Sphere packing and Fourier interpolation

M. Viazovska

École Polytechnique Fédérale de Lausanne

ICM 2018

Let $\mathbb{R}^d$ be Euclidean vector space.
For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_{>0}$ we denote by $B_d(x, r)$ the ball in $\mathbb{R}^d$ with center x and radius r.

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Let $X \subset \mathbb{R}^d$ be a set of points such that $\|x - y\| \geq 2$ for any distinct $x, y \in X$. Then the union

$$\mathcal{P} = \bigcup_{x \in X} B_d(x, 1)$$

is a *sphere packing*.

Sphere packing constant

The *finite density* of a packing $\mathcal{P}$ is defined as

$$\Delta_{\mathcal{P}}(r) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0,r))}{\mathrm{Vol}(B_d(0,r))}, \quad r > 0.$$

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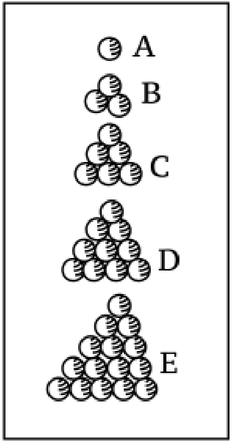
The *sphere packing constant* is the supremum over all possible packing densities

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

What is known about Δ_d?

$\Delta_1 = 1$ $\Delta_2 = \frac{\pi}{\sqrt{12}} \approx 0.9069$ $\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$



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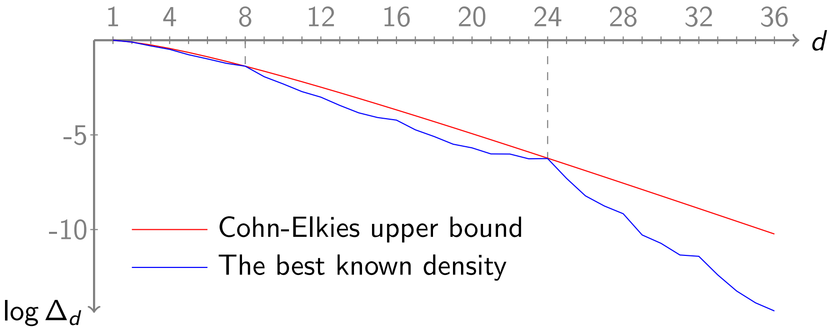

Dimension 8

The E_8-lattice $\Lambda_8 \subset \mathbb{R}^8$ is given by

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 | \sum_{i=1}^{8} x_i \equiv 0 \pmod 2\}.$$

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Theorem (V. 2016)

No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing. Therefore $\Delta_8 = \frac{\pi^4}{384} \approx 0.25367$.

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The Λ_{24}-lattice sphere packing is the packing of unit balls with centers at Λ_{24}.

Theorem (Cohn, Kumar, Miller, Radchenko, V. 2016)

No packing of unit balls in Euclidean space $\mathbb{R}^{24}$ has density greater than that of the Λ_{24}-lattice packing. Therefore $\Delta_{24} = \frac{\pi^{12}}{12!} \approx 0.00193$.

Linear programming bounds

Theorem (Cohn, Elkies 2003)

Suppose that $r_0 > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function such that:

$$f(x) \leq 0 \text{ for } \|x\| \geq r_0$$

$$\widehat{f}(x) \geq 0 \text{ for all } x \in \mathbb{R}^d$$

$$f(0) = \widehat{f}(0) = 1.$$

Then

$$\Delta_d \leq \mathrm{Vol}(B_d(0, r_0/2)).$$

Proof

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$$\geq \frac{\sharp(X/L)^2}{\mathrm{vol}(\mathbb{R}^d/L)} \cdot \widehat{f}(0).$$

Cohn-Elkies linear programming bounds

Theorem (Cohn, Elkies 2003)

$$\Delta_8 \leq 1.00016\, \Delta_{E_8},$$

$$\Delta_{24} \leq 1.019\, \Delta_{\Lambda_{24}}.$$

Solution of the Cohn-Elkies linear programming problem

Theorem (V 2016)

