Two-dimensional Stochastic Interface Growth

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ICM 2018, Rio
Random discrete interfaces and growth

- 2d discrete interfaces $\rightarrow$ random tilings, dimer model
- Stochastic growth (random deposition).
  - Large scales $\rightarrow$ non-linear PDEs, stochastic PDEs, ...
- An interesting story: Wolf’s conjecture on universality classes of 2d interface growth
Random discrete interfaces and growth

Links with:

- macroscopic shapes
- facet singularities
- massless Gaussian field (GFF)
Discrete monotone interface
Lozenge tiling of the plane
Dimer model (perfect matching of planar bipartite graph)

Link with spin systems: ground state of 3d Ising model
Tilings & interlaced particles

Lozenge tiling ⇔ Interlaced particle system

The whole interface/dimer/lozenge picture is still there
A stochastic deposition model

Continuous-time Markov process. Updates:

Jumps respect interlacing conditions
A stochastic deposition model

Continuous-time Markov process. Updates:

- Jumps respect interlacing conditions
  - symmetric case $p = 1/2$: uniform measure is stationary & reversible
  - $p \neq 1/2$: growth model, irreversibility. Interesting in infinite volume (or with periodic boundary conditions)
  - equivalent to zero temperature Glauber dynamics of 3d Ising
    - $p \leftrightarrow$ magnetic field

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Stochastic Interface Dynamics
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Interface growth: phenomenological picture

Speed of growth $v = v(\rho)$: asymptotic growth rate for interface of slope $\rho \in \mathbb{R}^d$ (for us, $d = 2$)
Interface growth: phenomenological picture

Speed of growth $v = v(\rho)$: asymptotic growth rate for interface of slope $\rho \in \mathbb{R}^d$

$$v(\rho) = \lim_{t \to \infty} \frac{h(x,t) - h(x,0)}{t}$$
Interface growth: phenomenological picture

- As \( t \to \infty \), law of gradients
  \[
  \nabla h \equiv (h(x + \hat{e}_i) - h(x)), \quad x \in \mathbb{Z}^d, i = 1, \ldots, d
  \]
  should tend to limit stationary, non-reversible measure \( \pi_\rho \)
  
  E. g. \( v(\rho) = p \times \pi_\rho(\text{□}) - (1 - p) \times \pi_\rho(\text{●}) \)

- Roughness exponent \( \alpha \): at large distances
  \[
  \sqrt{\text{Var}_{\pi_\rho}(h(x) - h(y))} \sim c_1 + c_2 |x - y|^\alpha
  \]

- Growth exponent \( \beta \): at large times,
  \[
  \sqrt{\text{Var}(h(x,t) - h(x,0))} \sim c_3 + c_4 t^\beta
  \]
Fluctuation field and link with the KPZ equation

Heuristics: large-scales behavior of fluctuations $\rightsquigarrow$ Kardar-Parisi-Zhang equation

relaxes large fluctuations \hspace{2cm} tunes strength of non-linearity. Useful in perturbation theory

$$\partial_t h(x, t) = \Delta h(x, t) + \lambda (\nabla h(x, t), H \nabla h(x, t)) + \xi_{\text{smooth}}(x, t)$$

$d \times d$ symmetric matrix \hspace{2cm} smoothed space-time white noise

Quadratic non-linearity from second-order Taylor expansion of hydrodynamic PDE.

$$H = D^2 v(\rho) \quad (\text{Hessian of speed of growth})$$
Fluctuation field and link with the KPZ equation

\[ \partial_t h(x, t) = \Delta h(x, t) + \lambda (\nabla h(x, t), H \nabla h(x, t)) + \xi_{\text{smooth}}(x, t) \]

- Linear case ($\lambda = 0$): Edwards-Wilkinson (EW) equation.
  Stationary state: **massless Gaussian field.**
  \[ \alpha_{EW} = \frac{2 - d}{2}, \quad \beta_{EW} = \frac{2 - d}{4}. \]

- $d = 1$: KPZ ’86 predicted **relevance of non-linearity.**
  \[ \beta = \frac{1}{3} \neq \beta_{EW} \]

  Confirmed by exact solutions (1-d KPZ universality class: universal non-Gaussian limit laws, ...)

- $d \geq 3$: predicted **irrelevance of small non-linearity**, transition at $\lambda_c$.
  \[ \Rightarrow \text{see Magnen-Unterberger ’17, Gu-Ryzhik-Zeitouni ’17 for } \lambda \ll 1 \]
The critical dimension $d = 2$ and Wolf’s conjecture

\[ \partial_t h(x, t) = \Delta h(x, t) + \lambda (\nabla h(x, t), H \nabla h(x, t)) + \xi_{\text{smooth}}(x, t) \]

Perturbative (in $\lambda$) Renormalization-Group analysis (D. Wolf ’91):
- if $\det(H) > 0$, non-linearity relevant, $\alpha \neq \alpha_{\text{EW}}, \beta \neq \beta_{\text{EW}}$;
- if $\det(H) \leq 0$, small non-linearity irrelevant. EW Universality class.

