Bogoliubov Excitation Spectrum for Bose-Einstein Condensates

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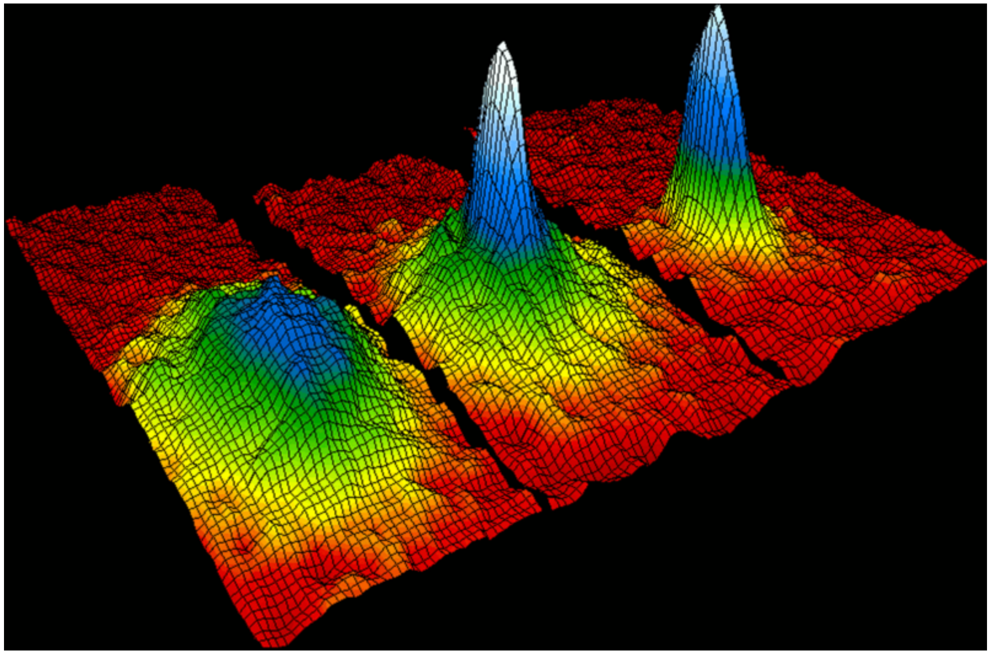
Introduction

Bose-Einstein condensates: in the last two decades, BEC have become accessible to experiments.

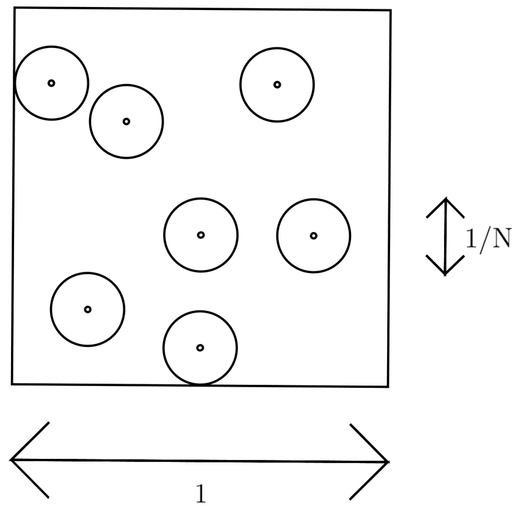
Goal of this talk: understand low temperature properties of trapped BEC, starting from microscopic description.



Gross-Pitaevskii regime: N bosons in $\Lambda = [0;1]^3$ described by **Hamilton operator**

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \kappa \sum_{i<j}^{N} N^2 V(N(x_i - x_j)) , \qquad \text{on } L^2_s(\Lambda^N)$$

$V \geq 0$ with **compact support**, $\kappa > 0$ coupling constant.



Dilute limit: range of interaction **much shorter** than typical distance among particles: **collisions are rare**.

