Phase transitions of Random Constraint Satisfaction Problems

Allan Sly, Princeton University

ICM, August 2018

Introduction:
random constraint satisfaction problems;

Combinatorics and Theoretical Computer Science

Constraint satisfaction problem (CSP): is it possible to assign values to a set of *variables* to satisfy a given set of *constraints*?

- Scheduling your appointments for the day
- System of linear equations.
- Colouring a graph or finding a large independent set.
- Satisfying a Boolean formula.

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Random CSPs

Our focus is to investigate properties when the constraints are chosen randomly.

Combinatorial properties of Random Graphs:

- Erdős-Rényi Random Graph: $G(n, \alpha/n)$ with n vertices and edges with probability α/n (average degree α).
- Random α-regular graph: Uniformly chosen from α-regular graphs on n vertices.

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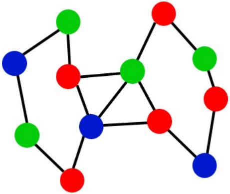
When is there a random proper k-colouring?



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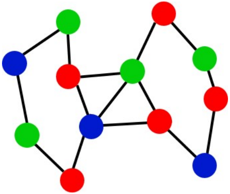

Is there an independent set of size βn?

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Variables: $x_1, \ldots, x_n \in \{\text{TRUE}, \text{FALSE}\} \equiv \{+, -\}$

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Example: A 3-SAT formula with 4 clauses:

$$\mathscr{G}(\underline{x}) = (+x_1 \text{ OR } +x_2 \text{ OR } -x_3) \text{ AND } \overbrace{(+x_3 \text{ OR } +x_4 \text{ OR } -x_5)}^{\text{clause}}$$
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clause $a \in F$, variable $v \in V$:
blue edge *(av)* if $+x_v$ in clause a
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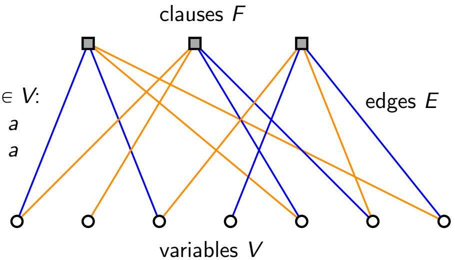

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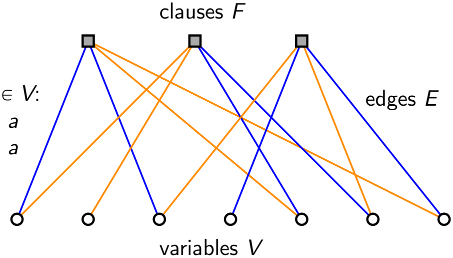

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The resulting random graph is locally tree-like, almost no short cycles and it's local distribution can be described completely.

Main Question:

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Other Question:

- *Free Energy*: How many solutions are there?
- *Local Statistics*: Properties of solutions such as how many clauses are satisfied only once?
- *Algorithmic*: Can solutions be found efficiently?

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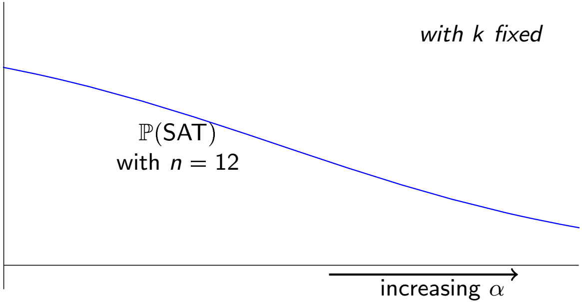

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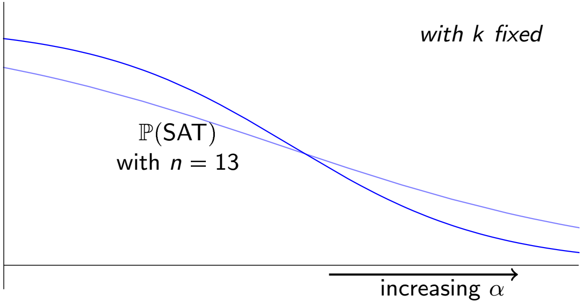