Conjecture: Two universality classes:
- Anisotropic KPZ (AKPZ) class: $\det(D^2v(\rho)) \leq 0$.
  Large-scale fixed point: EW equation. $\alpha_{\text{AKPZ}} = 0, \beta_{\text{AKPZ}} = 0$.
- KPZ class: $\det(D^2v(\rho)) > 0$. $\alpha_{\text{KPZ}} \neq 0, \beta_{\text{KPZ}} \neq 0$.

Numerics (Halpin-Healy et al.): in KPZ class, universal exponents $\alpha_{\text{KPZ}} \approx 0.39..., \beta_{\text{KPZ}} \approx 0.24...$.

Rest of the talk: new results for AKPZ class
Back to the deposition process

Envelope property: \( h(t = 0) = h^{(1)} \lor h^{(2)} \implies h(t) = h^{(1)}(t) \lor h^{(2)}(t) \)

Then, superadditivity argument (T. Seppäläinen, F. Rezakhanlou) implies that \( v(\cdot) \) exists and is convex.

Natural candidate for KPZ class. No math results on stationary states or critical exponents \( \alpha_{KPZ}, \beta_{KPZ} \)
A long-jump variant

rate = \( p \)

rate = 1 - \( p \)

Jumps constrained only by interlacement conditions

A. Borodin & P. Ferrari '08

Should the universality class change? not obvious a priori.

In fact, it does change

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**Theorem (F.T., 15)** Stationary states $\pi_\rho$ are “locally uniform”

- Stationary states free-fermionic (determinantal correlations)
- Roughness exponent: $\alpha = 0$
  - logarithmic fluctuations,
  - scaling to massless Gaussian field
- Growth exponent $\beta = 0$
  - $\text{Var}_{\pi_\rho}(h(x, t) - h(x, 0)) \xrightarrow{t \to \infty} O(\log t)$
**Theorem (M. Legras, F.T. ’17)**

If

$$\lim_{\epsilon \to 0} \epsilon h(\epsilon^{-1}x, t = 0) = \phi_0(x), \quad \forall x \in \mathbb{R}^2$$

with $\phi_0(\cdot)$ convex, then

$$\lim_{\epsilon \to 0} \epsilon h(\epsilon^{-1}x, \epsilon^{-1}t) = \phi(x, t), \quad t > 0$$

(with high probability as $\epsilon \to 0$) where $\phi$ solves

\[
\left\{ \begin{array}{l}
\partial_t \phi(x, t) = v(\nabla \phi(x, t)) \\
\phi(x, 0) = \phi_0(x).
\end{array} \right.
\]

Speed of growth $v(\rho)$: explicit and $\det D^2v(\rho) < 0$
Comments on hydrodynamic equation

- Non-linear Hamilton-Jacobi equation $\Rightarrow$ singularities in finite time
- Physically relevant solution: viscosity solution.

\[ v(\nabla \phi) \mapsto v(\nabla \phi) + \epsilon \Delta \phi, \quad \epsilon \to 0^+ \]

- $v(\cdot)$ non convex $\Rightarrow$ no variational formula (like “minimal action”) for viscosity solution.
  - For convex profile, variational formula.
- Technical difficulty: long jumps, possible pathologies
  (tools: from works of T. Seppäläinen)
Previous results on the model

**Theorem** (A. Borodin, P. Ferrari ’08)
For “triangular-array Gibbs-type initial conditions”, hydrodynamic limit and central limit theorem on scale $\sqrt{\log t}$.
Smooth phases and singularities of $v(\cdot)$

For equilibrium 2d discrete interface models, smooth (or “rigid”) (as opposed to: rough) phases at special slopes
Exponential decay of correlations, no fluctuation growth:

$$\sup_x \text{Var}(h(x) - h(0)) < \infty,$$

E.g. SOS model at low temperature; dimers (“gas phases”),...
Questions:

- AKPZ growth models with smooth stationary states?
- We implicitly assumed that speed $v(\cdot)$ is differentiable ($H = D^2v$ in KPZ Eq.)
  - What if it is not? Still Edwards-Wilkinson behavior?
Together with S. Chhita, we studied a growth model where:

- height function is $h : \mathbb{Z}^2 \ni x \mapsto h(x) \in \mathbb{Z}$
- Growth process in discrete time: $h_0(\cdot), h_1(\cdot), h_2(\cdot), \ldots$
- Local update rule: $h_n(x) \to h_{n+1}(x) \sim$ random function of neighboring values
  \[
  h_n(y), \quad |y - x| = 1
  \]

Dynamics is domino-shuffling algorithm with 2-periodic weights (J. Propp)

- Stationary states $\pi_\rho$ of $\nabla h$ are
  - logarithmically rough for $\rho \neq 0$, i.e. $\text{Var}_{\pi_\rho}(h(x) - h(y)) \sim \log |x - y|$ 
  - smooth for $\rho = 0$, i.e. $\text{Var}_{\pi_0}(h(x) - h(y)) = O(1)$
An AKPZ model with a smooth phase

**Theorem (S. Chhita, F.T. ’18)**

For $\rho \neq 0$, AKPZ signature:

