BÁLINT TÓTH
(Rényi Institute Budapest and University of Bristol)

SCALING LIMITS FOR RANDOM WALKS AND DIFFUSIONS WITH LONG MEMORY

ICM-2018, Probability Section
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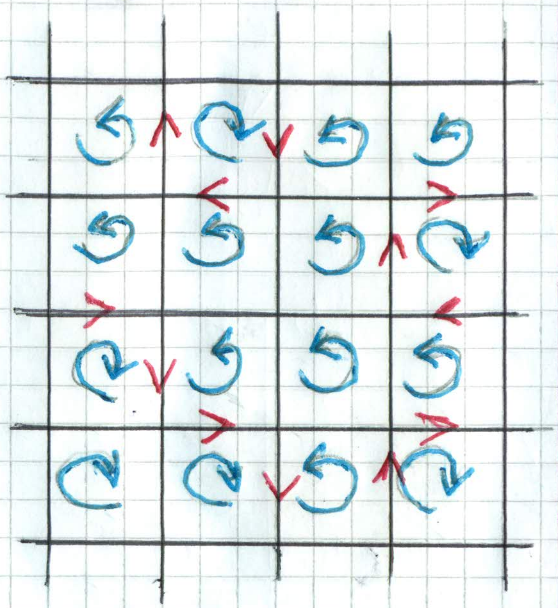
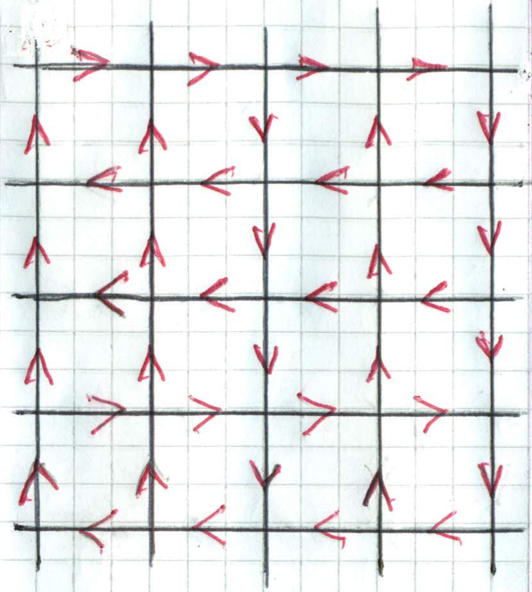
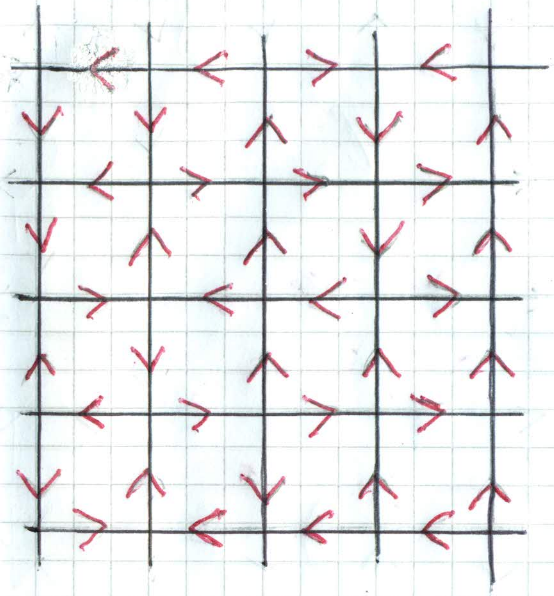
Small cycles with short range dependence



"Manhattan"



Six-vertex / "Square Ice"



$X(t)$: RW, at each step uniform choice between the allowed directions

Q: large scale behaviour? diffusive? super diff.?

A. RW in divergence-free random drift field:

$(\Omega, \pi, \tau_z : \Omega \to \Omega :: z \in \mathbb{Z}^d)$	probab. sp., ergodic $\mathbb{Z}^d$-action
$\mathcal{E} = \{k \in \mathbb{Z}^d : \lvert k\rvert = 1\}$	possible steps of the rw
$p : \Omega \to [0, \infty)^{\mathcal{E}}$,	jump rates of the rw

RWRE: Given $\omega \in \Omega$, $t \mapsto X(t) \in \mathbb{Z}^d$ cont. time Markov chain:

$$\mathbf{P}_\omega \left(X(t+dt) = x + k \middle| X(t) = x\right) = p_k(x, \omega)dt = p_k(\tau_x\omega)dt.$$

