Transfer operator approach to 1d random band matrices

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ICM 2018
Rio de Janeiro 06.08.2018

Band matrices: simplest model

H - hermitian or real symmetric $N \times N$ matrices with independent (up to the symmetry condition) entries H_{ij} such that

$$E\{H_{ij}\} = 0, \ \mathrm{Var}\{H_{ij}\} = (2W)^{-1} 1_{|i-j| \le W}$$

$$H = \begin{pmatrix}
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square
\end{pmatrix}$$

We are going to study the regimes $W \to \infty$, $W/N \to 0$, as $N \to \infty$,
Crossover in the local behavior is expected for $W^2 \sim N$

Band matrices: general definition

H - hermitian or real symmetric $N \times N$ matrices with independent (up to the symmetry condition) entries H_{ij} such that $E\{H_{ij}\} = 0$,

$$E\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}W^{-d}J((i-j)/W), \quad i,j \in \mathbb{Z}^d$$

and $J \in L_1(\mathbb{R}^d)$ is a piece-wise continuous function (with a finite number of jumps), satisfying the conditions

$$J(x) = J(|x|), \quad 0 \le J(x) \le C, \quad W^{-d}\sum_j J(j/W) \to 1, \text{u is continuous at} \, x = 0$$

Our model-1 (RBM)

$$\mathbb{E}\{H_{ij}H_{lk}\} = \delta_{ik}\delta_{jl}\big(-W^2\Delta + 1\big)^{-1}_{ij} \sim W^{-1}e^{-|i-j|/W},$$

Our model-2: 1d Wegner type band matrix (RBBM)

H is $N \times N$ hermitian block matrix which has n blocks of the size $W \times W$ $(N = nW)$ in the main diagonal. Only 3 block diagonals are non zero.

$$H = \begin{pmatrix}
\Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * \\
0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & \Box & \Box & \Box & \Box
\end{pmatrix}$$

where $\Box$s mean independent normal variables with the variance $(1-2\alpha)/W$ and $*$s mean independent normal variables with the variance α/W with some $0 < \alpha < \frac{1}{4}$. Crossover in the local behaviour is expected for $W^2 \sim N \Leftrightarrow W \sim n$

Global regime: results

The main objects of the global regime are linear eigenvalue statistics (LES), corresponding to the different test functions h:

$$\mathcal{N}_N[h] = \sum h(\lambda_i), \quad \text{where} \quad \{\lambda_i\}_{i=1}^N - \text{eigenvalues of } H$$

Limit of LES ([Molchanov,Khorunzhy,Pastur:92])

$$\lim_{N,W\to\infty} N^{-1}\mathcal{N}_N(h) = \int h(\lambda)\rho(\lambda)d\lambda,$$

$$\text{where} \quad \rho(\lambda) = 1_{[-2,2]}(2\pi)^{-1}\sqrt{4-\lambda^2}$$

CLT for LES [MS:15]

If $h \in \mathcal{H}_s$ with $s > 2$, then

$$\sqrt{W/N}(\mathcal{N}_N[h] - E\{\mathcal{N}_N[h]\}) \to V(J)\mathcal{N}(0,1)$$

Previous results:

L.Li, A. Soshnikov (2013), and I. Jana, K. Saha, and A.Soshnikov (2014): CLT for band matrices with $W^2 >> N$;

Local regime. Main objects

Spectral correlation functions

Let $p_n(\lambda_1, \ldots, \lambda_N)$ be a joint eigenvalue distribution of some random matrix. Then k point correlation function is the kth marginal density of p_n:

$$R_k(\lambda_1, \ldots, \lambda_k) = \int p_n(\lambda_1, \ldots, \lambda_N) d\lambda_{k+1} \ldots d\lambda_N$$

Local eigenvalue distribution in the bulk

Let $\rho(\lambda)$ be a limiting density which appears in the studies of global regime. The support of ρ is called the spectrum. Let $\rho(E) \neq 0$. To study local eigenvalue statistics near E means to study for any k

$$\lim_{N\to\infty} (\rho(E))^{-k} R_k(E + x_1/N\rho(E), \ldots, E + x_k/N\rho(E))$$

