De Giorgi-Nash-Moser and Hörmander theories: new interplay

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Part 1/4

Introduction:

Kinetic theory, motivation and conjectures

Statistical mechanics: the Boltzmann equation

- Thermodynamics and first concepts of entropy 18-19th centuries in relation to industrial revolution...
- New viewpoint on molecular dynamics and heat proposed by Maxwell: to describe the position variable and the "kinetic" variable (inaccessible to observation) in a statistical manner
- Boltzmann equation (Maxwell 1867, Boltzmann 1872):

$$\partial_t f + v \cdot \nabla_x f = Q(f,f) \quad \text{on } f = f(t,x,v) \geq 0$$

$$Q(f,f)(v) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(v - v_*, \omega) \Big[f(v') f(v'_*) - f(v) f(v_*) \Big] \, \mathrm{d}\omega \, \mathrm{d}v_*$$

- Nonlinear PDE: Q bilinear integral operator acting only on v
- For long-distance interactions Q has fractional ellipticity in v

The Boltzmann collision operator

- Statistical balance-sheet of gain/loss of particles with given velocity due to binary collisions $(v', v'_*) \mapsto (v, v_*)$
- In binary collision 4 scalar conservations (mass, momentum, energy) for 6 degrees of freedom: $\omega \in \mathbb{S}^2$ and

$$v' = v - \langle v - v_*, \omega \rangle \, \omega, \quad v'_* = v_* + \langle v - v_*, \omega \rangle \, \omega$$

- The unit vector ω removes ambiguity: in the case of hard spheres, direction of the line joining the two centers of the particles
- Kernel $B(v - v_*, \omega)$ describes the relative statistical frequency of binary collisions; it only depends on the modulus $|v - v_*|$ and the deflection angle θ between $v - v_*$ and $v' - v'_*$

The collision kernel (Maxwell 1867)

- Hard spheres: $B = |v - v_*| \sin\theta$
- Repulsive inverse power forces $r^{-\alpha}$: $B = |v - v_*|^\gamma b(\cos\theta)$ with

$$\begin{cases} \gamma = \dfrac{\alpha - 5}{\alpha - 1} \\ b(\cos\theta) \simeq_{\theta \to 0} \theta^{-(1+2s)} \quad \text{with} \quad 2s = \dfrac{2}{\alpha - 1} \end{cases}$$

- B non-integrable in θ for long-range interactions
- $\alpha = 5$, $\gamma = 0$ and $2s = 1/2$: Maxwell molecules
- $\alpha \in (5, +\infty)$, $\gamma > 0$ and $2s \in (0, 1/2)$: hard potentials
- $\alpha \in [3, 5)$, $\gamma \in [-1, 0)$, $2s \in (1/2, 1]$: moderately soft potentials
- $\alpha \in (2, 3)$, $\gamma \in (-3, -1)$, $2s \in (1, 2)$: very soft potentials
- Limit case $\alpha = 2$ ill-defined...

Plasmas physics: The Landau-Coulomb equation

- Limit case $\alpha \to 2$ (Coulomb interactions), Landau 1936

$$Q(f,f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} \mathbf{P}_{(v-v_*)^\perp} \Big(f(v_*) \nabla_v f(v) - f(v) \nabla_v f(v_*) \Big) |v - v_*|^{\gamma+2} \, \mathrm{d}v_* \right)$$

where $\mathbf{P}_{(v-v_*)^\perp}$ orthogonal projection on $(v - v_*)^\perp$ and $\gamma = -3$

- Rewrites as a nonlinear non-local drift-diffusion operator

$$Q(f,f) = \nabla_v \cdot \big(A[f] \nabla_v f + B[f] f \big)$$

$$\begin{cases} A[f](v) = a \displaystyle\int_{\mathbb{R}^3} \left(I - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{-1} f(t,x,v-w) \, \mathrm{d}w \\ B[f](v) = b \displaystyle\int_{\mathbb{R}^3} |w|^{-3} \, w \, f(t,x,v-w) \, \mathrm{d}w \end{cases}$$