Separate symmetric and skew-symmetric part of jump rates:

$$s_k(\omega) := \frac{p_k(\omega) + p_{-k}(\tau_k\omega)}{2} = s_{-k}(\tau_k\omega),$$
$$v_k(\omega) := \frac{p_k(\omega) - p_{-k}(\tau_k\omega)}{2} = -v_{-k}(\tau_k\omega).$$

Assumptions:

$$\sum_{k\in\mathcal{E}} p_{-k}(\tau_k\omega) \equiv \sum_{k\in\mathcal{E}} p_k(\omega) \quad \Leftrightarrow \quad \sum_{k\in\mathcal{E}} v_k(\omega) \equiv 0 \tag{DFR}$$

$$\int_\Omega s_k(\omega)^2 d\pi(\omega) < \infty \tag{L2}$$

$$s_k(\omega) =: s_* > 0, \qquad \pi\text{-a.s.} \tag{ELL}$$

$$\int_{[-\pi,\pi]^d} \frac{\sum_{k\in\mathcal{E}} \widehat{C}_{kk}(p)}{\sum_{i=1}^d (1-\cos(p_i))} dp < \infty \tag{H-1}$$

where

$$C_{kl}(x) := \int_\Omega v_k(\omega) v_l(\tau_x\omega) d\pi(\omega), \quad \widehat{C}_{kl}(p) := \sum_{x\in\mathbb{Z}^d} e^{\sqrt{-1}p\cdot x} C_{kl}(x)$$

First consequences of the assumptions?

- (DFR) $\Rightarrow$ The *environment process*:

$$t \mapsto \eta_t := \tau_{X(t)}\omega$$

is stationary and ergodic Markov process in (Ω, π).

- (H-1) $\Rightarrow$ zero overall drift of the walk:

$$\mathbf{E}\left(X(t)\right) := \int_\Omega \mathbf{E}_\omega \left(X(t)\right) d\pi(\omega) = 0.$$

- Thus, (DFR) & (H-1) $\Rightarrow$ (by the ergodic thm) LLN:

$$t^{-1}X(t) \to 0, \quad \text{a.s.}$$

- (L2) & (ELL) & (H-1) $\Rightarrow$ diffusive bounds:

$$0 < \varliminf_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) \le \varlimsup_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) < \infty.$$

Question left open: CLT?

Analogous diffusion problem: **diffusion in divergence-free random drift field** $t \mapsto X_t \in \mathbb{R}^d$ with infinitesimal generator

$$L := \frac{1}{2}\nabla \cdot a\nabla + v \cdot \nabla$$

$a = a_\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$; $v = v_\omega : \mathbb{R}^d \to \mathbb{R}^d$ are stationary, ergodic,

$$a = a^\dagger > 0, \qquad \operatorname{div} v \equiv 0, \qquad \pi\text{-a.s.}$$

with conditions analogous with (L2), (ELL), (H-1).

(One) Phyisical motivation: Diffusion of suspended particles in *incompressible turbulent flow*, in stationary regime.

History: From Lucretius Carus (cca 60 BC) ...... to ......
G Papanicolaou, SRS Varadhan (1981) ...... (more later)

Titus Lucretius Carus: De Rerum Naturae, Liber Secundus (cca 60 BC)

sic a principiis ascendit motus et exit // paulatim nostros ad sensus, ut moveantur // illa quoque, in solis quae lumine cernere quimus // nec quibus id faciant plagis apparet aperte.

Thus motion ascends from the primevals on, // And stage by stage emerges to our sense, // Until those objects also move which we // Can mark in sunbeams, though it not appears // What blows do urge them.

(Transl. William Ellery Leonard, 1916)

Proposition ("Helmholtz's theorem"). *Let* $v_k(x,\omega) := v_k(\tau_x\omega)$ *be an ergodic,* $\mathcal{L}^2$*, divergence-free flow on* $\mathbb{Z}^d$ *(as before).*
(i) $\exists!$ *tensor-field* $x \mapsto h(x,\cdot) \in \mathcal{L}^2(\Omega \to \mathbb{R}^{\mathcal{E}\times\mathcal{E}}, \pi)$ *such that*

$$h_{k,l}(y,\omega) - h_{k,l}(x,\omega) = h_{k,l}(y-x,\tau_x\omega) - h_{k,l}(0,\tau_x\omega),$$

$$h_{l,k}(x,\omega) = h_{-k,l}(x+k,\omega) = h_{k,-l}(x+l,\omega) = -h_{k,l}(x,\omega),$$

$$v_k(x,\omega) = \sum_{l\in\mathcal{E}} h_{k,l}(x,\omega).$$

($x \mapsto h(x)$ *is the* stream tensor.*)*

(ii) (H-1) *is valid (for* v*) if and only if for some* $h_{k,l} \in \mathcal{L}^2(\Omega,\pi)$

$$h_{k,l}(x,\omega) = h_{k,l}(\tau_x,\omega).$$

Theorem. *Assume* (DFR), *(*L2*)*, (ELL), (H-1)*. The non-degenerate covariance matrix* $(\sigma^2)_{ij} := \lim_{T\to\infty} T^{-1}\mathbf{E}\left(X_i(T)X_j(T)\right)$ *exists.*
For any $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}_+$ *and* $f : \mathbb{R}^{md} \to \mathbb{R}$ *cont & bdd*