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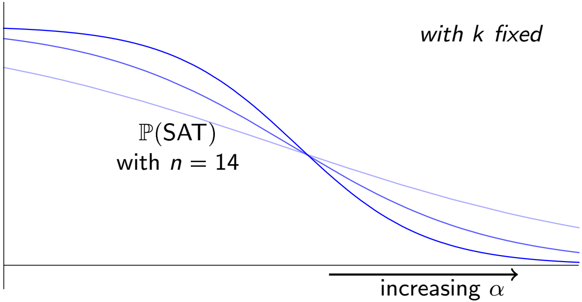

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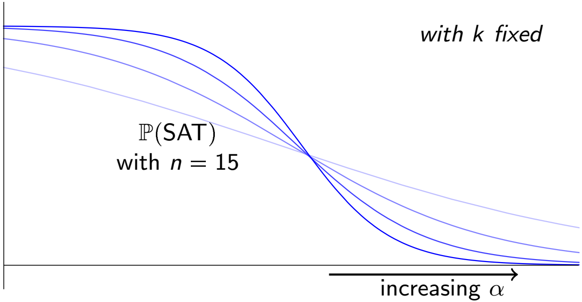

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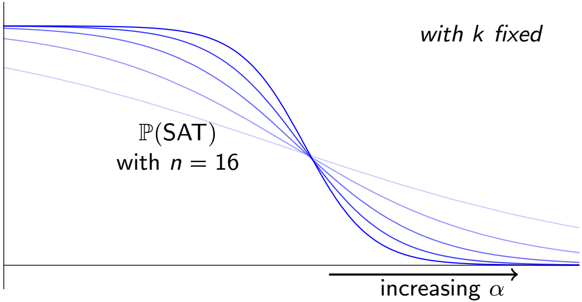

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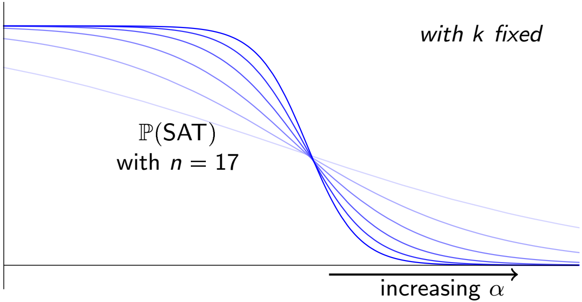

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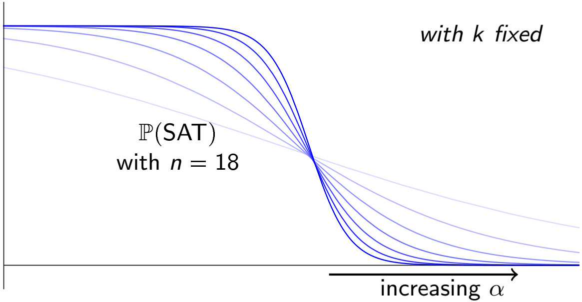

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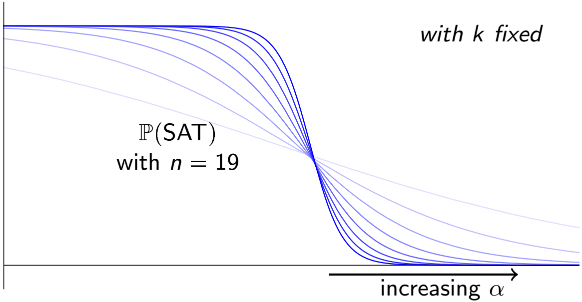

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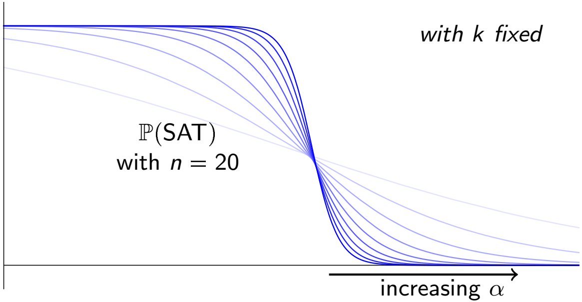

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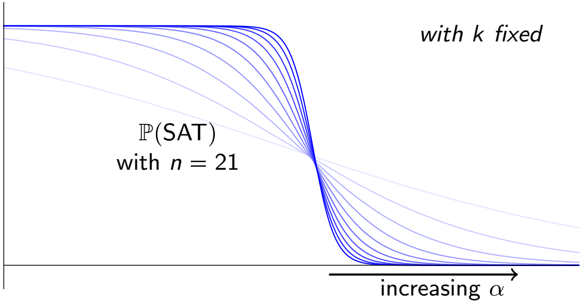