GUE and Poisson local eigenvalue statistics

GUE type of local eigenvalue statistics

We say that the model possesses GUE local eigenvalue statistic, if

$$\lim_{N\to\infty}(\rho(E))^{-k}R_k(E+x_1/N\rho(\lambda),\dots E+x_k/N\rho(\lambda))=\det\left\{\frac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)}\right\}$$

The famous universality conjecture, formulated by Dyson in 60s and proven first for the matrix models ([PS:97] and [DKVZ:98]), then for Wigner and many other models with independent entries by Tao-Vu and Erdos-Yao with co-authors in 09-18.

Poisson type of local eigenvalue statistics

We say that the model possesses Poisson local eigenvalue statistic, if

$$\lim_{N\to\infty}(\rho(\lambda))^{-k}R(E+x_1/N\rho(\lambda),\dots,E+x_k/N\rho(\lambda))=1$$

Localization of eigenvectors of random matrices

Localized and delocalized eigenvectors

Let $\Psi(\lambda) = \{\Psi_j(\lambda)\}_{j=1}^N$ be an eigenvector of the matrix H, corresponding to the eigenvalue λ. We say that Ψ is delocalized, if $\|\Psi\| = 1$ but for some $s > 2$

$$\sum_j |\Psi_j(\lambda)|^s \to 0, \quad N \to \infty.$$

Otherwise $\Psi(\lambda)$ is called localized. For most classical models of random matrices the eigenvectors are delocalized.

Exponentially localized eigenvectors

We say that $\Psi(\lambda)$ is exponentially localized, if there is j_0 such that

$$|\Psi_j| \sim Ce^{-|j-j_0|/\ell}$$

Then ℓ is the localization length.
The most famous example of the model with localized eigenvectors is 1d Anderson model

Link between eigenvector's localization and local eigenvalue statistics

Unproven conjecure

It is widely believed that for random matrices the Poisson local eigenvalue statistic corresponds to the case of the localization of eigenvectors, while the GUE local statistic corresponds to the case of delocalized eigenvectors.

"Anderson transition" for random band matrices (conjectures)

Let ℓ be a typical localization length of eigenvectors of H.

Conjecture (in the bulk of the spectrum):

$d = 1:$	$\ell \sim W^2$	$W^2 \gg N$	Delocalization, local GUE statistics
		$W^2 \ll N$	Localization, Poisson statistics
$d = 2:$	$\ell \sim e^{W^2}$	$W^2 \gg \log N$	Delocalization, local GUE statistics
		$W^2 \ll \log N$	Localization, local Poisson statistics
$d \geq 3:$	$\ell \sim N$	$W \geq W_0$	Delocalization, local GUE statistics

Previous results: $d = 1$

- Fyodorov, Mirlin (1991) – existence of the crossover for $W^2 \sim N$ (on the level of rigour of theoretical physics)
- Schenker (2009) $\ell \le W^8$ – localization techniques;
- Erdős, Knowles (2011): delocalization for $W >> N^{6/7}$;
- Erdős, Knowles, Yau, Yin (2012): delocalization for $W >> N^{4/5}$;
- T.Shcherbina (2013): GUE statistics for Wegner band matrix (fixed n);
- Bourgade, Erdős, Yau, Yin (2016) GUE statistics for $W \sim N$.
- Bourgade, Yang, Yau, Yin (2018) GUE statistics and delocalization for $W >> N^{3/4}$

- S.Sodin (2010): Edge universality iff $W >> N^{5/6}$

Main objects

"Generalised" correlation functions

$$\mathcal{R}_1(z_1, z_1') := \mathbb{E}\Big\{\frac{\det(H - z_1')}{\det(H - z_1)}\Big\}$$

$$\mathcal{R}_2(z_1, z_1'; z_2, z_2') := \mathbb{E}\Big\{\frac{\det(H - z_1')\det(H - z_2'))}{\det(H - z_1)\det(H - z_2))}\Big\}$$

We study these functions for

$$z_{1,2} = E + \xi_{1,2}/\rho(E)N, \quad z_{1,2}' = E + \xi_{1,2}'/\rho(E)N.$$

For our purposes it is sufficient to take $\xi_1 = -\xi_2 = \xi$ and $\xi_1' = -\xi_2' = \xi'$.