(i) [G Kozma, BT (AoP, 2017)] Annealed CLT:

$$\mathbf{E}_\omega\left(f(..., T^{-1/2}X(Tt_j), ...)\right) \xrightarrow{\pi\text{-prob}} \mathbf{E}\left(f(..., W_\sigma(t_j), ...)\right).$$

(ii) [BT (AoP, 2018+)] Quenched CLT:
If the marginally stronger conditions

$$\int_\Omega \left(s_k(\omega) + \left|h_{k,l}(\omega)\right|\right)^{2+\varepsilon} d\pi(\omega) < \infty \qquad (\{\texttt{L2\&H-1}\}\texttt{turbo})$$

hold, then

$$\mathbf{E}_\omega\left(f(..., T^{-1/2}X(Tt_j), ...)\right) \xrightarrow{\pi\text{-a.s.}} \mathbf{E}\left(f(..., W_\sigma(t_j), ...)\right).$$

Historical comments (sketchy, far from complete):

[SM Kozlov (1979)], [G Papanicolaou, SRS Varadhan (1981)]:
$s \in \mathcal{L}^\infty$ $v \equiv 0$, self-adjoint, diffusion, initiation of the problem

[H Osada (1983)], [SM Kozlov (1985)]:
$s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^\infty$, O: quenched diffusion; K: annealed walk

[K Oelschläger (1988)], [A Fannjiang, G Papanicolaou (1996)]:
$s =$ const., $h \in \mathcal{L}^2$, annealed, diffusion, with some restrictions

[A Fannjiang, T Komorowski (1997)]:
$s =$ const., $h \in \mathcal{L}^{d+\varepsilon}$, quenched diffusion.

[T Komorowski, S Olla (2003)], [J-D Deuschel, H Kösters (2008)]:
$s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^\infty$, K,O: annealed walk, D,K: quenched walk

[T Komorowski, C Landim, S Olla (2012)]:
$s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^d$, annealed walk, + diffusion with s, h Gaussian

Elements of proof:

- **Quenched tightness:** Nash moment bound
  extended from $s, h \in \mathcal{L}^{\infty}(\Omega, \pi)$ to $s, h \in \mathcal{L}^{2+\varepsilon}(\Omega, \pi)$:

$$\overline{\lim_{t\to\infty}}\, t^{-1/2}\mathbf{E}_{\omega}\left(|X(t)|\right) \leq C < \infty, \qquad \pi\text{-a.s.}$$

- **Harmonic coordinates:** Kozlov, Osada . . .
  extended from $s, h \in \mathcal{L}^{\infty}(\Omega, \pi)$ to $s, h \in \mathcal{L}^{2+\varepsilon}(\Omega, \pi)$:
  Find $\theta \in \mathcal{L}^{2}(\Omega \to \mathbb{R}^{\mathcal{E}}, \pi)$? for given $\varphi \in \mathcal{H}_{-1}$:

$$\sum_{k\in\mathcal{E}} p_k(\omega)\theta_k(\omega) = \varphi,$$

$$\theta_k(\omega) + \theta_l(\tau_k\omega) = \theta_l(\omega) + \theta_k(\tau_l\omega)$$

  Let $x \mapsto \Theta(x, \omega)$ be the $\mathbb{Z}^d$-cocycle with $\operatorname{grad}\Theta(x,\omega) = \theta(\tau_x\omega)$

Elements of proof (ctd):

◦ **"Relaxed Sector Condition":** Prove that

$|\Delta|^{-1/2}(L - L^*)|\Delta|^{-1/2}$ is **skew-self-adjoint** over $\mathcal{L}^2(\Omega, \pi)$.

◦ **Quenched martingale CLT** for

$$t \mapsto X(t) - \Theta(X(t), \omega)$$

◦ **Control the compensator:**

$$\lim_{t\to\infty} \mathbf{P}_\omega \left(t^{-1/2} |\Theta(X(t), \omega)| > \delta\right) = 0, \qquad \pi\text{- a.s.}$$

relying on [Zygmund (1951)]'s
unrestricted a.s. ergodic theorem (over $\mathbb{Z}^d$).

