The method of hypergraph containers

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UIUC IMPA

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This talk is all based on joint work with:

Wojciech Samotij

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David Conlon and Tim Gowers (2016), and Mathias Schacht (2016)

- Resolved many longstanding extremal problems in random sets, and inspired us to consider general hypergraphs.

Containers: what are they good for?

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- counting sum-free and AP-free sets;
- a construction in discrete geometry.

Extremal graph theory

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- Much less is known for bipartite graphs, despite enormous effort.

We will be interested in a similar question in the setting of random graphs.

Mantel's theorem in $G(n,p)$

Definition

The extremal number of a graph H with respect to the Erdős-Rényi random graph $G(n,p)$ is defined to be

$$\text{ex}\big(G(n,p), H\big) := \max\big\{e(G) : G \subset G(n,p) \text{ and } H \not\subset G\big\}.$$

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We will sketch a proof of the following theorem.

Theorem (Frankl–Rödl, 1986; Haxell–Kohayakawa–Łuczak, 1996)

If $p \gg 1/\sqrt{n}$, then

$$\text{ex}\big(G(n,p),K_3\big) = \left(\frac{1}{2} + o(1)\right)p\binom{n}{2}$$

with high probability as $n \to \infty$.

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Lower bound: consider the intersection of $G(n, p)$ with a copy of $K_{n/2,n/2}$.

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Idea

Calculate the expected number of triangle-free subgraphs of $G(n,p)$ with

$$m = \left(\frac{1}{2} + \varepsilon\right) p \binom{n}{2}$$

edges.

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Calculate the expected number of triangle-free subgraphs of $G(n,p)$ *with*

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edges. Maybe it's small!

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Bad Idea

Calculate the expected number of triangle-free subgraphs of $G(n,p)$ *with*

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edges. But it's big, so this approach fails.

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Hand-wavy idea

If we can understand this clustering, perhaps we can ‘group’ the triangle-free graphs on n vertices into a relatively small number of ‘bunches’, and deal with a whole bunch in a single step!

The container theorem for triangle-free graphs

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To be more precise, we'd like to prove the following 'container' theorem:

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There is a trade-off between ‘small’ and ‘few’.

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For example, if we wish each $G \in \mathcal{G}$ to contain only $o(n^3)$ triangles,

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For each $n \in \mathbb{N}$, there exists a collection $\mathcal{G}$ of graphs on n vertices with the following properties:

(a) $|\mathcal{G}| \leqslant n^{O(n^{3/2})}$.

(b) *Each $G \in \mathcal{G}$ contains $o(n^3)$ triangles.*

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For example, if we wish each $G \in \mathcal{G}$ to contain only $o(n^3)$ triangles, then we need about $n^{O(n^{3/2})}$ graphs in order to cover all triangle-free graphs.

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The deduction uses an easy application of Chernoff's inequality, and a 'supersaturation' theorem, to control the containers.

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We use the following 'supersaturation' theorem to control the containers.

Supersaturation theorem for triangle-free graphs

A graph with $o(n^3)$ triangles has at most $n^2/4 + o(n^2)$ edges.

The container lemma for 3-uniform hypergraphs

The Hypergraph Container Lemma for 3-uniform hypergraphs

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For every $c > 0$*, there exists* $\delta > 0$ *such that the following holds.*

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For every $c > 0$, there exists $\delta > 0$ such that the following holds.
Let $\mathcal{H}$ be a 3-uniform hypergraph with average degree d, and such that

$$\Delta_1(\mathcal{H}) \leqslant c \cdot d \qquad \textit{and} \qquad \Delta_2(\mathcal{H}) \leqslant c \cdot \sqrt{d}.$$

$\Delta_1(\mathcal{H}) = \max\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$ is the max degree of a vertex in $\mathcal{H}$.

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Then there exists a collection $\mathcal{C}$ of subsets of $V(\mathcal{H})$, with

$$|\mathcal{C}| \leqslant \binom{v(\mathcal{H})}{\tau \cdot v(\mathcal{H})},$$

where $\tau := 1/\sqrt{d}$, such that

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(a) *For every $I \in \mathcal{I}(\mathcal{H})$ there exists $C \in \mathcal{C}$ such that $I \subset C$,*

We write $\mathcal{I}(\mathcal{H})$ for the collection of independent sets of $\mathcal{H}$.

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where $\tau := 1/\sqrt{d}$, such that

- (a) *For every $I \in \mathcal{I}(\mathcal{H})$ there exists $C \in \mathcal{C}$ such that $I \subset C$,*
- (b) $|C| \leqslant (1 - \delta)v(\mathcal{H})$ *for every $C \in \mathcal{C}$.*

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Applying the container lemma

Definition (The hypergraph encoding triangles)

The *hypergraph encoding triangles in* K_n is the 3-uniform hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = E(K_n)$, whose edges are the triangles of K_n.

Applying the container lemma

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The triangle-free graphs $G \subset K_n$ are exactly the independent sets of $\mathcal{H}$.