The spectral correlation functions R_1 and R_2 can be easily obtained from $\mathcal{R}_1$ and $\mathcal{R}_2$.

Correlation function of the characteristic polynomials:

$$\mathcal{R}_0(\lambda_1, \lambda_2) = \mathbb{E}\Big\{\det(H - \lambda_1)\det(H - \lambda_2)\Big\}, \quad \lambda_{1,2} = E \pm \xi/n.$$

Integral representations for $\mathcal{R}_{0,1,2}$

There are a scalar kernel $\mathcal{K}_0(X_1,X_2)$, 2×2 matrix kernel $\mathcal{K}_1(X_1,X_2)$, and 70×70 matrix kernel $\mathcal{K}_2(X_1,X_2)$ (containing $z_{1,2}, z'_{1,2}$ as parameters) such that

$$\mathcal{R}_\delta = C_{N,W}\int g_\delta(X_1)\mathcal{K}_\delta(X_1,X_2)\dots\mathcal{K}_\delta(X_{N-1},X_N)f_0(X_N)\prod dX_i, \quad \delta=0,1,2$$

where

$$\begin{array}{llll} \text{for } \delta=0 & X_j=(x_j,y_j,U_j), & x_j,y_j\in\mathbb{R}, & U_j\in\mathring{U}(2) \\ \text{for } \delta=1 & X_j=(x_j,y_j), & x_j,y_i\in\mathbb{R}, & \\ \text{for } \delta=2 & X_j=(x_j,y_j,U_j,S_j,), & x_j,y_j\in\mathbb{R}^2 & U_j\in\mathring{U}(2),\ S_j\in\mathring{U}(1,1) \end{array}$$

dX means an integration over the Haar measure of X,

Recall that

$$S\in\mathring{U}(1,1) \quad \Leftrightarrow \quad S^*LS=L, \quad L=\mathrm{diag}\{1,-1\}$$

Idea of the transfer operator approach

Observation

Let $\mathcal{K}(X, Y)$ be the p-dimensional matrix kernel of the compact integral operator in $\oplus_{i=1}^{p} L_2[X, d\mu(X)]$. Then

$$\int g(X_1)\mathcal{K}(X_1, X_2) \ldots \mathcal{K}(X_{n-1}, X_n) f(X_n) \prod d\mu(X_i) = (\mathcal{K}^{n-1} f, \bar{g})$$
$$= \sum_{j=0}^{\infty} \lambda_j^{n-1}(\mathcal{K}) c_j, \quad \text{with} \quad c_j = (f, \psi_j)(g, \tilde{\psi}_j) \tag{1}$$

Here $\{\lambda_j(\mathcal{K})\}_{j=0}^{\infty}$ are the eigenvalues of $\mathcal{K}$ ($|\lambda_0| \geq |\lambda_1| \geq \ldots$), ψ_j are corresponding eigenvectors and $\tilde{\psi}_j$ are the eigenvectors of $\mathcal{K}^*$

Main technical difficulties

- $\mathcal{K}_{0,1,2}$ are not self adjoint operators, hence we can not use a standard perturbation theory;
- $\mathcal{R}_0$ contains the integration over unitary group $\mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$, and $\mathcal{R}_2$, contains the integration over unitary and hyperbolic $(\mathrm{U}(1,1)/\mathrm{U}(1) \times \mathrm{U}(1))$ groups, hence we need to work with corresponding special functions;
- $\mathcal{K}_1$ is a 2×2 Jordan type matrix, and $\mathcal{K}_2$ is a $2^8 \times 2^8$ matrix kernel, containing 4×4 Jordan type matrix in the main block.
  Using the symmetry of the problem, $\mathcal{K}_2$ could be replaced by 70×70 matrix kernel.