There exists a radial Schwartz function $f_{E_8} : \mathbb{R}^8 \to \mathbb{R}$ which satisfies:

$$f_{E_8}(x) \leq 0 \text{ for } \|x\| \geq \sqrt{2}$$
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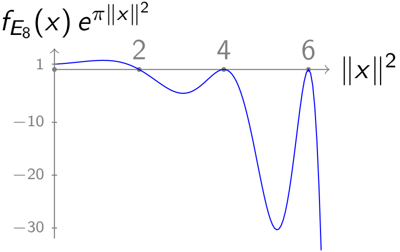
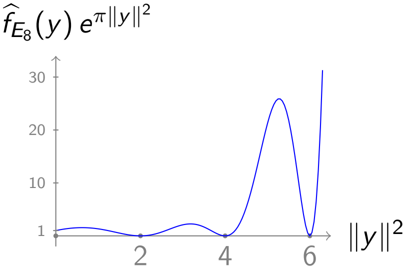
Plot of the "magic" function f_{E_8} and its Fourier transform $\widehat{f}_{E_8}$





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By the Poisson summation formula we have
$f_{E_8}(0) \geq \sum_{\ell\in\Lambda_8} f_{E_8}(\ell) = \sum_{\ell\in\Lambda_8} \widehat{f}_{E_8}(\ell) \geq \widehat{f}_{E_8}(0)$. This can happen only if
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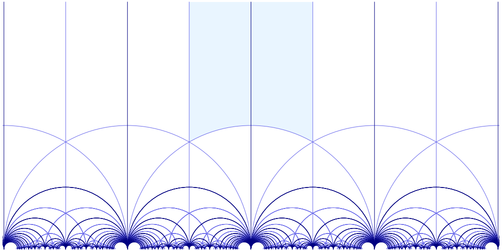
Remark 3

We have constructed the function f_{E_8} in the form
$f_{E_8}(r) = \sin(\pi r^2/2)^2 \int_0^\infty \varphi(it)\, e^{-\pi r^2 t} dt$ where φ is a holomorphic function on the upper half-plane.

Hidden symmetry

Let $\mathbb{H}$ be the upper half-plane $\{z \in \mathbb{C} | \Im(z) > 0\}$. Consider the modular group $\Gamma_1 := \mathrm{PSL}_2(\mathbb{Z})$. The group Γ_1 acts on $\mathbb{H}$ by linear fractional transformations

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z := \frac{az+b}{cz+d}.$$



Fourier interpolation

The idea behind our construction of f_{E_8} and $f_{\Lambda_{24}}$ is the hypothesis that a radial Schwartz function p can be uniquely reconstructed from the values $\{p(\sqrt{2n}), p'(\sqrt{2n}), \widehat{p}(\sqrt{2n}), \widehat{p}'(\sqrt{2n})\}_{n=0}^{\infty}$.

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The proof of this statement is a goal an ongoing project of the author in collaboration with H. Cohn, A. Kumar, S. D. Miller, and D. Radchenko.

New interpolation formula

Theorem (Radchenko, V. 2017)

There exists a collection of Schwartz functions $c_0, a_n \colon \mathbb{R} \to \mathbb{R}$ with the property that for any Schwartz function $p \colon \mathbb{R} \to \mathbb{R}$ and any $x \in \mathbb{R}$ we have

$$p(x) = c_0(x)\, p'(0) + \sum_{n \in \mathbb{Z}} a_n(x)\, p(\mathrm{sign}(n)\sqrt{|n|}) + \widehat{c}_0(x)\, \widehat{p}'(0) + \sum_{n \in \mathbb{Z}} \widehat{a}_n(x) \widehat{p}(\mathrm{sign}(n)\sqrt{|n|}),$$

where the right-hand side converges absolutely.

