Sixty years of percolation

How do we prove the existence of a phase transition?

Hugo Duminil-Copin, IHES and Unige

Rio 2018
PERCOLATION PROCESSES
I. CRYSTALS AND MAZES

BY S. R. BROADBENT AND J. M. HAMMERSLEY

Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a ‘medium’ influence the percolation of a ‘fluid’ through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.
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3. Random mazes. 3.1. Suppose that in an infinite set of atoms joined by bonds some (or all) of the bonds are dammed in a random manner. Fluid is supplied to a (finite, countable or uncountable) subset of atoms called source atoms, and then percolates the set in the following way. An atom of the set is said to be wet by the fluid either if it is a source atom or if there exists a walk to the atom from a source atom, the walk traversing undammed bonds only and in the permitted directions. All atoms not wet are said to be dry. We are interested in the properties of the wet atoms, and these naturally depend on the structure and connexions of the given set, on the manner in which bonds are dammed, and on the source atoms.
Mathematical definition

- Percolation is a model of random subgraph $\omega = (V_\omega, E_\omega)$ of a given (unoriented) graph $G = (V_G, E_G)$ with $V_\omega = V_G$ and $E_\omega \subset E_G$. 
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Definition (Bernoulli percolation)

Edges of $G$ are independently

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\begin{cases} 
\text{open with probability } p, \\
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\[ P_p(\omega) \overset{\text{def}}{=} p \# \text{open edges} (1 - p) \# \text{closed edges} \]
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- **Continuum percolation models (Voronoi, Boolean)**

- \( \{ x : \varphi_x \geq h \} \) for some (random) continuous function such as random homogeneous polynomials, random sums of eigenfunctions of the Laplacian, or the Gaussian Free Field:
  \[ dP_{GFF}[\varphi] \propto \exp\left[ -\frac{1}{2} \sum_{xy \in E_G} (\varphi_x - \varphi_y)^2 \right] d\varphi \]
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**Proposition (Hammersley)**

On $\mathbb{Z}^d$, if $p\mu(\mathbb{Z}^d) < 1$, then $\theta(p) = 0$, where $\mu(\mathbb{Z}^d)$ is the connective constant defined by $\mu(\mathbb{Z}^d) = \lim_{n \to \infty} (\# \text{ SAW of length } n)^{1/n}$.
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The fluid will be able to flow from one point to another if and only if there is a connexion without dams between them, and this will be so if and only if there is an undammed self-avoiding walk connecting them (i.e. a walk which visits no intermediate point more than once). It is, therefore, appropriate to study the self-avoiding walks.

💡 A path of length $n$ is open with probability $p^n$. Then, use union bound.
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### Proposition (Peierls)

On $\mathbb{Z}^d$ with $d \geq 2$, if $(1 - p)\mu(\mathbb{Z}^2) < 1$, then $\theta(p) > 0$.  

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It is sufficient to treat the $d = 2$ case. A minimal set of $n$ blocking edges is closed with probability $(1 - p)^n$. Then, use union bound.
Proposition

For every $d \geq 2$, there exists $p_c(Z^d) \in (0, 1)$ such that

- $\theta(p) = 0$ if $p < p_c(Z^d)$,
- $\theta(p) > 0$ if $p > p_c(Z^d)$.

Fix $p \leq p'$. Need to construct a coupling, i.e. a probability space $(\Omega, F, P)$ on which

- $\omega$ is a Bernoulli percolation of parameter $p$,
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- the probability of $\omega \subset \omega'$ is 1.

Consider a family of independent uniform $[0, 1]$ random variables $(U_e)$ associated with the edges of $Z^d$. Set $e$ open in $\omega$ if $U_e \leq p$ and $e$ open in $\omega'$ if $U_e \leq p'$. Then $\theta(p) = P[0 \leftrightarrow \infty \text{ in } \omega] \leq P[0 \leftrightarrow \infty \text{ in } \omega'] = \theta(p')$.  

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RSW theory (1978)

Consider Bernoulli percolation of parameter $p = \frac{1}{2}$. What is the probability of crossing from left to right a $n + 1$ by $n$ box?
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Theorem (Russo, Seymour-Welsh, 78)

Fix $p = \frac{1}{2}$. For every topological rectangle $(R, a, b, c, d)$, there exists $\delta > 0$ such that for every $n \geq 1,
\delta \leq P_{p^n} \leq 1 - \delta
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Many applications! For instance, $\theta(\frac{1}{2}) = 0$ and therefore $p_c(Z_2) \geq \frac{1}{2}$.

The limit should exist and be equal to Cardy-Smirnov's formula (conformally invariant answer).