Connection to fluid mechanics and irreversibility

- Maxwell 1867: particular solution cancelling the collision operator $f(v) = \rho\,(2\pi T)^{-3/2}\, e^{-\frac{|v-u|^2}{2T}}$, where $\rho > 0$, $u \in \mathbb{R}^3$ and $T > 0$ are the local density, mean velocity, and temperature of the fluid
- Boltzmann 1872: "it has not yet been proved that, for any initial state of the gas, it must approach the limit distribution discovered by Maxwell"
- *H*-theorem of Boltzmann with entropy $-\int f \ln f$
- Boltzmann (and Landau) dynamics connected to classical fluid mechanical equations on ρ, u and T
- Rigorous connexion by scaling limits: Bardos-Golse-Levermore, Grenier-Lions-Masmoudi, Golse-Saint-Raymond...

Derivation from first principles

- At the other end of the scales, ideas of Maxwell (statistical description) combined with that of Boltzmann (molecular chaos) connect formally Newton equations to Boltzmann equation (Grad)
- Lanford 1975: breakthrough in mathematical physics, rigorous limit for small time for hard spheres (see also King, Gallagher-Saint-Raymond-Texier, Pulvirenti-Saffirio-Simonella)
- Note that equivalent of Lanford theorem for the Boltzmann equation with long-range interactions and for the Landau equation is missing
- Hilbert 6th problem: "axiomatisation of mechanics" still wide open (incompatible time scales)
- Boltzmann equation "fundamental" - Landau equation obtained by "grazing collision limit" thus still considered "fundamental"

The Cauchy problem

- Short-time solutions have been constructed, as well as global solutions close to the trivial stationary solution or with space homogeneity
- Weak renormalised solutions shown to exist for BE (DiPerna-Lions 1989), even weaker for LE (Alexandre-Villani)
- Construction of strong solutions (with a uniqueness principle) "in the large" remains a formidable open problem
- Similarities with millenium problem of the regularity of solutions to 3D incompressible Navier-Stokes equations (where DiPerna-Lions solution play the role of Leray solutions 1933)

Condition relaxation (I)

- Given difficulty of Cauchy problem, Truesdell-Muncaster 1980:
"*Much effort has been spent toward proof that place-dependent solutions exist for all time. [...] The main problem is really to discover and specify the circumstances that give rise to solutions which persist forever. Only after having done that can we expect to construct proofs that such solutions exist, are unique, and are regular.*"
- Cercignani 1982 gave precise conjecture on the entropy production:
"The present contribution is intended as a step toward the solution of the first main problem of kinetic theory, as defined by Truesdell and Muncaster, i.e. 'to discover and specify the circumstances that give rise to solutions which persist forever'."
- Resolution of Cercignani's conjecture lead to new quantitative theories and proof of optimal relaxation rates in physical spaces, conditionnally to regularity+moments Carlen-Carvalho,Toscani,Wennberg,Villani...

Condition relaxation (II)

- In view of Cercignani's conjecture on entropy production and partial linearized result, one arrives at the conjecture:
- **Conjecture of conditional relaxation.** Any a priori solution to the Boltzmann (resp. Landau) equation in $L^\infty_x(\mathbb{T}^3; L^1_v(\mathbb{R}^3, (1+|v|)^k \, \mathrm{d}v))$, $k > 2$, converges to thermodynamical equilibrium with the optimal rate dictated by the linearized equation
- Ukai, Arkeryd, Desvillettes-Villani...
  Baranger-CM, CM, Gualdani-Mischler-CM (Mémoire de la SMF 2017) final answer hard spheres (factorization of non-symmetric semigroups)
- In progress for Landau equation and Boltzmann equation with long-range interactions: Carrapatoso, Mischler, Tristani, Wu...