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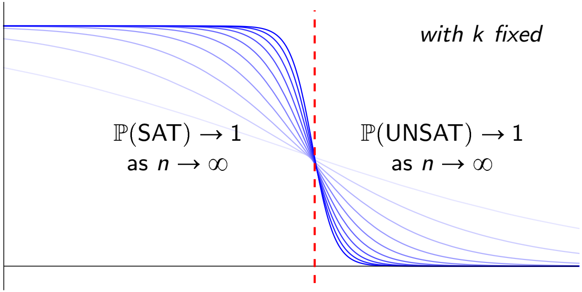

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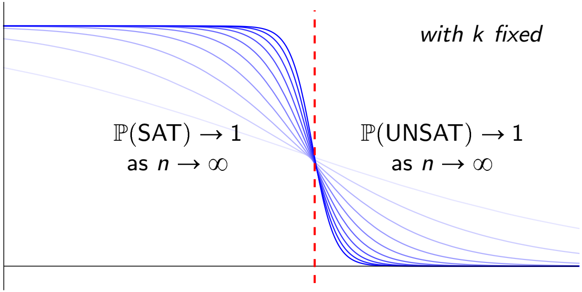

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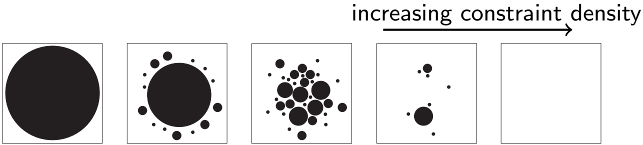
For general k, Friedgut ('99) proved the transition sharpens around a (possibly non-convergent) *threshold sequence* $\alpha_{\text{sat}}(n)$

(whereas conjecture requires $\alpha_{\text{sat}}(n) \to \alpha_{\text{sat}}$ as $n \to \infty$)

Theoretical Physics

Disordered systems such as *spin glasses* are models of interacting particles/variables with frustrated interactions. Many random constraint satisfaction problems can be recast as dilute mean-field spin glasses.



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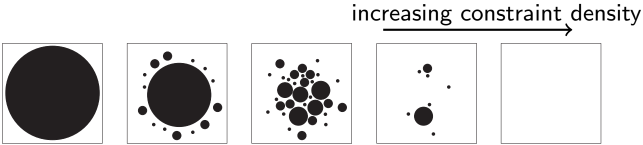

One-step Replica Symmetry Breaking Predictions:
Developed to study dense spin-glasses such as the Sherrington-Kirkpatrick model.

- **Replica Symmetry Breaking**: Clustering of assignments.
- **Cavity Method**: Heuristic for analyzing adding one variable.

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For random colourings and NAE-SAT, second moment method succeeds up to $\alpha_2 = \alpha_{\text{sat}} - O(1)$.

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1. *condensation and replica symmetry breaking (RSB)* —
   $\mathbb{E}Z$ dominated by **atypical clusters** for $\alpha > \alpha_{\mathsf{sat}} - \epsilon$

2. *neighborhood profile fluctuations* —
   $\mathbb{E}Z$ dominated by **atypical graphs** for all $\alpha > 0$

Some physics perspective:
condensation and replica symmetry breaking

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Some remarkable predictions proved for *dense* graphs

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and for optimization on complete graphs with random edge weights:

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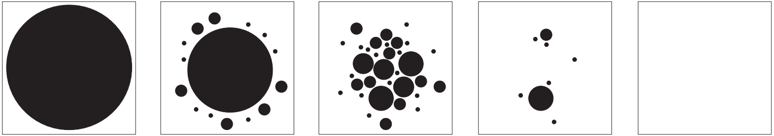
More recently a set of predictions for *sparse* random systems emerged: Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová '07, Montanari–Ricci-Tersenghi–Semerjian '08

Phase Diagram

Two solutions are connected if they differ by one bit.