Interpolating basis functions

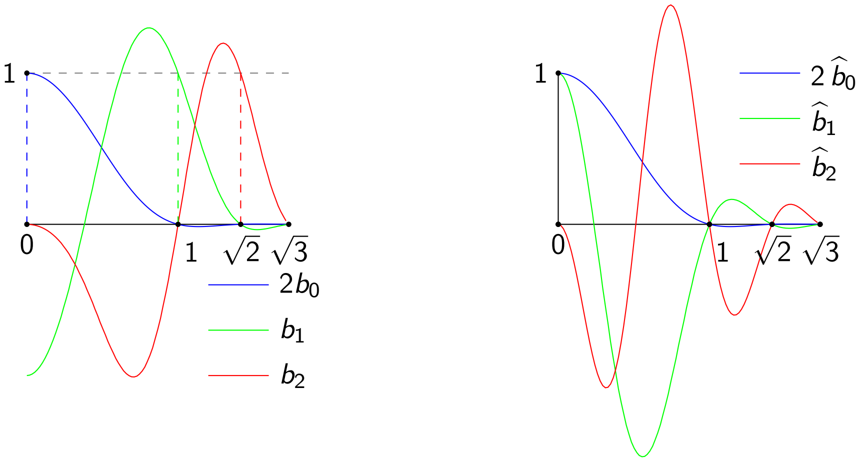

Figure: Plots of $b_n(x) := a_n(x) + a_n(-x)$ and $\widehat{b}_n$ for $n = 0, 1, 2$.

Crystalline measures

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Example

Dirac comb

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The interpolation formula implies that there exists a continuous family of *exotic* crystalline measures

$$\mu_x := \delta_x - \sum_{n=0}^{\infty} b_n(x)\, \delta_{\sqrt{n}}.$$

Proof of the interpolation formula: explicit construction of the interpolation basis

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For $x \in \mathbb{R}$ and $\tau \in \mathbb{H}$ we define

$$G(x,\tau) := \sum_{n=0}^{\infty} b_n(x)\, e^{\pi i n \tau}$$

$$\widetilde{G}(x,\tau) := \sum_{n=0}^{\infty} \widehat{b}_n(x)\, e^{\pi i n \tau}.$$

Functional equation

The interpolation formula applied to the Gaussian $e^{\pi i x^2 \tau}$ gives

$$e^{\pi i x^2 \tau} = G(x,\tau) + \frac{1}{\sqrt{-i\tau}}\widetilde{G}(x, \frac{-1}{\tau}).$$

We solve this functional equation explicitly using the Eichler cohomology and the theory of modular integrals.

Warning

⚠ Work in progress

- Interpolation for the Goursat problem
  Joint work with Andrew Bakan, Haakan Hedenmalm, Alfonso Montes-Rodriguez, Danilo Radchenko
- Universal optimality of E_8 and Λ_{24}
  Joint work with Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko

Interpolation for the Goursat problem

Goursat problem

Find a function $U : \mathbb{R}^2 \to \mathbb{R}$ such that

$$U_{xy} + U = 0, x, y \in \mathbb{R}; \quad U(x,0) = \varphi(x), U(0,y) = \psi(y), x, y \in \mathbb{R}, \tag{1}$$

for given $\varphi, \psi \in C(\mathbb{R})$ satisfying $\varphi(0) = \psi(0)$.

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$|U(x,y)| \leq C \exp(\,\theta(|x| + |y|)\,), \quad \theta \in [0,1).$

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Theorem(H. Hedenmalm, A. Montes-Rodriguez)

Suppose that $U(x,y) = \int_{\mathbb{R}} e^{ixt+iy/t} a(t)dt$ for some $a \in L_1(\mathbb{R})$. Then the equalities $U(\pi n, 0) = U(0, -\pi n) = 0$, $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ imply $U(x, -y) = 0$ for all $x, y \geq 0$ and therefore, such U is uniquely determined by its values at the points $\{(\pi n, 0)\}_{n \in \mathbb{N}_0}$ and $\{(0, -\pi n)\}_{n \in \mathbb{N}}$.

Interpolation for the Goursat problem

Let U be a real-valued continuous solution of the canonical telegraph PDE

$$U_{xy} + U = 0 \ , \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}_- \ ,$$

satisfying

$$|U(x, y)| \leq C\, e^{\theta\,(\,|x|+|y|\,)} \ , \ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_- \ ,$$

for some $\theta \in [0, 1)$ and $C \in (0, \infty)$. Suppose that $U(x, y) = \int_{\mathbb{R}} e^{ixt+iy/t} a(t)dt$ for some $a \in L_1(\mathbb{R})$.