When (H-1) fails: expect superdiffusive behaviour.

E.g. diffusion in the curl of 2-dim Gaussian Free Field:
$v : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $\quad v = \operatorname{curl} U * GFF$, $\quad U$: local regularization.

$dX(t) = dB(t) + v(X(t))dt$, $\qquad$ Expect : $\mathbf{E}\left(|X(t)|^2\right) \asymp t\sqrt{\log t}$

Theorem. [BT, B Valkó (JSP, 2012)]

$$t \log \log t \ll \mathbf{E}\left(|X(t)|^2\right) \ll t \log t$$

in the sense of Laplace transform (modulo Tauberian inversion).

Other superdiffusive bounds:

- RW on random Manhattan lattice, $d = 2, 3$
  [S Ledger, BT, B Valkó (ECP, 2018)].
- RW on $2d$ *dimer model* (a la Kasteleyn)
  [BT, B Valkó (in progress 2018+)].

B. Self-repelling Walks and Diffusions: Pushed by the negative gradient of its own occupation time measure.

Self-repelling random walk (TSAW): $t \mapsto X(t) \in \mathbb{Z}^d$,

$$\ell(t,x) := \ell(0,x) + |\{0 < s \leq t : X(s) = x\}|$$

$$\mathbf{P}\left(X(t+dt) = x + k | X(t) = x\right) = w(\ell(t,x) - \ell(t,x+k))dt$$

$w : \mathbb{R} \to (0,\infty)$ increasing

Self-repelling diffusion (SRBP): $t \mapsto X(t) \in \mathbb{R}^d$

$$\ell(t,A) := \ell(0,A) + |\{0 < s \leq t : X(s) \in A\}|$$

$$dX(t) = dB(t) - \text{grad}\left(V * \ell(t,\cdot)\right)(X(t))dt$$

$V : \mathbb{R}^d \to [0,\infty)$ approximate-δ, $\quad \widehat{V}(p) := \int_{\mathbb{R}^d} e^{ip\cdot x} V(x)dx \geq 0$

Roots:

TSAW, physics:

[D Amit, G Parisi, L Peliti (1983)]
[S Obukhov, L Peliti (1983)]
[L Peliti, L Pietronero (1987)]
...

SRBP, probability:

[J Norris, C Rogers, D Williams (1987)]
[R Durrett, C Rogers (1992)]
[M Cranston, Y Le Jan (1995)]
[M Cranston, T Mountford (1996)]
...

Conjectures, based on RG and scaling arguments ("physics"):

- $\mathbf{d = 1}$: $X(t) \sim t^{2/3}$, intricate, non-Gausssian scaling limit. (Limit distributions not identified.)

- $\mathbf{d = 2}$: $X(t) \sim t^{1/2}(\log t)^{\zeta}$, Gaussian scaling limit. (Controversy about the value of ζ.)

- $\mathbf{d \geq 3}$: $X(t) \sim t^{1/2}$, Gaussian scaling limit.

Some results: . . .

- $\mathbf{d=1}$: ◦ **Limit theorem** in some particular cases, [BT (1995)]:

$$t^{-2/3}X(t) \Rightarrow \mathcal{X}.$$

◦ Construction of the **scaling limit process** (TSRM, the Brownian Web, ...), [BT, W Werner (1998)]

$$t \mapsto \mathcal{X}(t)$$

◦ **"Robust" bounds**, [P Tarrés, BT, B Valkó (2012)]:

$$C_1 t^{5/4} \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_2 t^{3/2}.$$

(and more bounds for more general self-interactions)

◦ **Missing:** fully robust proofs.

- $\mathbf{d = 2}$: ∘ **Super diffusive bounds**, [BT, B Valkó (2012)]:

$$C_1 t \log\log t \le \mathbf{E}\left(|X(t)|^2\right) \le C_2 t \log t.$$

  ∘ Expected order: $\mathbf{E}\left(X(t)^2\right) \sim t\sqrt{\log t}$

- $\mathbf{d \ge 3}$: ∘ **CLT**, [I Horváth, BT, B Vető (2012)]:

$$t^{-1/2} X(t) \Rightarrow \mathcal{N}(0, \sigma).$$

Precise conditions and statements later.