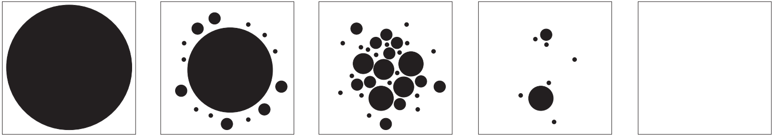


→ increasing α

KMRSZ '07, MRS '08

Phase Diagram

well-connected



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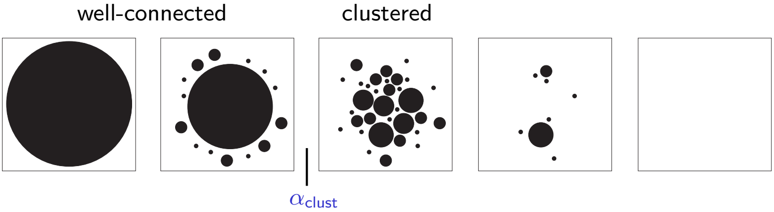

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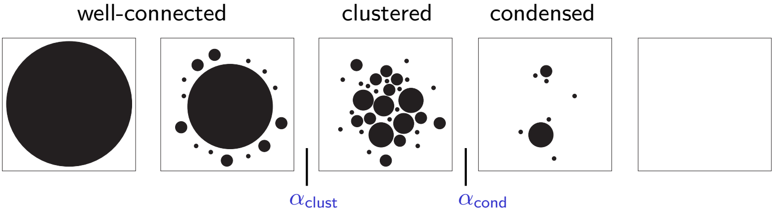

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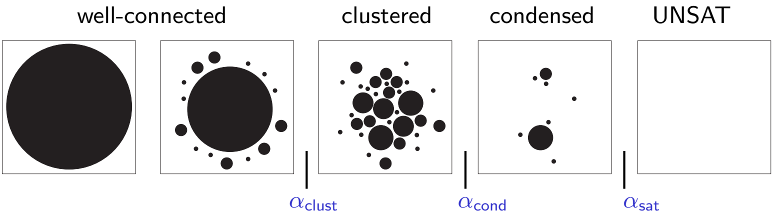

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After α_{sat}, no solutions w.h.p.

Condensation (in regular models)

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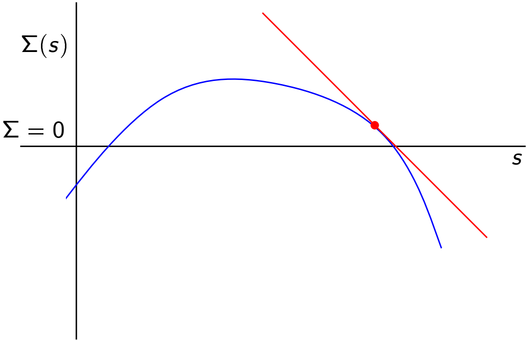

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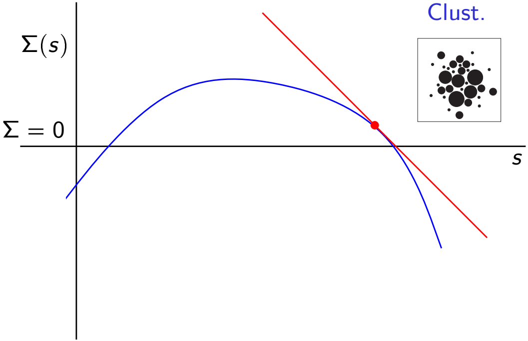

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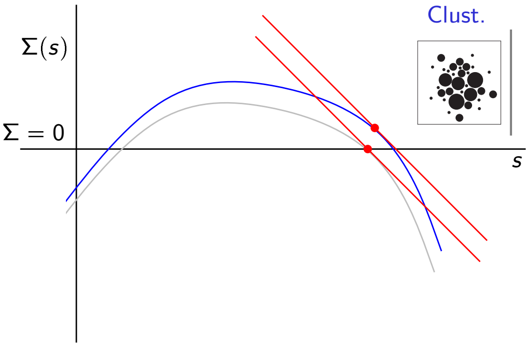

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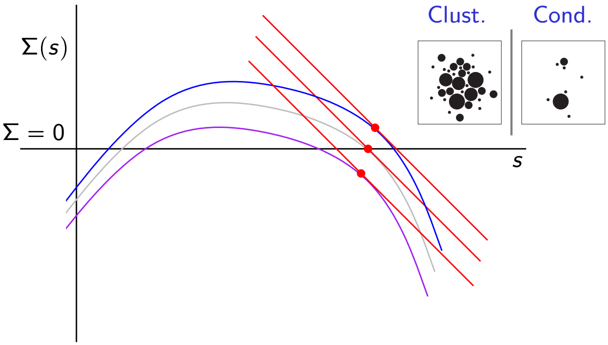

