Extremal theory of vertex and edge ordered graphs

Gábor Tardos
Rényi Institute, Budapest
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Turán-type (classical) extremal graph theory

\[ \text{ex}(n,H) := \text{max number of edges in an} \]
- n vertex
- simple graph
- with no subgraph \( \cong H \)
Turán-type (classical) extremal graph theory

$\text{ex}(n,H) := \text{max number of edges in an}$
- $n$ vertex
- simple graph
- with no subgraph $\cong H$

Early examples:
- Mantel 1903 $\text{ex}(n,K_3)$
- Erdős 1938 $\text{ex}(n,C_4)$
- Turán 1941 $\text{ex}(n,K_k)$
Turán-type (classical) extremal graph theory

ex(n,H) := max number of edges in an
- n vertex
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Early examples:
- Mantel 1903 \( ex(n,K_3) \)
- Erdős 1938 \( ex(n,C_4) \)
- Turán 1941 \( ex(n,K_k) \)

General result: Erdős-Stone-Simonovits
\[
\text{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1}\right) \cdot \frac{n^2}{2} + o(n^2)
\]
satisfactory (gives asymptotics) for \( \chi(H) \geq 3 \)
weak (gives only \( o(n^2) \)) for bipartite \( H \)
several forbidden graphs \( \approx \) single worst forbidden graph
(unless it is bipartite)
(vertex) ordered graphs
(vertex) ordered graphs

a subgraph
(vertex) ordered graphs

a subgraph
(vertex) ordered graphs

a subgraph

YES, it is $P_5$, but with a particular vertex-order
extremal theory of ordered graphs

\[ \text{ex}(n, H_<) := \max \text{ number of edges in an} \]

- n vertex
- ordered graph
- with no subgraph
  order-isomorphic to \( H_< \)
examples: orderings of $P_4$

- $\text{ex}(n, \circlearrowright)$
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$\text{ex}(n, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots) = \text{ex}(n, P_4) = n \text{ or } n-1$
examples: orderings of $P_4$

- $\text{ex}(n, \quad \quad \quad \quad) = \text{ex}(n, K_4) = n^2/3 + O(1)$

- $\text{ex}(n, \quad \quad \quad \quad) = \text{ex}(n, K_3) + O(n) = n^2/4 + O(n)$

- $\text{ex}(n, \quad \quad \quad \quad) = \text{ex}(n, \quad \quad \quad \quad) = 2n-3$

- $\text{ex}(n, \quad \quad \quad \quad) = n \log n + O(n)$

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- $\text{ex}(n, \underbrace{\quad \quad \quad}_{n \text{ times}}) = \text{ex}(n, K_4) = \frac{n^2}{3} + O(1)$

- $\text{ex}(n, \underbrace{\quad \quad \quad}_{n \text{ times}}) = \text{ex}(n, K_3) + O(n) = \frac{n^2}{4} + O(n)$

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- $\text{ex}(n, \underbrace{\quad \quad \quad}_{n \text{ times}}) = n \log n + O(n)$

- $\text{ex}(n, \ldots, \ldots, \ldots, \ldots) = \text{ex}(n, P_4) = n \text{ or } n-1$
Related results

Extremal theory of 0-1 matrices ($\approx$ only consider ordered bipartite graphs)

- Füredi: The maximum number of unit distances in a convex n-gon, 1990
- Bienstock, Győri: An extremal problem on sparse 0-1 matrices, 1991
- Füredi, Hajnal: Davenport-Schinzel theory of matrices, 1992
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Extremal theory of **convex geometric graphs** (= cyclic order on the vertices)

- Braß, Károlyi, Valtr: A Turán-type extremal theory of convex geometric graphs, 2003
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Ramsey theory of ordered graphs

- Balko, Cibulka, Král, J. Kynčl: Ramsey numbers of ordered graphs, 2015
- Conlon, Fox, Lee, Sudakov: Ordered Ramsey numbers, 2017
ordered Erdős-Stone-Simonovits theorem

\[ \text{ex}(n, H_\prec) = (1 - \frac{1}{\chi(H_\prec) - 1}) \cdot \frac{n^2}{2} + o(n^2) \]

(also for several forbidden ordered graphs)
ordered Erdős-Stone-Simonovits theorem

(Pach – T.)

\[ \text{ex}(n, H_{\prec}) = \left(1 - \frac{1}{\chi(H_{\prec}) - 1}\right) \cdot \frac{n^2}{2} + o(n^2) \]

(also for several forbidden ordered graphs)

\[ \chi(H) = \text{minimal number of independent sets to cover the vertices of } H \]

\[ \chi(H_{\prec}) = \text{minimal number of independent intervals to cover the vertices of } H_{\prec} \]
ordered Erdős-Stone-Simonovits theorem 
(Pach – T.)