Understanding regularity

- For long-range interactions, the Boltzmann and Landau-Coulomb operators show local ellipticity conditionnally to pointwise bounds on the local hydrodynamical fields and entropy:

$$\rho(t,x) := \int_{\mathbb{R}^3} f \,\mathrm{d}v, \quad e(t,x) := \int_{\mathbb{R}^3} f\,|v|^2 \,\mathrm{d}v, \quad h(t,x) := \int_{\mathbb{R}^3} f \ln f \,\mathrm{d}v$$

- Clear locally for the Landau-Coulomb operator, and understood two decades ago in the case of the Boltzmann collision operator: Desvillettes, Lions, Villani, Alexandre, Wennberg...
- This had lead colleagues working on non-local operators and fully nonlinear elliptic problems like Silvestre and Gualdani/Guillen to attempt to use barriers' techniques reminiscent to the Krylov-Safonov theory in order to obtain pointwise bounds for solutions

Conditional regularity

- These first attempts, while unsuccessful, later proved crucial in attracting the attention of a larger community on this problem, and lead to reformulate the initial goal into the conjecture:
- **Conditional regularity.** Consider any solution to the Boltzmann equation with long-range interactions (resp. Landau equation) on a time interval $[0, T]$ such that its hydrodynamical fields are bounded:

$$\forall\, t \in [0, T],\ x \in \mathbb{T}^3, \quad m_0 \le \rho(t, x) \le m_1, \quad e(t, x) \le e_1, \quad h(t, x) \le h_1$$

  then the solution is bounded and smooth on $(0, T]$
- The conjecture can be strenghtened by relaxing first inequality into $\int \rho(t, x)\,\mathrm{d}x \ge M_0 > 0$ (possibility of vacuum)
- It can also be weakened by assuming more at $t = 0\ldots$
- Related to understanding super-criticality (cf. Beale-Kato-Majda. . .)
- Shows that development of singularity, if any, must happen at the level of hydrodynamics
- (For hard spheres: conditional propagation of regularity open)

Results obtained so far

on the conditional regularity conjecture

First breakthrough

Theorem (Golse-Imbert-CM-Vasseur, Ann. Scuol. Norm. Pisa 2018)

Given f bounded weak solution in $B_1 \times B_1 \times (-1, 0]$ so that hydrodynamical field are bounded on $[0, T] \times \mathbb{T}^3$, then f is α-Hölder continuous with respect to $(x, v, t) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2}, 0]$ and

$$\|f\|_{C^\alpha(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (-\frac{1}{2},0])} \leq C \left(\|f\|_{L^2(B_1 \times B_1 \times (-1,0])} + \|f\|^2_{L^\infty(B_1 \times B_1 \times (-1,0])} \right)$$

- First breakthrough that relies on extending De Giorgi-Nash-Moser theory to hypoelliptic operators, see later
- Shortcomings: local, pointwise bounds, higher regularity
- Previous premonitory non-published study by Villani 2003 (Cours Peccot at Collège de France) with Nash techniques on BE

State of the art on the conjectures

- Cameron-Silvestre-Snelson 2017: LE for moderately soft potentials $\gamma \in (-2, 0]$, proof of pointwise bounds from the hydrodynamical controls
- Henderson-Snelson 2017: LE for moderately soft potentials, proof of higher regularity for a priori solutions with Gaussian decay
- Silvestre 2016: BE with moderately soft potentials $\gamma + 2s > 0$, proof of pointwise bounds from the hydrodynamical controls
- Imbert-Silvestre 2018: BE with moderately soft potentials, proof of local Hölder regularity from the hydrodynamical controls
- Imbert-CM-Silvestre 2018: BE with moderately soft potentials or hard potentials, proof of propagation and appearance (hard potentials) of pointwise polynomial decay in v
- Use of: isoperimetric argument from the first breakthrough (see later), nonlinear maximum principle techniques, modified Schauder estimates
- Conjecture $\sim$ complete in moderately soft, still open for very soft...