Environment process: $t \mapsto \eta(t,\cdot)$

$$\eta(t,x) := -\text{grad}\,\Big(V * \ell(t,\cdot)\Big)(X(t)+x).$$

Markov process with continuous sample path in

$$\Omega := \Big\{\omega \in C^\infty(\mathbb{R}^d \to \mathbb{R}^d) \;:\; \text{rot}\,\omega \equiv 0, \quad ||\omega||_{k,m,r} < \infty\Big\}$$

$$||\omega||_{k,m,r} := \sup_{x\in\mathbb{R}^d} \Big(1+|x\,|\Big)^{-1/r} \Big|\partial^{|m\,|}_{m_1,\dots,m_d}\omega_k(x)\Big|$$

"Miracle": **Gradient of** (mollified) **Gaussian Free Field = stationary & ergodic** for the Markov process $t \mapsto \eta(t)$:

$$\langle\omega_k(x)\rangle = 0, \qquad \langle\omega_k(x)\omega_l(y)\rangle = -\partial^2_{kl}V * \mathfrak{G}(y-x)$$

where $\mathfrak{G}$ is the (Laplacian) Green's function, $\Delta\mathfrak{G}(x) = \delta(x)$.

Proof 1: Itô-calculus. Proof 2: Functional analytic.

Theorem. [I Horváth, BT, B Vető (PTRF 2012)] *SRBP,* $d \geq 3$*:* *In the stationary regime* $\sigma^2 := \lim_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) \in (0,\infty)$*, and*

$$T^{-1/2}X(Tt) \overset{f.d.m.}{\Rightarrow} W_\sigma(t).$$

Theorem. *The following bounds hold for SRBP in the stationary regime, in the sense of Laplace transforms:*

(i) [P Tarrés, BT, B Valkó (AoP, 2012)] $d = 1$*:*

$$C_1 t^{5/4} \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_2 t^{3/2},$$

(ii) [BT, B Valkó (JSP, 2012)] $d = 2$*:*

$$C_1 t \log\log t \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_2 t \log t,$$

(+ Similar results for particular cases of the lattice versions.)

Elements of proofs:

Martingale decomposition of the displacement:

$$X(t) = M(t) + \int_0^t \varphi(\eta_s) ds, \qquad \text{where} \quad \varphi_l(\omega) = \omega_l(0).$$

- $\circ$ $t \mapsto M(t)$ well understood (cf. martingale CLT)
- $\circ$ long time asymptotics of $\int_0^t \varphi(\eta_s) ds$: tricky.

Goals:

$d \geq 3$: martingale approximation of $\int_0^t \varphi(\eta_s) ds$ by non-reversible "Kipnis-Varadhan theory"

$d = 2$: superdiffusive bounds on $\mathbf{E}\left(\left|\int_0^t \varphi(\eta_s) ds\right|^2\right)$ by "resolvent method" intiated by [Landim-Quastel-Salmhofer-Yau (2004)]

Analysis in the Hilbert space (Fock space / Wiener space)

$$\mathcal{L}^2(\Omega,\pi) =: \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$$

The infinitesimal generator acting on $\mathcal{L}^2(\Omega,\pi)$:

$$L = \Delta + \sum_{l=1}^{d} \Big(\nabla_l a_l + a_l^* \nabla_l \Big) = -S + A_- + A_+,$$

where

$$a_l^* \, {:}\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n){:} \; = \; {:}\omega_l(0)\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n){:}$$

$$a_l \, {:}\omega_{k_1}(x_1)\cdots\omega_{k_n}(x_n){:} \; = \sum_{m=1}^{n} K_{lk_m}(x_m) \, {:}\omega_{k_1}(x_1)\cdots\cancel{\omega_{k_m}(x_m)}\cdots\omega_{k_n}(x_n){:}$$

Proof: careful use of commutation relations, plus "directional derivative" identity (a la Malliavin calculus).

$\mathbf{d} \geq 3$: [Sethuraman, Varadhan, Yau (2000)]-turbo

- H_{-1}-bound: $\left\| \, |\Delta|^{-1/2} \varphi \right\| < \infty$
- Graded sector condition:

$$\sum_{n=1}^{\infty} \left\| \, |\Delta|^{-1/2} A_{\pm,n} |\Delta|^{-1/2} \right\|^{-1} = \infty.$$

$\mathbf{d} = 2$: [Landim, Quastel, Salmhofer, Yau (2004)]-turbo

- Variational formula for resolvent $R_\lambda := (\lambda I - L)^{-1}$:

$$(\varphi, R_\lambda \varphi) = \sup_{\psi \in \mathcal{H}} \left\{ 2(\psi, \varphi) - (\psi, (\lambda + |\Delta|)\psi) - (A\psi, (\lambda + |\Delta|)^{-1} A\psi) \right\}$$

- Choose appropriate test function $\psi \in \mathcal{H}$
  $+$ solve a subtle variational problem.