Hypergraph Matchings and Designs

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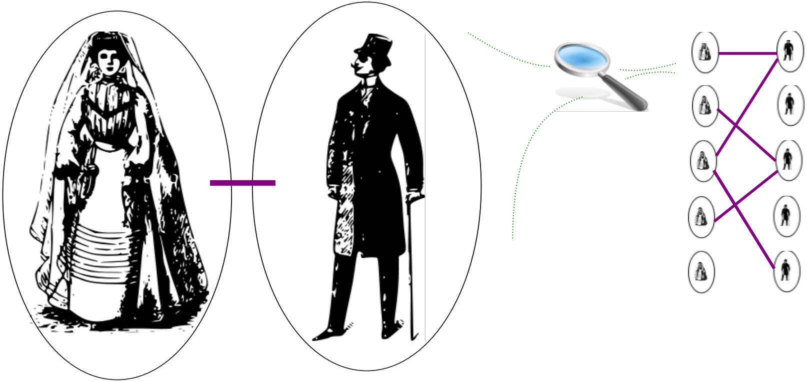
Introduction: the marriage problem



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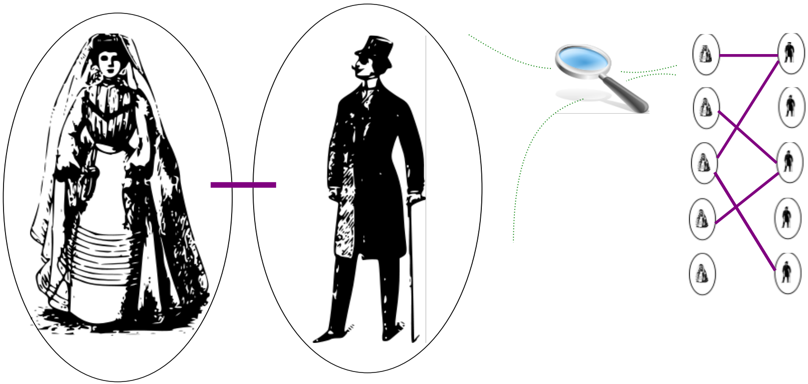

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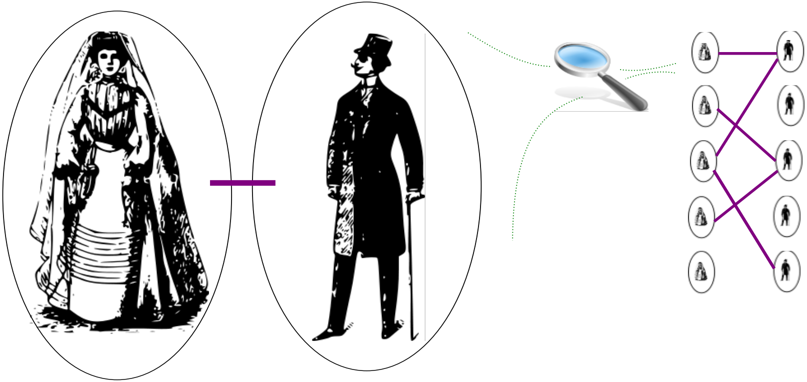

Can everyone be matched up? Is there a good solution? Is there a modern solution? Same-sex couples? Larger & more varied groups?

Matchings in graphs, existence theorems

Hall's Theorem

Let G be a bipartite graph with both parts X and Y of size n.
Then G has a perfect matching if and only if
$\forall\ S \subseteq X$, $\geq |S|$ vertices in Y adjacent to some vertex in S.

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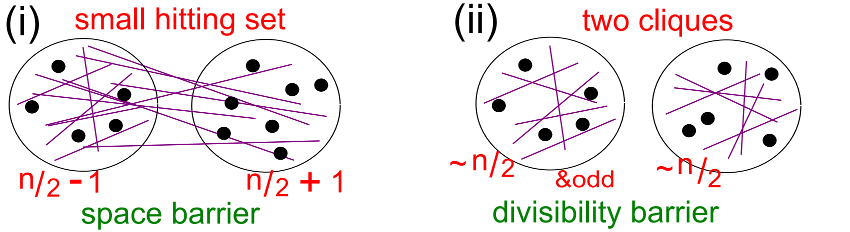

Hypergraph matching

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Hypergraph G: vertices $V(G)$, edges $E(G)$, $e \in E(G) \Rightarrow e \subseteq V(G)$.

r-graph G: every edge has size r.

Matching $M \subseteq E(G)$: every vertex is in ≤ 1 edge of M.

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For hypergraphs (even for 3-graphs) it is NP-complete.

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Theorem (K., Knox and Mycroft)

$\forall\ c > 1/k\ \exists$ polynomial time algorithm: given a k-graph G on $[n]$ with $\delta_{k-1}(G) \geq cn$, finds a perfect matching or a certificate that none exists.

