Liouville quantum gravity
as a *metric space and a scaling limit*

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*joint with*
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Overview

How does one make sense of the uniform measure on surfaces homeomorphic to the sphere?

- **Approach 1:** Random planar maps
  - Rooted in the combinatorics literature from the 1960s
- **Approach 2:** Liouville quantum gravity (LQG)
  - Rooted in the physics literature from the 1980s
- Relationship

Schramm-Loewner evolution, percolation, Eden growth model, diffusion limited aggregation
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Combinatorics: enumeration formulas

Physics: statistical physics models: percolation, Ising, UST...

Probability: “uniformly random surface,” Brownian surface
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What is the structure of a typical quadrangulation when the number of faces is large? How many are there? Tutte:

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\[
\frac{2 \times 3^n}{(n+1)(n+2)} \binom{2n}{n}.
\]
Random quadrangulation with 25,000 faces

(Simulation due to J.F. Marckert)
Structure of large random planar maps

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The Brownian map (TBM) comes equipped with an area measure which is the limit of the rescaled measure on RPM which assigns unit mass for each face.
Picking a surface at random in the continuum

**Uniformization theorem:** every Riemannian surface homeomorphic to the unit disk $D$ can be conformally mapped to the disk.

Isothermal coordinates: the metric for the surface takes the form $e^\rho(z)(dx^2 + dy^2)$ for some smooth function $\rho$ where $dx^2 + dy^2$ is the Euclidean metric.

$\Rightarrow$ Can parameterize the surfaces homeomorphic to $D$ with smooth functions on $D$.

$\overset{\psi}{\rightarrow}$ If $\rho = 0$, get $D$.

$\overset{\Delta \rho = 0}{\Rightarrow}$ If $\Delta \rho = 0$, i.e. if $\rho$ is harmonic, the surface described is flat.

Question: Which measure on $\rho$? If we want our surface to be a perturbation of a flat metric, natural to choose $\rho$ as the canonical perturbation of a harmonic function.
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![Diagram of a conformal mapping from a surface to a disk]

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LQG and TBM

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This talk is about endowing each of these objects with the other’s structure and showing they are equivalent.
Canonical embedding of TBM into $S^2$

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Discrete approach: take a uniformly random planar map and embed it conformally into $S^2$ (circle packing, uniformization, etc...), then in the $n \to \infty$ limit it converges to a form of $\sqrt{8/3}$-LQG.
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Main result

- Liouville quantum gravity (LQG): $e^{\gamma h(z)}(dx^2 + dy^2)$, $h$ a GFF
- The Brownian map (TBM): Gromov-Hausdorff scaling limit of uniformly random quadrangulations

Theorem (M., Sheffield)

*TBM and $\sqrt{8/3}$-LQG are equivalent. More precisely, there is a way to endow $\sqrt{8/3}$-LQG with a metric so that it is isometric to TBM.*
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Comments

1. Construction is purely in the continuum
2. Ideas are connected to aggregation models, such as the Eden model and diffusion limited aggregation
Schramm-Loewner evolution (SLE)

- Random fractal curve in a planar domain

![Critical percolation, hexagonal lattice]

Each hexagon is colored red or black with prob. $\frac{1}{2}$
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- Introduced by Schramm in ’99 to describe limits of interfaces in discrete models

- Characterized by conformal invariance and domain Markov property
- Indexed by a parameter $\kappa > 0$
- Simple for $\kappa \in (0, 4]$, self-intersecting for $\kappa \in (4, 8)$, space-filling for $\kappa \geq 8$

- Dimension: $1 + \frac{\kappa}{8}$ for $\kappa \leq 8$

Some special $\kappa$ values:
- $\kappa = 2$: LERW
- $\kappa = 8/3$: Self-avoiding walk
- $\kappa = 3$: Ising
- $\kappa = 16/3$: FK-Ising
- $\kappa = 4$: GFF level lines
- $\kappa = 8$: Percolation
- $\kappa = 12$: Bipolar orientations

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  (Lawler-Schramm-Werner, Smirnov, Schramm-Sheffield, ...)

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Loewner's equation: if $\eta$ is a non self-crossing path in $\mathbb{H}$ with $\eta(0) \in \mathbb{R}$ and $g_t$ is the Riemann map from the unbounded component of $\mathbb{H} \setminus \eta([0, t])$ to $\mathbb{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)).$$

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\( SLE_{\kappa} \) in \( \mathbb{H} \): The random curve associated with (\( \star \)) with \( W_t = \sqrt{\kappa} B_t \), \( B \) a standard Brownian motion.
**SLE}_κ**

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**SLE}_κ in \mathbb{H}**: The random curve associated with (\star) with $W_t = \sqrt{\kappa}B_t$, $B$ a standard Brownian motion. Other domains: apply conformal mapping.
Detour: Eden growth model (1961)

- Growth on a graph where at each time step, add a vertex uniformly at random from those adjacent to the cluster at the previous step.

- Question: Large scale behavior of the growth?

- Cox and Durrett (1981) showed that the macroscopic shape is convex.

- Computer simulations show that it is not a Euclidean disk.

- $\mathbb{Z}_2$ has preferential directions.

- But a random planar map does not...
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- RPM. Grow Eden cluster. (Angel’s peeling process).

Important observations:
- Conditional law of map given growth at time $n$ only depends on the boundary lengths of the outside components.