Theorem(Bakan, Hedenmalm, Montes-Rodriguez, Radchenko, V. 2018+)

1. There exist continuous solutions $\{R_n\}_{n\geq 0}$ of the canonical telegraph PDE such that each R_n is uniformly bounded on $\mathbb{R}_+ \times \mathbb{R}_-$,

$$\begin{aligned} &(a) \quad R_0(\pi m, 0) = \delta_{0m}\ , \quad R_0(0, -\pi m) = \delta_{0m}\ , \quad m \geq 0\ , \\ &(b) \quad R_n(\pi m, 0) = \delta_{nm}\ , \quad R_n(0, -\pi m) = 0\ , \quad m \geq 0\ , \quad n \geq 1\ , \end{aligned}$$

and the functions $R_n(t, -y)$, $R_n(x, -t)$ of t belong to $S(\mathbb{R}_+)$ for every $n \geq 0$ and $x, y \geq 0$.

2. If $\sum_{n\geq 1} \sqrt{n}\,(\,|U(\pi n, 0)| + |U(0, -\pi n)|\,) < +\infty$ then for arbitrary $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_-$ we have the following absolutely convergent expansion

$$U(x, y) = U(0, 0)\, R_0(x, y) + \sum_{n\geq 1} \Big[\, U(\pi n, 0)\, R_n(x, y) + U(0, -\pi n)\, R_n(-y, -x) \,\Big].$$

Potential energy

Given a *potential function* $p\colon (0,\infty) \to \mathbb{R}$, we define the *potential energy* of a finite subset $\mathcal{C}$ of $\mathbb{R}^d$ to be

$$\frac{1}{|\mathcal{C}|} \sum_{\substack{x,y \in \mathcal{C} \\ x \neq y}} p\big(|x-y|\big).$$

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Let $\mathcal{C}$ be a discrete, closed subset of $\mathbb{R}^d$. We say $\mathcal{C}$ has *density* ρ if

$$\lim_{r\to\infty} \frac{\big|\mathcal{C}\cap B_d(0,r)\big|}{\mathrm{vol}\big(B_d(0,r)\big)} = \rho.$$

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The *lower p-energy* of a nonempty, discrete, closed subset $\mathcal{C}$ of $\mathbb{R}^d$ is

$$E_p(\mathcal{C}) := \liminf_{r\to\infty} \frac{1}{\big|\mathcal{C} \cap B_d(0,r)\big|} \sum_{\substack{x,y \in \mathcal{C}\cap B_d(0,r) \\ x \neq y}} p\big(|x-y|\big).$$

If the limit of the above quantity exists then we call $E_p(\mathcal{C})$ the *p-energy* of $\mathcal{C}$.

Potential energy minimization

Let $\mathcal{C}$ be a discrete subset of $\mathbb{R}^d$ with density ρ, where $\rho > 0$, and let $p\colon (0, \infty) \to \mathbb{R}$ be any function. We say that $\mathcal{C}$ *minimizes energy for p* if its p-energy $E_p(\mathcal{C})$ exists and every configuration in $\mathbb{R}^d$ of density ρ has lower p-energy at least $E_p(\mathcal{C})$. We also call $\mathcal{C}$ a *ground state* for p.

Universal optimality

Let $\mathcal{C}$ be a discrete subset of $\mathbb{R}^d$ with density ρ, where $\rho > 0$. We say $\mathcal{C}$ is *universally optimal* if it minimizes p-energy whenever $p\colon (0,\infty) \to \mathbb{R}$ is a completely monotonic function of squared distance.

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Theorem (Cohn, Kumar)

The lattice $\mathbb{Z}$ is universally optimal.

Universal optimality

Let $\mathcal{C}$ be a discrete subset of $\mathbb{R}^d$ with density ρ, where $\rho > 0$. We say $\mathcal{C}$ is *universally optimal* if it minimizes p-energy whenever $p\colon (0,\infty) \to \mathbb{R}$ is a completely monotonic function of squared distance.