\[ \text{ex}(n, H_<) = \left(1 - \frac{1}{\chi(H_<) - 1}\right) \cdot \frac{n^2}{2} + o(n^2) \]

(also for several forbidden ordered graphs)

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ordered Erdős-Stone-Simonovits theorem

\((Pach – T.)\)

\[
ex(n, H_\prec) = (1 - \frac{1}{\chi(H_\prec) - 1}) \cdot \frac{n^2}{2} + o(n^2)
\]

(also for several forbidden ordered graphs)

satisfactory (gives asymptotics) for \(\chi(H_\prec) \geq 3\)
w\text{weak} (gives only} o(n^2)) for ordered bipartite \(H_\prec\)
Zoltán Füredi – Péter Hajnal’s conjecture:
\[ \text{ex}(n, H_{<}) \approx \text{ex}(n, H) \quad \text{if } H_{<} \text{ is bipartite} \]
trivial: \[ \text{ex}(n,H_{\prec}) \geq \text{ex}(n,H) \]

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False (Pach-T.):
\[
\text{ex}(n, H_{<}) = \Omega(n^{4/3})
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Exists ordered bipartite \( G_< \) with \( n \) vertices and \( \Omega(n^{4/3}) \) edges with no SUCH cycle of any length.
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Construction: \( n/2 \) points, \( n/2 \) “rich” lines, edge = incidence
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False (Pach-T.):
\[ \text{ex}(n, \text{G}_{<}) = \Omega(n^{4/3}) \]

Exists ordered bipartite \( G_{<} \) with \( n \) vertices and \( \Omega(n^{4/3}) \) edges with no \textbf{SUCH} cycle of any length.

Construction: \( n/2 \) points, \( n/2 \) “rich” lines, edge = incidence points ordered left to right, lines ordered by slope
How far is \( \text{ex}(n, H_<) \) from \( \text{ex}(n, H) \) for ordered bipartite \( H_\prec \)?

- \( H_\prec = \text{“special” ordered } C_{2k} \) (bipartite)
  \( \text{ex}(n, H_<) = \Omega(n^{4/3}) \)
- \( H = C_{2k} \) \( \text{ex}(n, H) = O(n^{1+1/k}) \)

\[
\frac{\text{ex}(n, H_<)}{\text{ex}(n, H)} = \Omega(n^{1/3-1/k})
\]
How far is $\text{ex}(n,H_\prec)$ from $\text{ex}(n,H)$ for ordered bipartite $H_\prec$?

- $H_\prec = \text{“special” ordered } C_{2k} \text{ (bipartite)}$
  
  $\text{ex}(n,H_\prec) = \Omega(n^{4/3})$

- $H = C_{2k}$
  
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$$\frac{\text{ex}(n,H_\prec)}{\text{ex}(n,H)} = \Omega(n^{1/3-1/k})$$

- No bipartite $H_\prec$ is known with

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\frac{\text{ex}(n,H_\prec)}{\text{ex}(n,H)} = \Omega(n^{1/3-1/k})
\]

- No bipartite $H_\prec$ is known with

\[
\frac{\text{ex}(n,H_\prec)}{\text{ex}(n,H)} = \Omega(n^{1/3})
\]

Can the ratio be as high as $n^{1-\varepsilon}$?
Füredi – Hajnal conjecture for forests:

\[ \text{ex}(n, H_{<}) \approx \text{ex}(n, H) = \Theta(n) \]

If true, characterizes ordered bipartite forests:

- \( H_{<} \) is not ordered bipartite: \( \text{ex}(n, H_{<}) = \Theta(n^2) \)
- \( H \) is not a forest: \( \text{ex}(n, H) \geq \text{ex}(n, H_{<}) \) \( \geq n^{1+\epsilon} \)
Füredi – Hajnal conjecture for forests:

$$\text{ex}(n, H_<) \approx \text{ex}(n, H) = \Theta(n)$$

- $\text{ex}(n, H_<) = o(n^{1+\varepsilon})$ for all $\varepsilon > 0$
- $\text{ex}(n, H_<) = n \log^{O(1)} n$
- $\text{ex}(n, H_<) = O(n \log n)$
Füredi – Hajnal conjecture for forests:

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If true, characterizes almost linear extremal functions:

- \( H_\prec \) is not ordered bipartite: \( \text{ex}(n,H_\prec) = \Theta(n^2) \)
- \( H \) is not a forest: \( \text{ex}(n,H_\prec) \geq \text{ex}(n,H) \geq n^{1+\varepsilon} \)
Füredi – Hajnal conjecture for forests:

$$\text{ex}(n,H_<) \approx \text{ex}(n,H) = \Theta(n)$$

$$\text{ex}(n,H_<) = o(n^{1+\varepsilon}) \text{ for all } \varepsilon > 0 \quad \text{OPEN}$$

$$\text{ex}(n,H_<) = n \log^{O(1)} n \quad \text{OPEN}$$

$$\text{ex}(n,H_<) = O(n \log n) \quad \text{FALSE}$$

If true, characterizes almost linear extremal functions:

- $H<$ is not ordered bipartite: $\text{ex}(n,H_<) = \Theta(n^2)$
- $H$ is not a forest: $\text{ex}(n,H_<) \geq \text{ex}(n,H) \geq n^{1+\varepsilon}$
Füredi – Hajnal conjecture for forests:

\[ \text{ex}(n, H_{<}) \approx \text{ex}(n, H) = \Theta(n) \]

\[ \text{ex}(n, H_{<}) = o(n^{1+\varepsilon}) \text{ for all } \varepsilon > 0 \quad \text{OPEN} \]

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\[ \text{ex}(n, H_{<}) = O(n \log n) \quad \text{FALSE} \]

Pettie: A bipartite tree \( H_{<} \) with

\[ \text{ex}(n, H_{<}) = \Omega(n \log n \log \log n) \]

Park and Shi: \( \text{ex}(n, H_{<}) = \Omega(n \log n \log \log n \log \log \log n \ldots) \)
Füredi – Hajnal conjecture for forests:

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Pach, T.: \( \text{ex}(n,H_\prec) = n \log^{O(1)} n \) for very small \( H_\prec \).
Füredi – Hajnal conjecture for forests:

\[ \text{ex}(n, H_{<}) \approx \text{ex}(n, H) = \Theta(n) \]

\[ \text{ex}(n, H_{<}) = o(n^{1+\varepsilon}) \text{ for all } \varepsilon > 0 \quad \text{OPEN} \]

\[ \text{ex}(n, H_{<}) = n \log^{O(1)} n \quad \text{OPEN} \]

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**Korándi, T., Tomon, Weidert**: \( \text{ex}(n, H_{<}) = o(n^{1+\varepsilon}) \) for degenerate \( H_{<} \).
Korándi, T., Tomon, Weidert:

$$\text{ex}(n,H_{\prec}) = o(n^{1+\varepsilon})$$ for **degenerate** $H_{\prec}$

Degenerate = adjacency matrix can be recursively split

$$
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}
$$
Korándi, T., Tomon, Weidert: \( \text{ex}(n,H_\prec) = o(n^{1+\varepsilon}) \) for degenerate \( H_\prec \)

Degenerate = adjacency matrix can be recursively split

Each split has at most 1 column with 1 above and below the split.
Korándi, T., Tomon, Weidert:
\( \text{ex}(n, H_\prec) = o(n^{1+\varepsilon}) \) for degenerate \( H_\prec \)

**Degenerate** = adjacency matrix can be recursively split

- Each split has at most 1 column with 1 above and below the split.

- Density increment argument
- Bound obtained:
  \[ \text{ex}(n, H_\prec) = 2^{\log^c n} \]
  
  \( c < 1 \) depending on
  
  # of sequential splits needed
Korándi, T., Tomon, Weidert: 

\[ \text{ex}(n, H_{\prec}) = o(n^{1+\varepsilon}) \text{ for degenerate } H_{\prec} \]

**Degenerate** = adjacency matrix can be recursively split

\[ \begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
\end{array} \]

Each split has at most 1 column with 1 above and below the split

\[ \text{ex}(n, H_{\prec}) = o(n^{1+\varepsilon}) \text{ is not known for ANY ordered graph } H_{\prec} \text{ that is NOT degenerate.} \]

\[ \begin{array}{cccc}
1 & . & 1 & . \\
. & 1 & . & 1 \\
. & . & 1 & . \\
1 & . & . & 1 \\
\end{array} \]
OPEN: What $H_<$ have linear extremal fn:

$$\text{ex}(n,H_>) = O(n)$$
OPEN: What $H_<$ have linear extremal fn:

$$\text{ex}(n,H_<) = O(n)$$

Marcus – T. (another Füredi – Hajnal conj.):
Bipartite matchings have linear extremal fn.
OPEN: What $H_<$ have linear extremal fn:

$$\text{ex}(n, H_<) = O(n)$$

Marcus – T. (another Füredi – Hajnal conj.):
Bipartite **matchings** have linear extremal fn.