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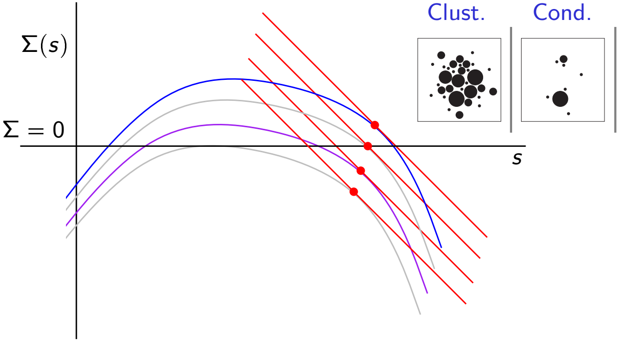

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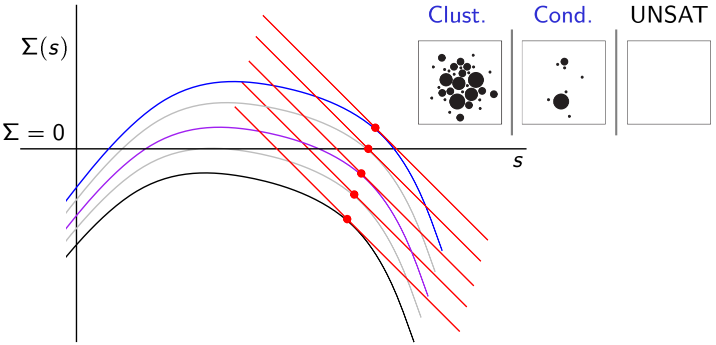

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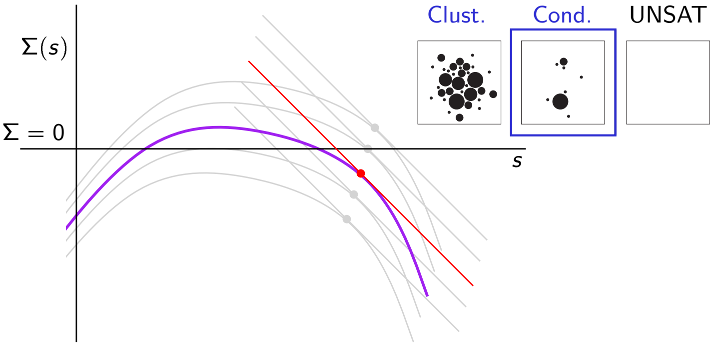

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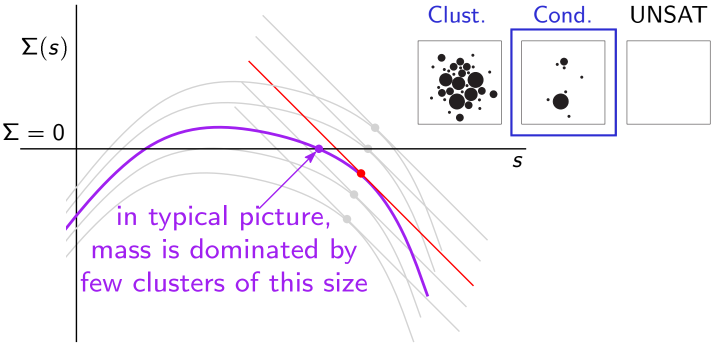
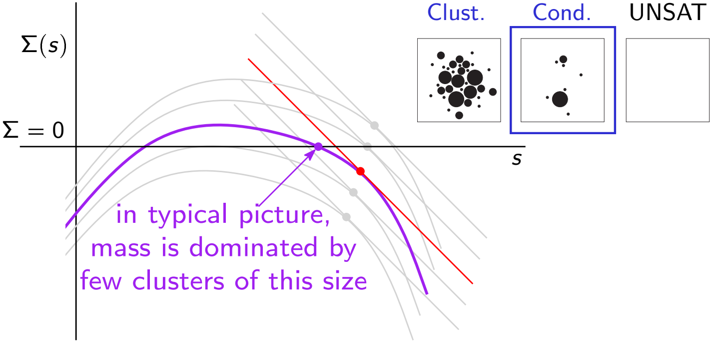


Condensation and non-concentration



The 1-RSB prediction:

- Satisfiability Threshold

$$\alpha_{\text{sat}} = \sup\{\alpha : \sup_s \Sigma(s) \geqslant 0\}$$

- Condensation Threshold and free energy

$$\alpha_{\text{cond}} = \sup\left\{\alpha : \sup_s s + \Sigma(s) = \sup_{s:\Sigma(s)\geqslant 0} s + \Sigma(s)\right\}$$

$$\Phi = \lim_{n\to\infty} \frac{1}{n}\log Z = \sup\{s + \Sigma(s) : \Sigma(s) > 0\} = \sup\{s : \Sigma(s) > 0\}$$

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Maximum Independent Set *d*-Regular, large *d* [Ding, S', Sun]

Regular SAT, large *k* [Coja-Oghlan, Panagiotou]

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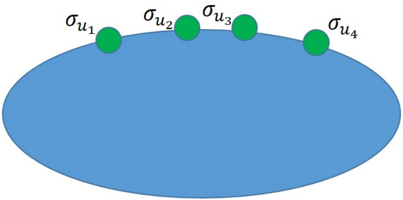
Locally rigid resulting in no clustering.

Cavity Method: Adding a new vertex v (or clause).



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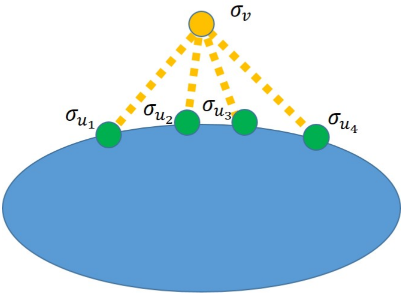

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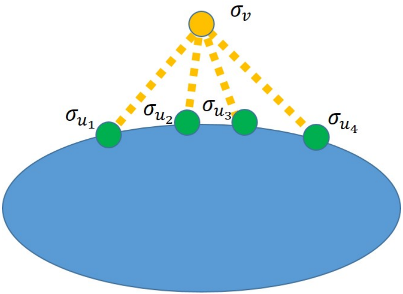

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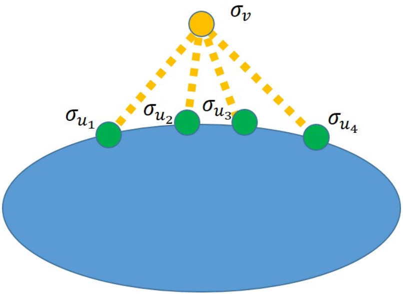

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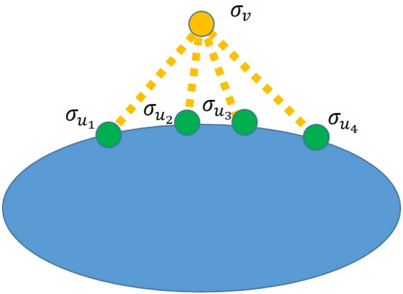

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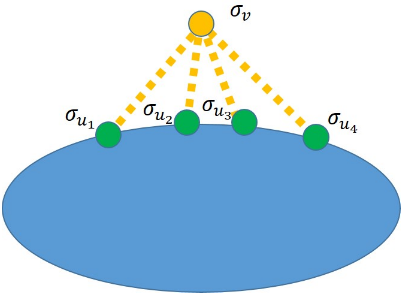

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Then the 1RSB prediction $\alpha_{\text{sat}} \approx 2^k \ln 2 - (1 + \ln 2)/2$ is the root of $\Phi(\alpha) = 0$.

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Theorem.(Ding, S., Sun) *For $k \geqslant k_0$ (absolute constant), random k-SAT has a sharp satisfiability threshold, with explicit value* $\alpha_{\mathsf{sat}} = \alpha_\star$ *matching the one-step replica symmetry breaking prediction of Mertens–Mézard–Zecchina '06.*

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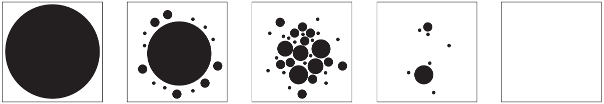

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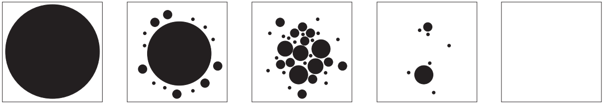

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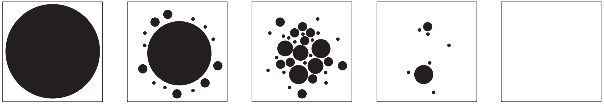

Models at finite temperature? Clusters no longer have rigid combinatorial description.

Models with full Replica Symmetry breaking, e.g. MAX-CUT?

Thanks!