Applying the container lemma

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The triangle-free graphs $G \subset K_n$ are exactly the independent sets of $\mathcal{H}$.

Applying the container lemma to this hypergraph gives:

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There exists a collection $\mathcal{G}$ *of graphs on* n *vertices, with*

$$|\mathcal{G}| \leqslant n^{O(n^{3/2})},$$

such that

(a) *Every triangle-free graph is a subgraph of some* $G \in \mathcal{G}$,

(b) *Each* $G \in \mathcal{G}$ *has at most* $(1 - \delta)e(K_n)$ *edges.*

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B., M. and Samotij (2015); Saxton and Thomason (2015)

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$$\Delta_\ell(\mathcal{H}) \leqslant c \cdot \tau^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}$$

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for every $1 \leqslant \ell \leqslant k$. Then there exists a collection $\mathcal{C}$ of subsets of $V(\mathcal{H})$, and a function $f \colon \mathcal{P}(V(\mathcal{H})) \to \mathcal{C}$ such that:

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$$|S| \leqslant \tau \cdot v(\mathcal{H}) \qquad \text{and} \qquad I \subset f(S).$$

(b) $|C| \leqslant (1-\delta)v(\mathcal{H})$ *for every $C \in \mathcal{C}$.*

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- Erdős:
  Existence of a (special) point set in the plane without a large subset in general position.

The Cameron–Erdős problem: number of sum-free sets

Exercise

At most how many integers from $[n]$ can we select without creating a solution of

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- $\{n/2+1, n/2+2, \ldots, n\}$ *is sum-free.*

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- *Any subset of a sum-free set is sum-free.*

Cameron–Erdős problem: number of sum-free sets

Cameron–Erdős [1990]

Is the number of sum-free subsets of $[n]$ $\quad O(2^{n/2})$*?*

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Is the number of sum-free subsets of $[n]$ $O(2^{n/2})$*?*

Green [2004], Sapozhenko [2003]

There are constants c_e *and* c_o *s.t. the number of sum-free subsets of* $[n]$ *is*

$$(1 + o(1))c_e 2^{n/2}, \quad (1 + o(1))c_o 2^{n/2}$$

depending on the parity of n*.*

Cameron–Erdős: number of maximal sum-free sets

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Is there a $c > 0$ *that the number of* **maximal** *sum-free subsets of* $[n]$ *is*

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There are at least $2^{n/4}$ **maximal** *sum-free subsets of* $[n]$*.*

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Łuczak–Schoen (2001), Wolfowitz (2009)

$c = 2^{-28}$, $\quad c = 1/8 \quad$ *works.*

Cameron–Erdős: number of maximal sum-free sets

Balogh–H. Liu–Sharifzadeh–Treglown [2015+]

There are c_0, c_1, c_2, c_3 *s.t. the number of maximal sum-free subsets of* $[n]$ *is*

$$(c_i + o(1))2^{n/4},$$

where $i \equiv n \pmod 4$.

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- Delicate computations + Theorems of Green–Morris, Hujter–Tuza.

Arithmetic Progression-free sets

$$\mathsf{AP}(\mathbf{n}, \mathbf{k}) := \max\{|A| : \; A \subset [n] \text{ is } k\text{-AP-free}\}.$$

Szemerédi [1976]

$$\mathsf{AP}(n, k) \; = \; o(n).$$

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- Containers contain at most $n \cdot \log^{10} n$ k-APs!

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- Varnavides [1959]: Szemerédi's Theorem implies:

  For every a and k there exists b that if n is large, $A \subset [n]$, and $|A| > a \cdot n$ then A contains $b \cdot n^2$ different k-APs.

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- How smooth is $\text{AP}(n,k)$ for fixed k?

$(3,4)$-problem in Discrete Geometry

Erdős [1986, 1986, 1989]

Let S be a point set in the plane, without **four** *points in a line. Denote $\alpha(S)$ the size of the largest subset of S without* **three** *points in a line. Let*

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$\alpha(n) > \sqrt{n \log n}$

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[density Hales–Jewett Theorem, Fürstenberg–Katznelson 1989.]

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"True order of magnitude is much closer to the upper bounds,..., because it is very difficult to realize such system on the plane."

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Balogh–Solymosi [2018]

$$\alpha(n) \leqslant n^{5/6+o(1)}.$$

- **Hypergraph** approach:

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- **Lower bound** [Füredi]:

  Used lower bounds on independence number of these (geometric) hypergraphs.

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- $I \subset C_i \cap [m]^3_p \quad \rightarrow \quad |I| \leqslant m^{5/3}$.
- $n \approx pm^3 = m^2$ points that every $n^{5/6}$ of them has a collinear 3-tuple.

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- Using random subset of $[m]^3$

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  No four points on a line,
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- Lower bound is $n^{1/2}$; from this construction we expect $n^{3/4}$.

Balogh–Solymosi [2018]

Conjecture: $n^{3/4}$ *is best possible.*

$(3, 4)$-problem in Discrete Geometry

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- More creative ideas!!!