsible for the $k+2$ term arithmetic progressions.

The famous Host-Kra theorem gives a geometric characterization of the factors $\mathcal{B}_k$.

Theorem (Host-Kra): The measure preserving system on the factor $\mathcal{B}_k$ is the inverse limit of nil-systems.

Nil-systems: Let L be a nilpotent Lie group and let $\Gamma \leq L$ be a co-compact subgroup. Let N be the left-coset space of Γ. Note that N is a finite dimensional compact manifold. We have that L acts on N in a measure preserving way. A nil-system is the action of a cyclic subgroup of L on N (ergodicity is assumed).

It was proved in a paper by Bergelson, Tao and Ziegler that Host-Kra seminorms can be generalized to other Abelian actions and furthermore it was mentioned that these norms can even be generalized to nilpotent actions. They analized the case of $\mathbb{Z}_p^\infty$ actions and they obtained finite characteristic inverse theorems for Gowers norms using their results. Bergelson, Tao, Ziegler and also Host and Kra prepared a program to study the above factors for general nilpotent actions. Our next results contribute to this program in ergodic theory.

Theorem (Candela-Sz. 2018) Characteristic factors of the Host-Kra seminorms of countable nilpotent group actions are nilspace systems.

Theorem (Candela-Sz. 2018) Characteristic factors of the Host-Kra seminorms of finitely generated nilpotent actions are inverse limits of nilsystems.

These theorems generalize the celebrated Host-Kra, Ziegler results from $\mathbb{Z}$ to arbitrary nilpotent group actions.

Remark: The above theorem gives a new proof for the Host-Kra, Ziegler theorems, and this new proof is based more on probability theory than on ergodic theory.

Some technicality: In the paper we also prove a variant of the previous theorem. We characterize affine cubic exchangeable systems with a certain conditional independence property. We call them cubic couplings. Although it seems like a technical notion, it turns out to be more useful in applications.

Back to additive limit theory

A sequence of functions $f_i : A_i \to \mathbb{R}$ on compact abelian groups is convergent if the density of every fixed linear configuration converges. A linear configuration is given by an integer matrix $M \in \mathbb{Z}^{k \times n}$. Its density in a function $f : A \to \mathbb{R}$ is given by the formula:

$$\int_{x_1, x_2, \ldots, x_n \in A} \prod_{i=1}^{k} f(M_{i,1}x_1 + M_{i,2}x_2 + \cdots + M_{i,n}x_k).$$

For example the 3 term arithmetic progression density is given by

$$\int_{x_1, x_2 \in A} f(x_1)f(x_1 + x_2)f(x_1 + 2x_2)$$

Back to additive limit theory

There is a theory of complexity of linear forms developed by Gowers, Green, Tao, Wolf and others. It turns out that if we restrict the limit theory to a given complexity then reduced limit theories can be obtained that are simpler than the general limit theory. For example the complexity 1 case was completely described by a paper of Sz. Limit objects here are again functions on compact abelian groups.

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Higher complexity linear forms have a rather complicated limit theory. Limit theory restricted to configurations given by $0-1$ matrices in the general setting was developed by Sz. in 2010. Limit objects are functions on nilspaces. Our current results give a refinement of this limit theory: We can take limits of functions already defined on nilspaces.

An interesting open problem related to exchangeability

Let $G = SL(3, \mathbb{Z})$ and Let $\Omega = \{(a, b, c) \in \mathbb{Z}^3 : \gcd(a, b, c) = 1\}$. G acts on Ω.

What about (G, Ω)-exchangeability? Can we characterize G invariant probability measures on $\{0, 1\}^{\Omega}$?

Two things may help: 1) G has Kazdhan property-T $\Rightarrow$ ergodic measures are closed (Glasner-Weiss) 2) New results in arithmetic exchangeability suggest that such a characterization is not completely out of reach.

Invese therems for Gowers norms using exchangeability

Candela-Sz (2019?)

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Moreover we get a generalization for the inverse theorem to nilmanifolds: Gowers norms can be defined for functions on nilmanifolds and this allows us to generalize the concept of higher order Fourier analysis to them.