The first breakthrough

De Giorgi-Nash-Moser meets Hörmander

Hilbert 19th problem

- Hilbert's 19th problem: analytic regularity of minimizers u of an energy functional $\int_U L(\nabla u)\,\mathrm{d}x$, where $L : \mathbb{R}^d \to \mathbb{R}$ Lagrangian satisfies growth, smoothness and convexity conditions
- Euler-Lagrange equations for the minimizers take the form

$$\nabla \cdot \Big[\nabla L(\nabla u)\Big] = 0 \quad \text{i.e.} \quad \sum_{ij} \underbrace{\big[(\partial_{ij} L)(\nabla u)\big]}_{b_{ij}} \partial_{ij} u = 0$$

- Dirichlet energy $L(p) = |p|^2$, minimal surfaces $L(p) = \sqrt{1 + |p|^2}$
- With suitable assumptions on L and the domain, control of ∇u
- However existence-uniqueness-regularity requires more: if $u \in C^{1,\alpha}$ with $\alpha > 0$ then $b_{ij} \in C^\alpha$ and Schauder estimates imply $u \in C^{2,\alpha}$ (then bootstrap yields higher regularity. . .)

De Giorgi and Nash result

- Equation on derivative $f := \partial_k u$ (divergence form):

$$\sum_{ij} \partial_i \Big[\underbrace{(\partial_{ij} L)(\nabla u)}_{a_{ij}} \partial_j f \Big] = 0$$

- De Giorgi 1956 – Nash 1958: with controls (but no regularity) on a_{ij} then $f = \nabla u$ is Hölder (Nash considered the parabolic case)
- Proof of De Giorgi: (1) iterative gain of integrability (2) "isoperimetric argument" to control oscillations
- Proof of Moser 1964: (1) iterative gain of integrability (2) relating positive and negative Lebesgue norms by studying $g := \ln f$
- Non-divergence theory by Krylov-Safonov not considered here: open problem to extend it to hypoelliptic operators

Hörmander's theory of hypoellipticity (1)

- Starting point: 3 pages note of Kolmogorov Annals of Math. 1934 "*Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung)*"
- This paper considered dimension $d = 1$ transport with constant drift and diffusion (thus sometimes called "Kolmogorov equation")

$$\partial_t f + v \cdot \partial_x f = \partial_v^2 f \quad \text{and fundamental solutions from } \delta_{x_0, v_0}$$

$$G(t, x, v) = \frac{\sqrt{3}}{2\pi t^2} \exp \left\{ - \frac{3|x - x_0 - t_{v_0} - t(v - v_0)/2|^2}{t^3} - \frac{|v|^2}{4t} \right\}$$

Hörmander's theory of hypoellipticity (2)

- Hörmander 1967's seminal paper starts from observing the regularisation of this fundamental solution and builds a general theory based on commutator estimates
- Regularisation Gevrey instead of analytic for parabolic equations
- Simpler case when no first order part and missing directions of diffusion ("Hörmander type I"): DGNM theory already extended
- Hörmander original theory is local but recently global estimates derived under the impulsion of hypocoercivity
- Example of commutator estimates in a (very) simple case:

$$\partial_t f + Bf + A^* A f = 0, \quad B = v \cdot \partial_x, \quad A = \partial_v$$

$$[A, B] = C = \partial_x, \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle Af, Cf \rangle = -\|Cf\|^2 + \dots$$

Extension of De Giorgi-Nash-Moser to hypoelliptic PDEs

Theorem (GIMV)

(Simplified) Equation $\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A(t,x,v)\nabla_v f)$

where the $d \times d$ *symmetric matrix* A *satisfies the ellipticity condition* $0 < \lambda Id \leq A \leq \Lambda Id$ *but is, besides that, merely measurable.*
We define for $z = (t,x,v)$ *the cube* $Q_r(z) = B_{r^3}(x) \times B_r(v) \times (t - r^2, t]$.
Then for $0 < r_1 < r_0$, *if* f *is a solution in* $Q_{r_0(z_0)}$ *then*

$$\|f\|_{L^\infty(Q_{r_1}(z_0))} + \|f\|_{C^\alpha(Q_{r_1}(z_0))} \leq C\|f\|_{L^2(Q_{r_0}(z_0))}$$

where C *depends on* $z_0, r_0, r_1, \lambda, \Lambda, d$ *and* $\alpha \in (0,1)$ *depends on* λ, Λ, d.