Resolvent version of the transfer operator approach

Observation 2

$$(\mathcal{K}^n f, \bar{g}) = -\frac{1}{2\pi i}\oint_L z^n (\mathcal{G}(z) f, \bar{g}) dz, \quad \mathcal{G}(z) = (\mathcal{K} - z)^{-1}$$

where L is any closed contour which contains all eigenvalues of $\mathcal{K}$.

Choose $L = L_1 \cup L_2$, where $L_2 = \{z : |z| = 1 - \log^2 n/n\}$, and L_1 is some special contour, enclosing the eigenvalues outside L_2.
Then, since $|z|^n \le e^{-\log^2 n}$ for $z \in L_2$, the integral over L_2 is neglectable.

Definition of asymptotically equivalent operators $(n, W \to \infty)$

$$\mathcal{A}_{Wn} \sim \mathcal{B}_{Wn} \quad \Leftrightarrow \oint_{L_1} z^n ((\mathcal{A}_{Wn} - z)^{-1} f, \bar{g}) dz = \oint_{L_1} z^n ((\mathcal{B}_{Wn} - z)^{-1} f, \bar{g}) dz + o(1)$$

for our L_1

Mechanism of the crossover for $\mathcal{R}_0$

Key technical step

$$\mathcal{K}_0 \sim \mathcal{K}_* \otimes \mathcal{A}, \quad \mathcal{K}_* : L_2(\mathring{U}(2)) \to L_2(\mathring{U}(2)), \quad \mathcal{A} : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$$

$$\mathcal{K}_*(\xi)(U_1, U_2) = e^{-i\xi\nu(U_1)/N} K_*(U_1 U_2^*) e^{-i\xi\nu(U_2)/N},$$

$$f = f_0(U) \otimes f_1(x, y), \quad g = g_0(U) \otimes g_1(x, y)$$

Then

$$\mathcal{R}_0 = (\mathcal{K}_*^{N-1}(\xi) \otimes \mathcal{A}^{N-1} f, \bar{g})(1 + o(1)) = (\mathcal{K}_*^{N-1}(\xi) f_0, f_0)(\mathcal{A}^{N-1} f_1, \bar{g}_1)(1 + o(1)).$$

If we introduce the normalization constant

$$D_2 = \mathcal{R}_0(E, E).$$

then

$$D_2^{-1} \mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(\mathcal{K}_*^{N-1}(\xi) f_0, g_0)}{(\mathcal{K}_*^{N-1}(0) f_0, g_0)}(1 + o(1))$$

Spectral analysis of $\mathcal{K}_*(\xi)$

A good news is that K_* with a kernel

$$K_* = t_* W^2 e^{-t_* W^2 |(U_1 U_2^*)_{12}|^2}$$

is a self-adjoint "difference" operator. It is known that his eigenfunctions are Legendre polynomials P_j. Moreover, it is easy to check that corresponding eigenvalues have the form:

$$\lambda_j = 1 - t_* j(j+1)/W^2 + O((j(j+1)/W^2)^2), \quad j = 0, 1 \dots .$$

Besides,

$$\mathcal{K}_*(\xi) = K_* - 2i\xi\hat{\nu}/N + O(N^{-2})$$

where $\hat{\nu}$ is the operator of the multiplication by ν. Thus the eigenvalues of $\mathcal{K}_*(\xi)$ are in the N^{-1}-neighbourhood of λ_j.

Mechanism of the Poisson behavior for $W^2 << N$

For $W^{-2} >> N^{-1}$ the spectral gap is much bigger than the norm of perturbation.

$$\lambda_0(\mathcal{K}_*(\xi)) = 1 - 2N^{-1} i\xi(\nu f_0, g_0) + o(N^{-1}),$$
$$|\lambda_1(\mathcal{K}_*(\xi))| \le 1 - O(W^{-2}) \quad \Rightarrow \quad |\lambda_j(\mathcal{K}_{*\xi})|^N \to 0, \ (j = 1, 2, \dots).$$

Since

$$(\nu f_0, g_0) = 0,$$

we obtain that

$$\lambda_0(\mathcal{K}_*(\xi)) = 1 + o(N^{-1}),$$

and

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{\lambda_0^N(\mathcal{K}_*(\xi))}{\lambda_0^N(\mathcal{K}_*(0))}(1 + o(1)) \to 1$$

The relation corresponds to the Poisson local statistics.