Belief:
- At large scales this is close to a ball in the graph metric (now proved by Curien).
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Goal: Make sense of the Eden model in the continuum on top of a LQG surface

- Explain a discrete variant of the Eden model that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces ($\text{SLE}_6$)
Eden model on random planar maps II

**Variant:**

- Pick two *edges* on outer boundary of cluster
Eden model on random planar maps II

**Variant:**

- Pick two edges on outer boundary of cluster
- Color vertices between edges blue and yellow
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- Pick two edges on outer boundary of cluster
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- Repeat

This exploration also respects the Markovian structure of the map. Expect that at large scales this growth process looks the same as the Eden model, hence the same as the graph metric ball.
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Continuum limit ansatz

- Sample a random planar map
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random

▶ Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE$_6$. 

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Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability $1/2$ and draw percolation interface

\[ \sqrt{\frac{8}{3}} \text{-LQG surface and the image of the interface converges to an independent SLE}_6. \]
Continuum limit ansatz

- Sample a random planar map and two edges uniformly at random
- Color vertices blue/yellow with probability 1/2 and draw percolation interface
- Conformally map to the sphere
Continuum limit ansatz

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**Ansatz** Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent $\text{SLE}_6$. 
Goal: make sense of percolation on a RPM in the continuum

(Number of subdivisions)
SLE exploration of LQG

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- **Goal**: make sense of percolation on a RPM in the continuum
- Start off with $\sqrt{8/3}$-LQG surface
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- Surfaces which are cut out have a Poissonian structure
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  - $\kappa = 6 \rightarrow \gamma = \sqrt{8/3}$

(Number of subdivisions)
Continuum analog of first passage percolation on LQG

- Start off with $\sqrt{8/3}$-LQG surface
- Fix $\delta > 0$ small and a starting point $x$
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- Start off with $\sqrt{8/3}$-LQG surface
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- Repeat

Know the conditional law of the LQG surface at each stage $\mathbb{QLE}_{(8/3, 0)}$ is the limit as $\delta \to 0$ of this growth process.

In the limit, this describes the growth of a metric ball in a metric space which is isometric to TBM.
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LQG-TBM I-III, An Axiomatic Characterization of TBM (M.-Sheffield)
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LQG-TBM I-III, An Axiomatic Characterization of TBM (M.-Sheffield)
Discrete approximation of $\text{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG
What is $\text{QLE}(\gamma^2, \eta)$?

$\text{QLE}(8/3, 0)$ is a member of a two-parameter family of processes called $\text{QLE}(\gamma^2, \eta)$

- $\gamma$ is the type of LQG surface on which the process grows
- $\eta$ determines the manner in which it grows

Eden model: $\eta = 0$

Diffusion limited aggregation: $\eta = 1$

$\eta$-dielectric breakdown model: general values of $\eta$
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Let $\mu_{\text{HARM}}$ (resp. $\mu_{\text{LEN}}$) be harmonic (resp. length) measure on a $\gamma$-LQG surface. The rate of growth (i.e., rate at which microscopic particles are added) is proportional to

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\left( \frac{d\mu_{\text{HARM}}}{d\mu_{\text{LEN}}} \right)^\eta d\mu_{\text{LEN}}.
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- $\eta$-dielectric breakdown model: general values of $\eta$
Simulation of Euclidean DLA (Witten and Sander, 1981)
Discrete approximation of QLE(2, 1). DLA on a $\sqrt{2}$-LQG.
QLE($\gamma^2, \eta$) processes we can construct

Each of the QLE($\gamma^2, \eta$) processes with ($\gamma^2, \eta$) on the orange curves is built from an SLE$_\kappa$ process using tip re-randomization.
Where are we now?

Convergence results for planar maps (RPM) decorated with a statistical physics model to SLE on a random surface.
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Gromov-Hausdorff topology

- Self-avoiding walk on RPM to SLE$_{8/3}$ on $\sqrt{8/3}$-LQG (Gwynne, M.)
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- FK-weighted RPM with \( q \in (0, 4) \)
  - Infinite volume (Sheffield)
  - finite volume (Gwynne, Mao, Sun and Gwynne, Sun)
- Bipolar orientation decorated RPM (Kenyon, M., Sheffield, Wilson)
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**Embedded planar maps**
- Mated-CRT maps (Gwynne, M., Sheffield)
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- Very little is understood about how the metric should behave or how to construct it for $\gamma \neq \sqrt{8/3}$. 

Watabiki prediction: 

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4\sqrt{2 + \gamma^2}} + \frac{16}{\gamma^2}.$$ 

Ding, Goswami, Gwynne, Zeitouni, Zhang.
Other $\gamma$?

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- Very little is understood about how the metric should behave or how to construct it for $\gamma \neq \sqrt{8/3}$.
- Hausdorff dimension of $\gamma$-LQG for $\gamma \neq \sqrt{8/3}$ is not known.
Other $\gamma$?

- Other $\gamma$ values correspond to random planar maps which are decorated by a statistical physics model (e.g., the Ising model).
- Very little is understood about how the metric should behave or how to construct it for $\gamma \neq \sqrt{8/3}$.
- Hausdorff dimension of $\gamma$-LQG for $\gamma \neq \sqrt{8/3}$ is not known.
  - Watabiki prediction:
    \[
d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4} \sqrt{(4 + \gamma^2)^2 + 16\gamma^2}.
    \]
  - Ding, Goswami, Gwynne, Zeitouni, Zhang.
Thanks!