Scattering length: defined by zero-energy **scattering equation**

$$\left[-\Delta + \frac{\kappa}{2} V(x)\right] f(x) = 0, \qquad \text{with} \qquad f(x) \to 1 \quad \text{as } |x| \to \infty$$

$$\Rightarrow \qquad f(x) = 1 - \frac{\mathfrak{a}_0}{|x|}, \qquad \text{for large } |x|$$

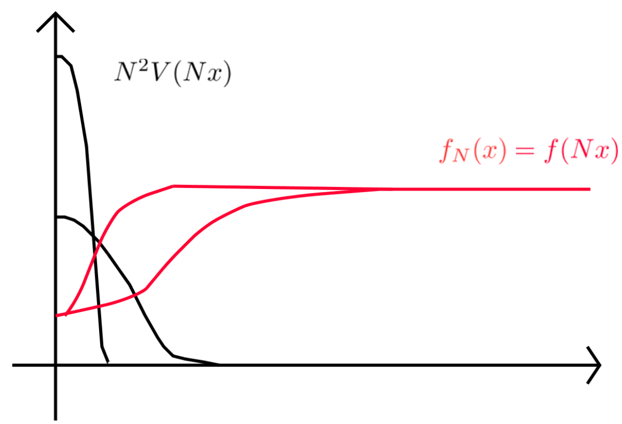
Equivalently,

$$8\pi \mathfrak{a}_0 = \kappa \int V(x) f(x) dx$$

By **scaling**,

$$\left[-\Delta + \frac{\kappa}{2} N^2 V(Nx)\right] f(Nx) = 0$$

Hence, rescaled potential has scattering length $\mathfrak{a}_0/N$.



Ground state energy: from [**Lieb-Yngvason '98**], ground state energy given to leading order by

$$E_N = 4\pi\mathfrak{a}_0 N + o(N)$$

Bose-Einstein condensation: from [**Lieb-Seiringer '02**], ground state ψ_N exhibits BEC, i.e. reduced density defined by

$$\gamma_N(x;y) = \int dx_2 \dots dx_N \, \psi_N(x, x_2, \dots, x_N)\overline{\psi}_N(y, x_2, \dots, x_N)$$

is such that

$$\langle \varphi_0, \gamma_N \varphi_0 \rangle \to 1$$

with $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Warning: this does not mean that $\psi_N \simeq \varphi_0^{\otimes N}$. In fact

$$\langle \varphi_0^{\otimes N}, H_N \, \varphi_0^{\otimes N} \rangle = \frac{(N-1)}{2} \kappa \widehat{V}(0) \gg 4\pi\mathfrak{a}_0 N$$

Correlations play crucial role!!

Main results

Theorem 1: Suppose $\kappa > 0$ is **small enough**. Then there exists $C > 0$ such that

$$|E_N - 4\pi\mathfrak{a}_0 N| \leq C$$

uniformly in N.

Furthermore, if $\psi_N \in L^2_s(\Lambda^N)$ such that

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + \zeta$$

we have

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq \frac{C(\zeta + 1)}{N}$$

Interpretation: in low-energy states, condensation holds with optimal rate, with **bounded** number of excitations.

Theorem 2: Suppose $\kappa > 0$ is **small enough**. Then

$$E_N = 4\pi\mathfrak{a}_N(N-1) - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] + \mathcal{O}(N^{-1/4})$$

where $\Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$ and

$$8\pi\mathfrak{a}_N = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty}\frac{(-1)^k\kappa^{k+1}}{(2N)^k} \times \sum_{p_1,\dots,p_k\in\Lambda_+^*}\frac{\widehat{V}(p_1/N)}{p_1^2}\left(\prod_{i=1}^{k-1}\frac{\widehat{V}((p_i-p_{i+1})/N)}{p_{i+1}^2}\right)\widehat{V}(p_k/N)$$

Remark: sum over p converges, gives contribution of order one.

Remark: for $\kappa > 0$ small enough, the **Born series** defining $\mathfrak{a}_N$ converges absolutely.