Mechanism of the GUE behavior for $W^2 >> N$

In the regime $W^{-2} << N^{-1}$ we have $K_*^N \to I$ in the strong vector topology, hence one can prove that

$$\mathcal{K}_*(\xi) \sim 1 + O(W^{-2}) - 2N^{-1} i\xi\nu \Rightarrow (\mathcal{K}_*(\xi)^N f_0, g_0) \to (e^{-2i\xi\hat{\nu}} f_0, g_0)$$

and

$$D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = \frac{(e^{-2i\xi\hat{\nu}} f_0, g_0)}{(f_0, g_0)}(1 + o(1)) \to \frac{\sin(2\pi\xi)}{2\pi\xi}.$$

The expression for $D_2^{-1}\mathcal{R}_0$ coincides with that for GUE.

Behavior $\mathcal{R}_0$ for $W^2 \sim N$

In the regime $W^{-2} = C_* N^{-1}$ we can write $\mathcal{K}_*(\xi)$ as

$$\mathcal{K}_*(\xi) \sim 1 - N^{-1}(C_* t_* \Delta_U + 2i\xi\nu) + o(N^{-1}) \Rightarrow (\mathcal{K}_*^N(\xi) f_0, g_0) \to (e^{-C_* t_* \Delta_U - 2i\xi\hat{\nu}} f_0, g_0)$$

where

$$\Delta_U = -\frac{d}{dx} x(1-x) \frac{d}{dx}, \quad x = |U_{12}|^2.$$

Results for $\mathcal{R}_0$

Theorem 1 [TS:14]

$N < W^{2-\theta}$, where $0 < \theta < 1$, and $E \in (-2, 2)$ we have

$$\lim_{N,W\to\infty} D_2^{-1}\mathcal{R}_0\Big(\lambda_0 + \frac{\xi}{N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{N\rho(\lambda_0)}\Big) = \frac{\sin(2\pi\xi)}{2\pi\xi},$$

i.e. the limit coincides with that for GUE. The limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$. Here

$$D_2 = \mathcal{R}_0(\lambda_0, \lambda_0).$$

Theorem 2 [TS,MS:16]

$N > CW^2 \log W$

$$\lim_{N,W\to\infty} D_2^{-1}\mathcal{R}_0\Big(\lambda_0 + \frac{\xi}{N\rho(\lambda_0)}, \lambda_0 - \frac{\xi}{N\rho(\lambda_0)}\Big) = 1$$

The limit is uniform in ξ varying in any compact set $C \subset \mathbb{R}$.

Theorem 3 [TS: in preparation]

For 1d RBM with $N = C_* W^2$, $E \in (-2, 2)$, we have

$$\lim_{N\to\infty} D_2^{-1}\mathcal{R}_0\Big(E + \frac{\xi}{N\rho(E)}, E - \frac{\xi}{N\rho(E)}\Big) = (e^{-C_* t_* \Delta_U - 2i\xi\hat{\nu}} f_0, g_0),$$

where $t^* = (2\pi\rho(E))^2$, and the limit is uniform in ξ varying in any compact subset of $\mathbb{R}$.

Result for $\mathcal{R}_1$

$$\mathcal{K}_1 \sim F(x_1,y_1)A_1(x_1,x_2)A_2(y_1,y_2)F(x_2,y_2)\begin{pmatrix} 1+L(\bar{x},\bar{y})/W^2 & -1 \\ -L(\bar{x},\bar{y}) & 1 \end{pmatrix}$$

Operators A_1 and A_2 contain a large parameter W in the exponent, hence only $W^{-1/2}$ neighbourhood of the stationary point (x^*,y^*) gives essential contribution. The function $L(\bar{x},\bar{y})$ here satisfies the relation

$$L(\bar{x},\bar{y}) = 0\Big|_{\bar{x}=\bar{x}^*,\bar{y}=\bar{y}^*}$$

Hence the main order of our operator is a Jordan matrix.
The spectral gap of $A_{1,2}$ is $O(W^{-1}) >> N^{-1}$, hence $A^N_{1,2} \sim \lambda_0^N(A_{1,2})P_{1,2}$ $(\mathrm{rank}\, P_{1,2} = 1)$