Obstructions to perfect matching

Space barriers

A space barrier G is a k-graph on $[n]$ such that for some $S \subseteq [n]$ with $|S| < in/k$ we have $|e \cap S| \geq i$ for all $e \in E(G)$.
Any matching M has $|M| \leq |S|/i < n/k$, so is not perfect.

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Divisibility barriers

Let $(V_1, \ldots, V_d)$ be a partition of V and $L \subseteq \mathbb{Z}^d$ be a lattice. For $S \subseteq V$ define $i(S) \in \mathbb{Z}^d$ by $i(S)_j = |S \cap V_j|$. A divisibility barrier G is a k-graph on V such that $i(e) \in L$ for all $e \in E(G)$ but $i(V) \notin L$.

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(Example) Theorem (K., Knox and Mycroft)

If G is a 3-graph on $[n]$ with $\delta_2(G) > n/3 + o(n)$, $3 \mid n$, no PM then $\exists$ partition (A, B) of $[n]$ st $|A|$ odd, $|e \cap A|$ even for all $e \in E(G)$.

Perfect matchings in simplicial complexes

Definitions

k-complex J: j-graphs J_j, $0 \leq j \leq k$, closed under taking subsets.

Degree sequence: $\delta'(J) = (\delta'_0, \ldots, \delta'_{k-1})$:
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Theorem (K. and Mycroft)

Suppose J is an k-complex on n vertices with $k|n$, and
$\delta'(J) \ge (n, (k-1)n/k - o(n), \ldots, n/k - o(n))$ pointwise.
Then J_k has a perfect matching, unless all but $o(n^k)$ edges are contained in a space barrier or divisibility barrier.
(Each coordinate is best possible.)

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Applications: KKM structure theorem, codegree threshold for perfect K_4^3-packing (KM), partite Hajnal-Szemerédi theorem (KM)... etc?

Decompositions

When can object A be decomposed into copies of object B?

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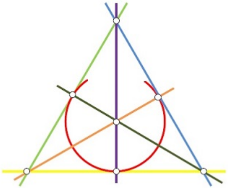
e.g. $A = E(K_n)$, $B = E(K_{q+1})$?

Fisher's $\leq$: need $n \geq q^2 + q + 1$.

Equality for projective planes.

Decomp = perfect matching in H:
$V(H) = A$, $E(H) = \{\text{copies of } B\}$.



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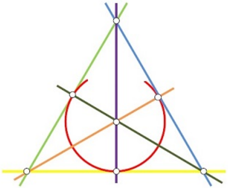

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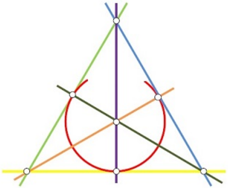

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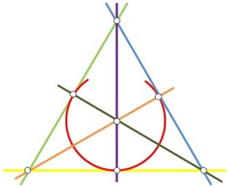

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Theorem (Kirkman 1848) 'Steiner Triple Systems'

K_n has a triangle decomposition iff n is 1 or 3 mod 6.

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Corollary (K.) / Conjecture (Wilson 1974)

If K_n is tridivisible then it has $(n/e^2 + o(n))^{n^2/6}$ triangle decompositions.

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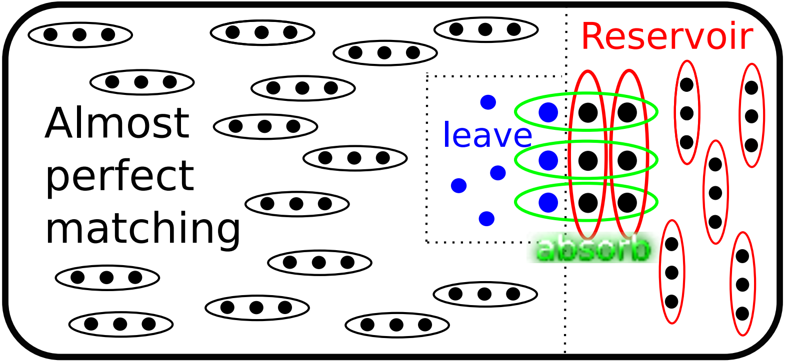
Theorem (Glock, Kühn, Lo, Osthus) (K. if $H = K_q^r$)

If G is H-divisible, 'dense' and 'typical' then G has an H-decomposition.