Gain of L^∞ in [Pascucci-Polidoro 2004]
Related Hölder regularity results in [Wang-Zhang 2011]

The De Giorgi-Moser iteration (elliptic case)

- We consider, with $f = f(v)$ and g source term nicely behaved:

$$\nabla_v \left(A(v) \nabla_v f \right) = g$$

- Core energy estimate (valid for subsolutions):

$$\|f\|_{H^1(Q_{r_1})} \lesssim \frac{1}{(r_0 - r_1)^2} \|f\|_{L^2(Q_{r_0})} + \|g\|_{L^2(Q_{r_1})}$$

- Sobolev embedding translates the gain H^1 into L^p, $p > 2$
- Iteration by applying the argument to any subsolution $f^{p/2}$, $p \geq 2$, for a sequence of radii $r_n \to r_\infty > 0$, to get finally L^∞ in Q_{r_∞}
- Uses the ellipticity of the operator in all directions $v \in \mathbb{R}^d$

The De Giorgi-Moser iteration (parabolic case)

- Parabolic case (one step closer to our setting) with $f = f(v,t)$:

$$\partial_t f = \nabla_v \left(A(v,t) \nabla_v f \right)$$

- Core energy estimate:

$$\left(\int_{v \in B_{r_1}} f^2 \, \mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{v \in B_{r_1}} |\nabla_v f|^2 \, \mathrm{d}v \, \mathrm{d}t \lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{v \in B_{r_0}} f^2 \, \mathrm{d}v \, \mathrm{d}t$$

- Similar iteration argument in both variables v, t
- Again uses ellipticity of the operator in all directions $v \in \mathbb{R}^d$

Iteration in the hypoelliptic case

- Coming back to our equation $\partial_t f + v \cdot \nabla_x f \leq \nabla_v (A \nabla_v f)$ we derive the corresponding energy estimate:

$$\left(\int_{x \in B_{r_1^3}} \int_{v \in B_{r_1}} f^2 \,\mathrm{d}x \,\mathrm{d}v \right)_{t=T} + \int_{T-r_1^2}^{T} \int_{x \in B_{r_1^3}} \int_{v \in B_{r_1}} |\nabla_v f|^2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t$$

$$\lesssim \frac{1}{(r_0 - r_1)^2} \int_{T-r_0^2}^{T} \int_{x \in B_{r_0^3}} \int_{v \in B_{r_0}} f^2 \,\mathrm{d}x \,\mathrm{d}v \,\mathrm{d}t$$

- Problem 1: control only on v-gradients, not x-gradients
- Key tool in kinetic theory to remedy this: averaging lemma Agoshkov 1984, Golse-Perthame-Sentis 1985
- Problem 2: the iteration requires to work on subsolutions ($f^{p/2}$, $p > 2$) for which averaging lemma do not hold in general

Averaging lemma

Theorem (Averaging lemma)

$$\partial_t f + v \cdot \nabla_x f = (1 - \Delta_{t,x})^\beta \nabla_v^k g, \quad f, g \in L^p_{t,x,v}, \quad k \geq 0, \ \beta \in (0, 1/2)$$

implies regularity (for $p > 1$) on $\int_v f \, \mathrm{d}v \in W^{s,p}_{t,x}$ *($s > 0$ small)*

- Averages "transversal" to cancellations of symbol of the hyperbolic transport operator (gain of regularity limited by order 1 of operator)
- It degenerates if RHS g not controlled $\Rightarrow$ problem for subsolutions

$$\partial_t f + v \cdot \nabla_x f \leq \nabla_v \cdot H_0 + H_1 \text{ with } H_0, H_1 \in L^2$$

- Comparison principle: $0 \leq f \leq F$ with true solution F on which energy estimate $L^2_{t,x} H^1_v$ and averaging lemma $H^s_{t,x} L^1_v$ imply $H^{s'}_{t,x,v}$ $(0 < s' < s)$ and thus some gain of integrability $L^{p>2}$ by Sobolev embedding, inherited by f

Control of oscillations

- De Giorgi's strategy: always consider oscillation as a whole without separating controls on suprema and infima, and control decrease of oscillation when reducing the size of the cube considered
- Main Lemma of decrease of oscillations: for f solution in Q_2 with $|f| \leq 1$ then $\mathrm{osc}_{Q_{1/2}} f \leq 2 - \delta$ for some $\delta > 0$
- It implies Hölder regularity at the point at which cubes shrink
- It is implied by the following Lemma of decrease of supremum bound: for f solution in Q_2 with $|f| \leq 1$ and $|\{f \leq 0\} \cap Q_1| \geq (1/2)|Q_1|$ then $\sup_{Q_{1/2}} f \leq 1 - \delta$
- This decrease of the supremum bound follows from...