Geneson – Keszegh:
Infinitely many minimal non-linear patterns.
Several forbidden patterns

\[ ex(n,G,H) \leq \min(ex(n,G), ex(n,H)) \quad \text{(trivially)} \]

G and H interact if

\[ ex(n,G,H) = o\left(\min(ex(n,G), ex(n,H))\right) \]

Does interaction exist?
Several forbidden patterns

\[ \text{ex}(n,G,H) \leq \min(\text{ex}(n,G),\text{ex}(n,H)) \] (trivially)

G and H interact if

\[ \text{ex}(n,G,H) = o(\min(\text{ex}(n,G),\text{ex}(n,H))) \]

unordered theory:

• not for \( \chi > 2 \)
• OPEN in general

(Faudree-Simonovits conjecture: YES)
Several forbidden patterns

\[ \text{ex}(n,G,H) \leq \min(\text{ex}(n,G),\text{ex}(n,H)) \]  \hspace{1cm} \text{(trivially)}

\( G \) and \( H \) interact if

\[ \text{ex}(n,G,H) = o(\min(\text{ex}(n,G),\text{ex}(n,H))) \]

Does interaction exist?

unordered theory:
- not for \( \chi > 2 \)
- OPEN in general
  (Faudree-Simonovits conjecture: YES)

ordered graphs:
- not for \( \chi > 2 \)
- LOTS of bipartite examples
  e.g.: short ordered paths
Edge ordered graphs

Simple graph
Edge ordered graphs

Simple graph + linear order on edges
Edge ordered graphs

Subgraph
Edge ordered graphs

Subgraph

4 1 7 9

= 2 4 3
Extremal theory of edge ordered graphs

Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer

Sporadic results

\[
\begin{align*}
\text{ex}(n, 1 & 3 & 2 ) & = 3(n-1)/2 \\
\text{ex}(n, 1 & 3 & 2 & 4 ) & = \Theta(n \log n) \\
\text{ex}(n, 1 & 3 & 4 & 2 ) & = \Omega(n \log n) \\
& = O(n \log^2 n) \\
\text{ex}(n, 2 & 4 & 1 & 3 ) & = \binom{n}{2} \\
\text{ex}(n, 1 & 2 & 3 & 4 ) & = \Omega(n^{3/2}) \\
& = O(n^{3/2} \log n)
\end{align*}
\]
Extremal theory of edge ordered graphs

Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer

General theory?
Extremal theory of edge ordered graphs

Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer

General theory?

\[ \begin{align*}
\text{ex}(n,H) & \geq \frac{n^2}{4} \text{ if } H \text{ is not in } \\
\text{lexicographicly } \text{ordered } K_{n,n} & = O(n^c) \text{ for some } c<2 \text{ otherwise }
\end{align*} \]
Extremal theory of edge ordered graphs

Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer

General theory?

\[ ex(n,H) \begin{cases} \geq n^2/4 & \text{if } H \text{ is not in lexicographically ordered } K_{n,n} \\ = O(n^c) & \text{for some } c<2 \text{ otherwise} \end{cases} \]

Tricky version of chromatic number makes Erdős-Stone-Simonovits theorem hold

\[ ex(n,H) = (1 - \frac{1}{\chi''(H)-1}) \frac{n^2}{2} + o(n^2) \]
Extremal theory of edge ordered graphs

Gerbner, Methuku, Nagy, Pálvölgyi, T., Vizer

General theory?

\[ \geq \frac{n^2}{4} \text{ if } H \text{ is not in } \text{lexicographic} \]

\[ \text{ex}(n,H) = \begin{cases} 
\text{lexicographically ordered } K_{n,n} \\
= O(n^c) \text{ for some } c < 2 \text{ otherwise}
\end{cases} \]

**Tricky version** of chromatic number makes Erdős-Stone-Simonovits theorem hold

\[ \text{ex}(n,H) = (1 - \frac{1}{\chi''(H) - 1}) \frac{n^2}{2} + o(n^2) \]

\( \chi'' \) is an integer or infinity
Ramsey for edge ordered graphs

∀ k ∈ n
∀ edge ordered $K_{n,n}$ contains lexicographic $K_{k,k}$

“Lexicographic is the only homogeneous order”

Similar result for edge ordered complete or complete multipartite graphs – but with several homogeneous structures.
Ramsey for edge ordered graphs

Balko, Vizer (2018++):
∀ edge ordered graph $S$
∃ edge ordered graph $T$
∀ 2-coloring of edges of $T$
∃ monochromatic subgraph of $T$
order isomorphic to $S$
Ramsey for edge ordered graphs

Balko, Vizer (2018++):
\[
\forall \text{ edge ordered graph } S \\
\exists \text{ edge ordered graph } T \\
\forall \text{ 2-coloring of edges of } T \\
\exists \text{ monochromatic subgraph of } T \text{ order isomorphic to } S
\]

Same statement
- Holds trivially for ordered graphs
- Fails for directed graphs