Absorbing Method



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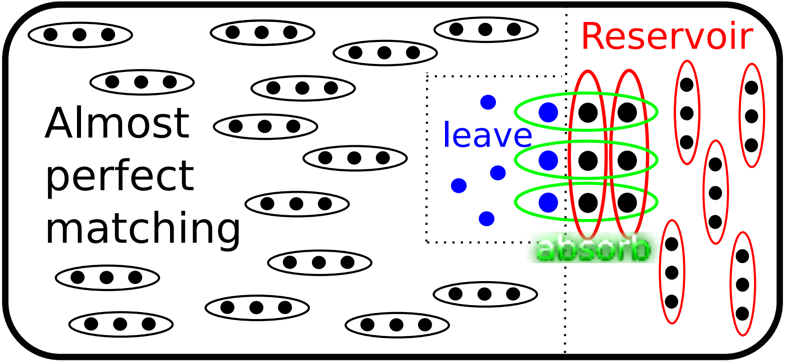

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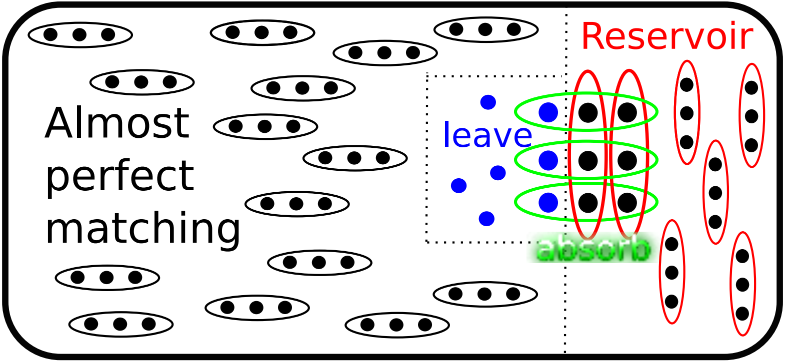

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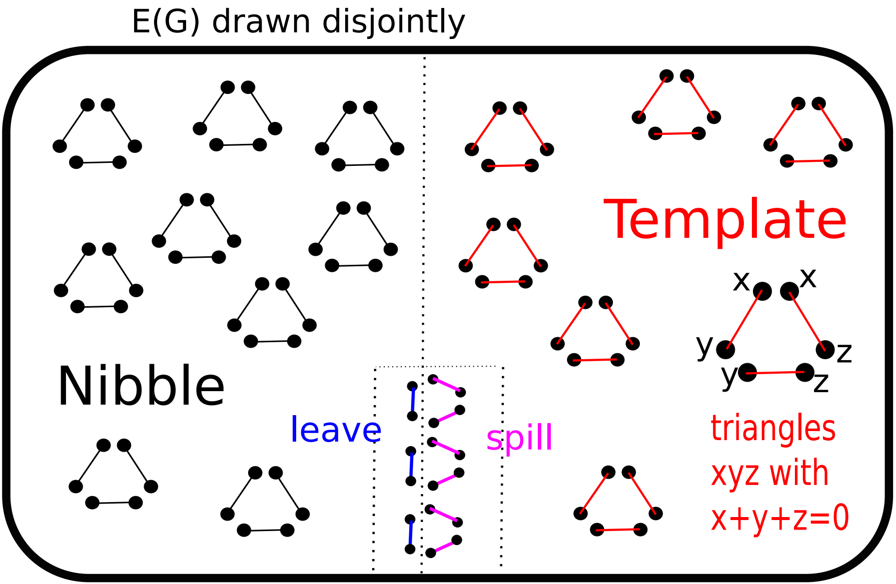
Iterative absorption (e.g. Glock-Kühn-Lo-Osthus designs): repeatedly reduce leave until each piece can use its 'private absorber'.

Randomised Algebraic Construction I



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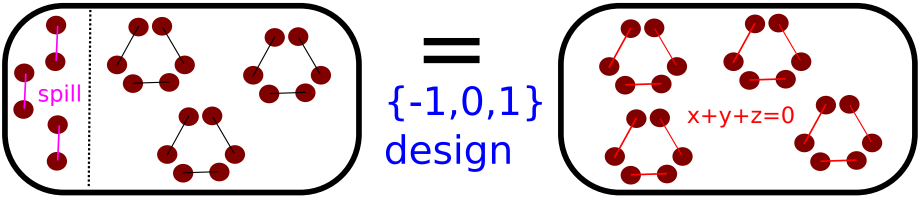
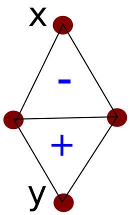
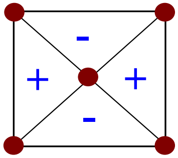
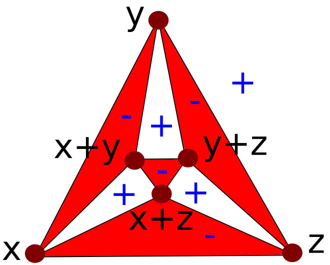




xy-transfer



alternating cycle



zero on edges

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- Hadamard Conj: $\exists\ H \in M_n(\pm 1)$: $H^t H = nI$ iff n is 1, 2 or div by 4?
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4. Perspectives.

- Extremal theory of designs/decomps: min degree thresholds?
- Probabilistic theory: (a) thresholds, e.g. for STS in $G^3(n,p)$?
  (b) random designs (e.g. STS) cf. theory of Random Graphs?