De Giorgi's isoperimetric argument (1)

Lemma (Intermediate-value)

Consider $f \in H^1$ on Q_2 with $f \leq 1$ and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 > 0 \quad \text{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2 > 0$$

then there is $\nu > 0$ depending on δ_1, δ_2 and the H^1 norm so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \nu$$

- Original statement is proved by constructive direct calculation
- In our setting we do not have an H^1 bound in all variables, and H^s with small $0 < s < 1/2$ seems insufficient
- We argue by contradiction for solutions to the equation

De Giorgi's isoperimetric argument (2)

Lemma (Hypoelliptic version of the intermediate-value lemma)

For all $\delta_1, \delta_2 > 0$ and $f \leq 1$ solution of our equation on Q_2 and

$$\left| \left\{ f \geq \frac{1}{2} \right\} \cap Q_1 \right| \geq \delta_1 \quad \text{and} \quad |\{f \leq 0\} \cap Q_1| \geq \delta_2$$

there is $\nu > 0$ depending on δ_1, δ_2 and the bounds on A so that

$$\left| \left\{ 0 < f < \frac{1}{2} \right\} \cap Q_1 \right| \geq \nu$$

- Contradiction argument: sequence f_k, A_k (oscillations of diff. matrix)
- We use refined regularity results to get compactness and ideas from homogeneisation to identify the limit
- Interesting open problem to get a constructive proof

Beyond conditional regularity:

Breaking super-criticality on a toy model

A toy nonlinear model

Toy nonlinear model for the Landau-Coulomb equation in $x \in \mathbb{T}^d$:

$$\partial_t f + v \cdot \nabla_x f = \rho[f] \nabla_v \cdot (\nabla_v f + v f)$$

Equiibrium solution $\mu(v) := e^{-|v|^2/2}$ and $\rho[f](t,x) := \int f(t,x,v) \mathrm{d}v$

Theorem (Imbert-CM 2018)

This equation is globally well-posed for $f_{in} \in H^k$, $k \geq d/2$, with $C_1 \mu \leq f_{in} \leq C_2 \mu$, and the unique solutions are C^∞ for positive times.

Goal: developping a methodology for future study
(1) Blow-up criterion by energy estimate with interpolation
** blow-up controlled by pointwise bound on v-derivative **
(2) Integral-to-pointwise bounds (iteration or barrier)
(3) Hölder regularity (oscillation)
(4) Schauder estimate (hypoelliptic trajectorial estimates)

Pointwise control

- Freeze ρ then solutions f to $\partial_t f + v \cdot \nabla_x f = \rho \nabla_v \cdot (\nabla_v f + vf)$ preserve sign (e.g. positive/negative parts are sub-solutions)
- Linearity of the equation and μ steady state implies that if $C_1\mu \leq f_{in}(\cdot,\cdot) \leq C_2\mu$ then $C_1\mu \leq f(t,\cdot,\cdot) \leq C_2\mu$
- Hence L^∞ bound free for this toy model without De Giorgi-Nash
- If solution satisfies $C_1\mu \leq f(t,\cdot,\cdot) \leq C_2\mu$ then bounds of ellipticity on the coefficient: $C_1 \leq \rho(t,\cdot) \leq C_2$
- The latter bound opens the way for the study of Hölder regularity along the line of our previous theorem

Energy estimates

Denote $\mu(v) := (2\pi)^{-d/2} e^{-|v|^2/2}$ and change unknown $g := f\mu^{-1/2}$:

$$\partial g + v \cdot \nabla_x g = r[g] L[g] \quad \text{with} \quad r[g] := \int_v g \mu^{1/2} \, \mathrm{d}v$$

$$\text{and} \quad L[g] := \mu^{-1/2} \nabla_v (\mu \nabla_v (\mu^{-1/2} g)) = \left(\Delta_v g + \frac{d}{2} g - \frac{|v|^2}{4} g \right)$$