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Use differential inequalities on thermodynamical quantities

$$\Pi'_n \geq C \log n \; \Pi_n (1 - \Pi_n),$$

where $\Pi_n(p) \overset{\text{def}}{=} \mathbb{P}_p[\text{rectangle crossed}]$
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\theta_n' \geq \frac{cn}{\sum_{k<n} \theta_k} \theta_n (1 - \theta_n),
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where \( \theta_n(p) \overset{\text{def}}{=} \mathbb{P}_p[0 \text{ connected to distance } n] \)

Theorem (Menshikov 86, Aizenman-Barsky 87, DC-Tassion 17)

For any \( d \geq 2 \) and \( p < p_c(\mathbb{Z}^d) \), there exists \( c > 0 \) such that for every \( n \geq 1 \),

\[
\theta_n(p) \leq \exp(-cn).
\]
Proposition

For every $d \geq 2$, $\frac{1}{\mu(\mathbb{Z}^d)} \leq p_c(\mathbb{Z}^d) \leq \frac{1}{2}$.
Percolation on $\mathbb{Z}^d$ with $d \geq 3$

**Proposition**

For every $d \geq 2$, $\frac{1}{\mu(\mathbb{Z}^d)} \leq p_c(\mathbb{Z}^d) \leq \frac{1}{2}$.

- Hara-Slade ($\approx$ 1990) used the *lace-expansion* technique to derive the following expansion of the critical value:

$$p_c(\mathbb{Z}^d) = \frac{1}{2d} + \frac{1}{(2d)^2} + \frac{7}{2} \frac{1}{(2d)^3} + O\left(\frac{1}{(2d)^4}\right).$$
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Quite apart from the fact that percolation theory had its origin in an honest applied problem (see Hammersley and Welsh (1980)), it is a source of fascinating problems of the best kind a mathematician can wish for: problems which are easy to state with a minimum of preparation, but whose solutions are (apparently) difficult and require new methods.
Consider the Cayley graph $G$ of a group $G$ with a symmetric and finite set of generators $S$:

$$V_G = G$$

$$E_G = \{\{x, y\} \mid xy^{-1} \in S\}.$$ 

It is easy to justify the existence of a critical point $p_c(G) \geq 1/\mu(G) \geq 1/|S|$, but it is unclear whether $p_c(G) \neq 1$.

If $G$ has linear growth, $p_c(G) = 1$.
Consider the Cayley graph $G$ of a group $G$ with a symmetric and finite set of generators $S$:

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Percolation beyond \( \mathbb{Z}^d \)

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**Conjecture 1** If $G$ is the Cayley graph of an infinite (finitely generated) group, which is not a finite extension of $\mathbb{Z}$, then $p_c(G) < 1$. 
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$p_c < 1$ and isoperimetry (1)

**Theorem (DC, Goswami, Raoufi, Severo, Yadin 18)**

Consider a graph $G$ with bounded degree. Assume that there exist $d > 4$ and $c > 0$ such that

$$|\partial K| \geq c|K|^{1-\frac{1}{d}}$$

for all finite $K \subset V_G$. (Isop$_d$)

Then, there exists $p < 1$ such that for every finite set $S \subset V_G$,

$$\mathbb{P}_p[S \text{ is connected to } \infty] \geq 1 - \exp[-\frac{1}{2}\text{cap}(S)].$$

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- Groups with polynomial growths are **virtually nilpotent** by Gromov's theorem. As a consequence, any Cayley graph $G$ of such a group with super-linear growth contains a graph which is quasi-isometric to $\mathbb{Z}^2$, and $p_c(G) < 1$. 
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- Cayley graphs having \( \liminf n^{-d} |B_x(n)| > 0 \) for some \( d \) satisfy \((\text{Isop})_d\) (Coulhon-Saloff-Coste 1993).
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\[ |\partial K| \geq c|K|^{1 - \frac{1}{d}} \] for all finite $K \subset V_G$. \textbf{(Isop$_d$)}

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**Corollary**

$p_c(G) < 1$ for every Cayley graph with super-linear growth.
$p_c < 1$ and isoperimetry (2)

- Use the Gaussian Free Field $(\varphi_x)_{x \in V_G}$ to write:

$$G(x, y) \asymp \mathbb{E}_{\text{GFF}}[\mathbb{P}_p(x \leftrightarrow y)],$$

where $p_{xy} = p_{xy}(\varphi) \overset{\text{def}}{=} 1 - \exp[-(\varphi_x)_+ (\varphi_y)_+]$. 

Integrate out the randomness coming from the GFF to prove that for some $p_c < 1$ large enough,

$$\mathbb{E}_{\text{GFF}}[\mathbb{P}_p(x \leftrightarrow y)] \leq \mathbb{P}_p[\mathbb{P}_p(x \leftrightarrow y)].$$

Since $G(x, y)$ does not decay too fast, this implies that $p \geq p_c(G)$. 

Percolation can help understanding geometry of groups, e.g.

A measurable group theoretic solution to von Neumann's problem

Gaboriau-Lyons Conjecture (Benjamini Schramm, 96)

A Cayley graph is non-amenable, i.e. satisfies (Isop∞), iff $p_c < p_u$, where $p_u \overset{\text{def}}{=} \inf \{p \in [0, 1] : \mathbb{P}_p[\exists \text{unique infinite connected component}] = 1\}$. 

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Percolation can help understanding geometry of groups, e.g.

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Percolation can help understanding geometry of groups, e.g. [A measurable group theoretic solution to von Neumann’s problem Gaboriau-Lyons]

**Conjecture (Benjamini Schramm, 96)**

A Cayley graph is non-amenable, i.e. satisfies $(\text{Isop}_\infty)$, iff $p_c < p_u$, where

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Thank you