Conjecture(Cohn, Kumar)

The lattices $\mathbb{Z}, A_2, \Lambda_8$ and Λ_{24} are universally optimal.

Theorem (Cohn, Kumar)

The lattice $\mathbb{Z}$ is universally optimal.

Theorem (Cohn, Kumar, Miller, Radchenko, V 2018+)

The lattices Λ_8 and Λ_{24} are universally optimal.

Idea of the proof: linear programming

Theorem(Cohn, Kumar)

Let $p\colon (0,\infty) \to \mathbb{R}$ be any function, and suppose $f: \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function. If $f(x) \le p\big(|x|\big)$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\widehat{f}(y) \ge 0$ for all $y \in \mathbb{R}^d$, then every subset of $\mathbb{R}^d$ with density ρ has lower p-energy at least $\rho\widehat{f}(0) - f(0)$.

Fourier interpolation of second order

Theorem (2018+)

Let (d, n_0) be $(8,1)$ or $(24,2)$. There exists a collection of radial Schwartz functions $a_n, b_n, \widetilde{a}_n, \widetilde{b}_n : \mathbb{R}^d \to \mathbb{R}$ such that for every $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$f(x) = \sum_{n=n_0}^{\infty} f(\sqrt{2n})\, a_n(x) + \sum_{n=n_0}^{\infty} f'(\sqrt{2n})\, b_n(x) + \sum_{n=n_0}^{\infty} \widehat{f}(\sqrt{2n})\, \widetilde{a}_n(x) + \sum_{n=n_0}^{\infty} \widehat{f}'(\sqrt{2n})\, \widetilde{b}_n(x),$$

and these series converge absolutely.

Construction of "Magic functions"

Let $p : (0, \infty) \to \mathbb{R}$ be a strictly monotonic potential function. The only possible "magic" function f that could prove a sharp bound for E_8 or the Leech lattice under a potential p:

$$f(x) = \sum_{n=n_0}^{\infty} p(\sqrt{2n})\, a_n(x) + \sum_{n=n_0}^{\infty} p'(\sqrt{2n})\, b_n(x). \tag{2}$$

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In order to prove that E_8 or the Leech lattice minimize the p-energy, it suffices to show that $f(x)\leq p(|x|)$ for all $x\in\mathbb{R}^d\setminus\{0\}$ and $\widehat{f}(y)\geq 0$ for all $y\in\mathbb{R}^d$.

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If a configuration is a ground state for every Gaussian $r \mapsto e^{-\alpha r^2}$, then the same is true for every completely monotonic function of squared distance.

Functional equation for the "magic function"

Consider the generating functions

$$F(\tau,x)=\sum_{n\geq n_0} a_n(x)\,e^{2\pi i n\tau}+2\pi i\tau\sum_{n\geq n_0}\sqrt{2n}\,b_n(x)\,e^{2\pi i n\tau}$$

and

$$\widetilde{F}(\tau,x)=\sum_{n\geq n_0} \widetilde{a}_n(x)\,e^{2\pi i n\tau}+2\pi i\tau\sum_{n\geq n_0}\sqrt{2n}\,\widetilde{b}_n(x)\,e^{2\pi i n\tau},$$

The interpolation formula for the complex Gaussian $x\mapsto e^{\pi i\tau|x|^2}$ is equivalent to

$$F(\tau,x)+(i/\tau)^{d/2}\widetilde{F}(-1/\tau,x)=e^{\pi i\tau|x|^2}.$$

Solving the functional equation

Using the methods developed in the theory of automorhic forms and by improving these techniques we can *explicitly* solve the functional equation

$$F(\tau+2,x)-2F(\tau+1,x)+F(\tau,x)=0$$
$$\widetilde{F}(\tau+2,x)-2\widetilde{F}(\tau+1,x)+\widetilde{F}(\tau,x)=0$$
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Moderate growth of F implies the interpolation formula

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Moderate growth of F implies the interpolation formula

The inequality $F(it,x)>0$ for $t\in(0,\infty)$ implies the universal optimality of Λ_8 and Λ_{24}.

Thank you for your attention.

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Please ask questions.