Natural space of symmetry: $L^2(\mathrm{d}x \, \mathrm{d}v)$.
Denote $h := \mu^{1/2} \nabla_v (\mu^{-1/2} g)$, and write energy estimates:
At the zero-th derivative level:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{x,v} |g|^2 \leq -C_1 \int_{x,v} |h|^2$$

where we used $\int L[g] g = \int \mu^{-1/2} \nabla_v (\mu \nabla_v (\mu^{-1/2} g)) g = -\int |h|^2$

Blow-up criterion (I)

Study high-order v derivative for $\ell \geq 1$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v}|\partial_{v_i}^{\ell}g|^2 = -\ell\int_{x,v}(\partial_{v_i}^{\ell-1}\partial_{x_i}g)\partial_{v_i}^{\ell}g - \int_{x,v} r[g]\left|\nabla_v\left(\frac{\partial_{v_i}^{\ell}g}{\sqrt{\mu}}\right)\right|^2\mu$$
$$+\frac{1}{4}\binom{\ell}{1}\int_{x,v} r[g]\left|\partial_{v_i}^{\ell-1}g\right|^2 + \frac{1}{2}\binom{\ell}{2}\int_{x,v} r[g]\left|\partial_{v_i}^{\ell-1}g\right|^2$$

Using $\int_{x,v}(\partial_{v_i}^{\ell-1}\partial_{x_i}g)\partial_{v_i}^{\ell}g \lesssim \int_{x,v}|\partial_{v_i}^{\ell}g|^2 + \int_{x,v}|\partial_{x_i}^{\ell}g|^2$ and the control $r[g] \lesssim 1$ this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v}|\partial_{v_i}^{\ell}g|^2 \lesssim_k \|g\|_{H^{\ell}_{x,v}}^2$$

Blow-up criterion (II)

Study high-order x derivatives for $k > d/2$: since x-derivatives commute with the operators $v \cdot \nabla_x$ and Fokker-Planck

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{x,v}|\partial_{x_i}^k g|^2 = \sum_{0\leq\beta\leq k}\binom{k}{\beta}\int_{x,v}\partial_{x_i}^{k-\beta} r[g]\partial_{x_i}^{\beta} L[g]\partial_{x_i}^k g$$

$$\lesssim_k -C_1\int_{x,v}|\partial_{x_i}^k h|^2 + \sum_{0\leq\beta_i<k}\int_{x,v}|\partial_{x_i}^{k-\beta_i} r[g]|.|\partial_{x_i}^{\beta_i} h||\partial_{x_i}^k h|$$

where we used

$$\int L[\partial_x^* g]\partial_x^{**} g = \int \mu^{-1/2}\nabla_v(\mu\nabla_v(\mu^{-1/2}\partial_x^* g))\partial_x^{**} g = -\int \partial_x^* h\partial_x^{**} h$$

Blow-up criterion (III)

Standard interpolation: given $\beta < k$, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ s.t.

$$\left\| \partial_x^{k-\beta} r \partial_x^\beta h \right\|_{L^2(\mathbb{T}^d)} \leq \varepsilon \|r\|_{L^\infty_x(\mathbb{T}^d)} \|h\|_{H^k_x(\mathbb{T}^d)} + C_\varepsilon \|r\|_{H^k_x(\mathbb{T}^d)} \|h\|_{L^\infty_x(\mathbb{T}^d)}.$$