Remark: recall $\mathfrak{a}_0$ defined through

$$8\pi\mathfrak{a}_0 = \int V(x)f(x)dx$$

where f solves

$$\left(-\Delta + \frac{V}{2}\right) f = 0$$

Definition of $\mathfrak{a}_N$ can be compared with **Born series**

$$8\pi\mathfrak{a}_0 = \kappa\widehat{V}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k \kappa^{k+1}}{2^k (2\pi)^{3k}} \times \int_{\mathbb{R}^{3k}} dp_1 \dots dp_k \, \frac{\widehat{V}(p_1)}{p_1^2} \left(\prod_{i=1}^{k-1} \frac{\widehat{V}(p_i - p_{i+1})}{p_{i+1}^2} \right) \widehat{V}(p_k)$$

for $\mathfrak{a}_0$. We find

$$|\mathfrak{a}_N - \mathfrak{a}_0| \leq CN^{-1}$$

At level of precision of Theorem, ground state energy sensitive to **finite volume** effects!

Theorem 3: Suppose $\kappa > 0$ is **small enough**. Then **spectrum** of $H_N - E_N$ below ζ consists of eigenvalues

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3))$$

where $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$.

Quasiparticle interpretation: an excitation with momentum $p \in \Lambda_+^*$ costs energy $\varepsilon(p) = \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2}$.

Remark: excitation spectrum is crucial to understand the physical properties of the system.

The **linear dependence** of $\varepsilon(p)$ on $|p|$ for small p can be used to explain the emergence of **superfluidity**.

Previous works: mathematically simpler models described by

$$H_N^\beta = \sum_{j=1}^N -\Delta_{x_j} + \frac{\kappa}{N}\sum_{i<j}^N N^{3\beta} V(N^\beta(x_i - x_j))$$

for $\beta \in [0;1)$.

In **mean field regime**, $\beta = 0$, excitation spectrum determined in [Seiringer, '11], [Grech-Seiringner, '13], [Lewin-Nam-Serfaty-Solovej, '14], [Derezinski-Napiorkowski, '14], [Pizzo, '16].

Dispersion of excitations given by $\varepsilon_{\text{mf}}(p) = \sqrt{|p|^4 + 2\kappa\widehat{V}(p)p^2}$.

For **intermediate regimes**, $\beta \in (0;1)$, excitations spectrum determined in [BBCS, '17].

Dispersion of excitations given by $\varepsilon_\beta(p) = \sqrt{|p|^4 + 2\kappa\widehat{V}(0)p^2}$.

Bogoliubov approximation

Fock space: define $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\Lambda^n)$.

Creation and annihilation operators: for $f \in L^2(\Lambda)$, set

$$(a^*(f)\Psi)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n f(x_j)\Psi^{(n-1)}(x_1,\dots,\not{x}_j,\dots,x_n)$$

$$(a(f)\Psi)^{(n)}(x_1,\dots,x_n) = \sqrt{n+1} \int \bar{f}(x)\Psi^{(n+1)}(x,x_1,\dots,x_n)dx$$

For $p \in \Lambda^* = 2\pi\mathbb{Z}^3$, set

$$a_p^* = a^*(f_p), \qquad a_p = a(f_p), \quad \text{with} \quad f_p(x) = e^{-ip\cdot x}$$

We obtain

$$[a_p, a_q^*] = \delta_{p,q}, \quad [a_p, a_q] = [a_p^*, a_q^*] = 0$$

Remark: $a_p^* a_p$ measures number of particles with momentum p. Hence

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \text{total } \textbf{number of particles} \text{ operator}$$

Hamilton operator: we write

$$H_N = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

Number substitution: **BEC** implies that

$$a_0, a_0^* \simeq \sqrt{N} \gg 1 = [a_0, a_0^*]$$

Hence, **Bogoliubov** replaced a_0^*, a_0 by factors of $\sqrt{N}$. He found

$$\begin{aligned} H_N \simeq & \frac{(N-1)}{2} \kappa \widehat{V}(0) + \sum_{p \neq 0} p^2 a_p^* a_p + \kappa \widehat{V}(0) \sum_{p \neq 0} a_p^* a_p \\ & + \frac{\kappa}{2} \sum_{p \neq 0} \widehat{V}(p/N) \left[2 a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p} \right] \\ & + \frac{\kappa}{\sqrt{N}} \sum_{p,q \neq 0} \widehat{V}(p/N) \left[a_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} a_{p+q} \right] \\ & + \frac{\kappa}{N} \sum_{p,q,r \neq 0} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned}$$