Theorem 3 [MS,TS:16]

Let $N >> W$. Then

$$|R_1(E) - \rho(E)| \le C/W$$

Sigma-model $\mathcal{R}_2^{(\sigma)}$

The model can be obtained by some scaling limit ($\alpha = \beta/\mathrm{W}$, $\mathrm{W} \to \infty$, β, n-fixed) from the expression for $\mathcal{R}_2$.
The crossover is expected for $\beta \sim \mathrm{n}$.

$$\mathcal{R}_2^{(\sigma)} = \int \exp\Big\{\frac{\beta}{4}\sum \mathrm{Str}\, \mathrm{Q_j Q_{j+1}} + \frac{\varepsilon + \mathrm{i}\xi}{4\mathrm{n}} \sum \mathrm{Str}\, \mathrm{Q_j}\Lambda\Big\} \\ \times \prod (1 - 2\rho_{1\mathrm{j}}\tau_{1\mathrm{j}}\rho_{2\mathrm{j}}\tau_{2\mathrm{j}}) \prod d\mathrm{Q_j}$$

Here $\mathrm{Q_j}$ is a 4×4 super matrix of some special form, depending on unitary matrix $\mathrm{U_j}$, hyperbolic matrix $\mathrm{S_j}$ and 4 Grassmann variables $\rho_{1\mathrm{j}}, \rho_{2\mathrm{j}}, \tau_{1\mathrm{j}}, \tau_{2\mathrm{j}}$.

Result for $\mathcal{R}_2^{(\sigma)}$

Theorem 4 [MS,TS:18] (published in JSP)

For the sigma-model in the regime $C\beta/\log^2\beta > n$

$$\lim_{n\to\infty} \mathcal{R}_2^{(\sigma)} = (\hat{F}_0\tilde{f}, \tilde{g})$$

where

$$\hat{F}_0 = F_0 \begin{pmatrix} 1 & F_1 & F_2 & F_1F_2 \\ 0 & 1 & 0 & F_2 \\ 0 & 0 & 1 & F_1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$F_0 \sim e^{\varphi(U,S)}, \quad F_{1,2} \sim \varphi_{1,2}(U,S)$$

$$\tilde{f} = (e_4 - e_1), \quad \tilde{g} = (e_1 - e_4)$$

Corollary

For $|E| \le \sqrt{2}$ the second order correlation function of RBBM with $\alpha = \beta/W$ in the limit $W \to \infty$ and then $\beta, n \to \infty$, $(\beta >> n)$ coincides with that for GUE.

Result for $\mathcal{R}_2$

After a rather involved asymptotic analysis we obtain

$$\mathcal{K}_2 \sim \tilde{\mathrm{F}}\hat{\mathrm{K}}_0\tilde{\mathrm{F}}$$

where $\hat{\mathrm{K}}_0$ and $\tilde{\mathrm{F}}$ are 4×4 matrices similar to that for sigma-model.

Theorem 5 [MS,TS:18] (in preparation)

For $|\mathrm{E}| \le \sqrt{2}$ and $\mathrm{W} > \mathrm{CN}^{1/2} \log \mathrm{N}$,

$$\lim_{n \to \infty} \mathcal{R}_2 = (\hat{\mathrm{F}}_0 \tilde{\mathrm{f}}, \tilde{\mathrm{g}})$$

where $\hat{\mathrm{F}}_0, \tilde{\mathrm{f}}, \tilde{\mathrm{g}}$ are the same as in Theorem 4.

Corollary

For $|\mathrm{E}| < \sqrt{2}$ the second order correlation function of 1d RBBM in the limit $\mathrm{N}, \mathrm{W} \to \infty$, $\mathrm{W}^2 / \log^2 \mathrm{W} > \mathrm{CN}$, coincides with that for GUE.