Using lower and upper bounds on $r[g]$ and the negative term in the previous estimate we get with ε small enough:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{x,v} |\partial_{x_i}^k g|^2 \leq -C_1 \int_{x,v} |\partial_{x_i}^k h|^2 + C_2 \varepsilon \|h\|_{H^k_x L^2_v} \left(\int_{x,v} |\partial_{x_i}^k h|^2 \right)^{1/2} \\ + C_\varepsilon \|r\|_{H^k_x} \|h\|_{L^\infty_x L^2_v} \left(\int_{x,v} |\partial_{x_i}^k h|^2 \right)^{1/2}\end{aligned}$$

which yields summing up to k and taking ε small:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g\|^2_{H^k_x L^2_v} \lesssim -\frac{C_1}{2} \|h\|^2_{H^k_x L^2_v} + \|h\|^2_{L^\infty_x L^2_v} \|g\|^2_{H^k_x L^2_v}$$

Local-in-time and continuation

Local well-posedness:
Standard with $\ell = k > d/2$ and Sobolev embedding on $\|h\|^2_{L^\infty_x L^2_v}$ (either decompose then in terms of g or use time-integrability from previous negative terms)

Continuation requires a $L^\infty_x L^2_v$ bound on $h = \nabla_v g + (v/2) g$

This is where we use the extension of the De Giorgi-Nash theory

We obtain Hölder regularisation first using the previous result

However we need the pointwise control of a full derivative hence we develop hypoelliptic Schauder estimates

Pointwise norms respecting hypoelliptic scaling

$$\text{Scaling:} \quad z := (t, x, v), \qquad rz := (r^2 t, r^3 x, rv).$$

(changing the operator $\partial_t - v \cdot \nabla_x - \Delta_v$ by a factor r^2)

Translation: $z_1 \circ z_2 = (t_1, x_1, v_1) \circ (t_2, x_2, v_2) := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)$

Scaled cylinder

$$\begin{aligned} Q_r(z_0) &:= \left\{ z : \frac{1}{r}(z_0^{-1} \circ z) \in Q_1 \right\} \\ &= \left\{ (t, x, v) : |t - t_0| \leq r^2, \, |x - x_0 - (t - t_0) v_0| \leq r^3, \, |v - v_0| \leq r \right\} \end{aligned}$$

Hölder norms $\mathcal{C}(\mathcal{Q})$ based on these cylinders and higher-order norm

$$[g]_{\mathcal{H}^\alpha(\mathcal{Q})} := [\partial_t g + v \cdot \nabla_x g]_{\mathcal{C}^\alpha(\mathcal{Q})} + [D_v^2 g]_{\mathcal{C}^\alpha(\mathcal{Q})}$$

Hypoelliptic estimates on trajectories (I)

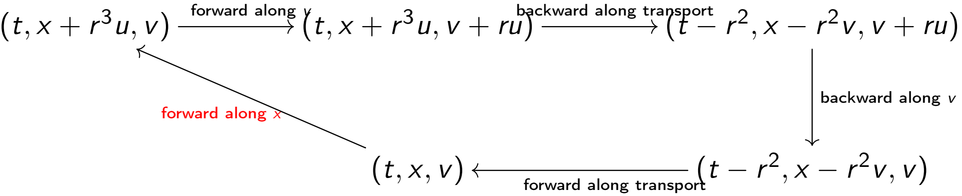
Lemma

The hypoelliptic Hölder regularity along free transport and v-diffusion allow to recover the following full (i.e. in all directions) pointwise controls:

$$[g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})},$$
$$[\nabla_v g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

The main difficulty is to obtain the Hölder regularity on the x and t directions from the higher regularity along the directions $\partial_t + v \cdot \nabla_x$ and ∇_v. This is an hypoelliptic commutator estimate in disguise. Take two points $z_1 \in Q_r(z_0) \subset \mathcal{Q}$ with $z_1 = z_0 + (0, r^3 u, 0)$ and $z_0 = (t, x, v)$ with $u \in \mathbb{S}^{d-1}$ and $r > 0$, and follow

Hypoelliptic estimates on trajectories (II)



(observe that all four points $(t, x + r^3 u, v)$, $(t, x + r^3 u, v + ru)$, $(t - r^2, x - r^2 v, v + ru)$, $(t - r^2, x - r^2 v, v)$ belong to $Q_r(z_0)$ with $z_0 = (t, x, v)$))

Write then the four Taylor expansions for g, and bootstrap a similar reasoning for the variations of $\nabla_v g$