- Logarithmic growth of fluctuations:

  \[
  \text{Var}_{\pi_\rho}(h(x, t) - h(x, 0)) = O(\log t)
  \]

- Twice differentiable speed and

  \[
  \det(D^2 v(\rho)) < 0.
  \]

For $\rho = 0$, new picture:

- bounded fluctuations:

  \[
  \text{Var}_{\pi_0}(h(x, t) - h(x, 0)) = O(1)
  \]

- Non-differentiability of $v(\cdot)$ at 0
Smooth phases, facets and singularities of $v(\cdot)$

**Non-differentiability** related to facets of macroscopic shapes

\[ v(\rho) \xrightarrow{\rho \to 0} |\rho| f_1(\theta) + |\rho|^3 f_2(\theta) \]

related to “facet singularities”

non-differentiability

\[ h(x_0 + \varepsilon) \sim \varepsilon^{3/2} \]

Pokrovsky-Talapov law
Smooth phases, facets and singularities of $v(\cdot)$

**Non-differentiability related to facets** of macroscopic shapes

$$v(\rho) \overset{\rho \to 0}{\approx} |\rho| f_1(\theta) + |\rho|^3 f_2(\theta)$$

related to “facet singularities”

$h(x_0 + \epsilon) \sim \epsilon^{3/2}$

Pokrovsky-Talapov law
A more general AKPZ class

- Fluctuation & hydrodynamic results have been extended to other AKPZ models
- Puzzling points:
  - explicit computation of speed $\implies \det(D^2v) < 0$ without clear connection to Wolf’s heuristics.
  - speed is harmonic w.r.t. suitable complex structure

Any pattern behind?
AKPZ growth and Euler-Lagrange equation

A geometric argument behind $\det(D^2v(\rho)) \leq 0$ for AKPZ models (A. Borodin, F.T., '18)

- Common feature of most known AKPZ growth models: stationary, non-reversible Gibbs measures $\pi_\rho$:
  \[ \nabla h(t = 0) \sim \pi_\rho \implies \nabla h(t) \sim \pi_\rho \]

- Gibbs states $\pi$: probability measures such that
  law of $h(x)$ given $h|_{\mathbb{Z}^2\setminus\{x\}}$ depends only on $\{h(y)\}_{|y-x|=1}$.

In many examples, $\pi_\rho$ locally uniform, free-fermionic
\textbf{AKPZ growth and Euler-Lagrange equation}

- \exists \text{ continuum of non-translation-invariant Gibbs measures and } \nabla h(t = 0) \sim \pi^{(0)} \Rightarrow \nabla h(t) \sim \pi^{(t)}.

- Macroscopically, typical height profile sampled from Gibbs state is minimizer \( \phi \) of surface tension functional
  \[
  \int_{\mathbb{R}^2} \sigma(\nabla \phi) \, dx
  \]
  with \( \sigma(\cdot) \) convex, i.e. solution of Euler-Lagrange equation
  \[
  \sum_{i,j=1}^{2} \sigma_{ij}(\nabla \phi) \partial_{x_i x_j}^2 \phi = 0, \quad (\sigma_{ij}(\rho) := \partial_{\rho_i \rho_j}^2 \sigma(\rho)).
  \]
AKPZ growth and Euler-Lagrange equation

Preservation of Gibbs property $\implies$ hydrodynamic PDE

$$\partial_t \phi = v(\nabla \phi)$$

preserves solutions of Euler-Lagrange:

$$\phi(t = 0) \text{ solves Euler-Lagrange} \implies \phi(t) \text{ does too}$$

**Theorem** (A. Borodin, F.T.) This gives a non-linear relation between $D^2 v$ and $D^2 \sigma$, that implies $\det(D^2 v) \leq 0$.

For dimer models, solutions of Euler-Lagrange parametrized by complex variable

$$z = z(\nabla \phi) \ (R. \ Kenyon \ & \ A. \ Okounkov \ '07)$$

**Theorem** (A. Borodin, F.T.) Hydrodynamic PDE preserves Euler-Lagrange equation $\iff$ speed $v(\cdot)$ is harmonic function of $z$. 
Things that were left out

- Bounds $O(L^{2+\epsilon})$ on mixing time in finite $L \times L$ domain (P. Caputo, B. Laslier, F. Martinelli, F.T.)
- Convergence to non-linear parabolic PDE for long-jump symmetric dynamics (B. Laslier, F.T. ’17)
Summary

We discussed Wolf’s conjecture on universality classes of 2d stochastic interface growth.

New results on AKPZ growth models:

- hydrodynamic limits
- logarithmic bounds on fluctuation growth, $\alpha_{AKPZ} = \beta_{AKPZ} = 0$
- singularities of $v(\cdot) \leftrightarrow$ smooth steady states, facets
- origin of $\det D^2 v \leq 0$: preservation in time of Gibbs property
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Open problems:
- Full convergence to Edwards-Wilkinson fixed point? (proven in limiting regimes: A. Borodin, I. Corwin & F.T. ’17, A. Borodin, I. Corwin & P. Ferrari ’17)
- Results for the KPZ class?
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Thanks!