Diagonalization: neglecting cubic and quartic terms, he used **Bogoliubov transformation**

$$\widetilde{T} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda^*_+} \tau_p\left(a^*_p a^*_{-p} - a_p a_{-p}\right)\right]$$

satisfying

$$\widetilde{T}^*\, a_p\, \widetilde{T} = a_p \cosh(\tau_p) + a^*_{-p} \sinh(\tau_p)$$
$$\widetilde{T}^*\, a^*_p\, \widetilde{T} = a^*_p \cosh(\tau_p) + a_{-p} \sinh(\tau_p)$$

to write

$$\begin{aligned}\widetilde{T}^* H_N \widetilde{T} \simeq \frac{(N-1)}{2}\kappa \widehat{V}(0) - \frac{\kappa^2}{2}\sum_{p\neq 0}\frac{\widehat{V}^2(p/N)}{2p^2} \\ - \frac{1}{2}\sum_{p\neq 0}\left[p^2 + \kappa\widehat{V}(0) - \sqrt{|p|^4 + 2\kappa\widehat{V}(0)p^2} - \kappa^2\frac{\widehat{V}(0)^2}{2p^2}\right] \\ + \sum_{p\neq 0}\sqrt{|p|^4 + 2\kappa\widehat{V}(0)p^2}\, a^*_p a_p\end{aligned}$$

Scattering length: replacing $\kappa\widehat{V}(0) \to 8\pi\mathfrak{a}_0$ and also

$$\kappa\widehat{V}(0) - \frac{\kappa^2}{N}\sum_p \frac{\widehat{V}^2(p/N)}{2p^2} \to 8\pi\mathfrak{a}_0$$

as suggested by **Landau**, Bogoliubov obtained

$$\begin{aligned}\widetilde{T}^* H_N \widetilde{T} \simeq\ & 4\pi\mathfrak{a}_0(N-1) \\ & + \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right] \\ & + \sum_{p\neq 0}\sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}\, a_p^* a_p\end{aligned}$$

which implies

$$E_N \simeq 4\pi\mathfrak{a}_0(N-1) + \frac{1}{2}\sum_{p\neq 0}\left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2}\right]$$

and that excitation spectrum consists of eigenvalues given approximately by

$$\sum_{p\neq 0} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2}$$

Some ideas from proof

Orthogonal excitations: for $\psi_N \in L^2_s(\Lambda^N)$ and $\varphi_0 \equiv 1$ on Λ, write

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes (N-1)} + \alpha_2 \otimes_s \varphi_0^{\otimes (N-2)} + \cdots + \alpha_N$$

where $\alpha_j \in L^2_{\perp \varphi_0}(\Lambda)^{\otimes_s j}$.

As in [**Lewin-Nam-Serfaty-Solovej, '12**], define unitary map

$$U : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N} := \bigoplus_{j=0}^{N} L^2_{\perp \varphi_0}(\Lambda)^{\otimes_s j}$$
$$\psi_N \to U\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$$

Excitation Hamiltonian: we use unitary map U to define

$$\mathcal{L}_N = U H_N U^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

For $p, q \in \Lambda^*_+ = 2\pi\mathbb{Z}^3 \backslash \{0\}$, we have

$$U\, a_p^* a_q\, U^* = a_p^* a_q, \qquad U\, a_0^* a_0\, U^* = N - \mathcal{N}_+$$
$$U\, a_p^* a_0\, U^* = a_p^* \sqrt{N - \mathcal{N}_+}, \qquad U a_0^* a_p\, U^* = \sqrt{N - \mathcal{N}_+}\, a_p$$

Hence, similarly to **Bogoliubov substitution**,

$$\begin{aligned}
\mathcal{L}_N = & \frac{(N-1)}{2N} \kappa \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\kappa \widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+) \\
& + \sum_{p \in \Lambda^*_+} p^2 a_p^* a_p + \sum_{p \in \Lambda^*_+} \kappa \widehat{V}(p/N) a_p^* \frac{N - 1 - \mathcal{N}_+}{N} a_p \\
& + \frac{\kappa}{2} \sum_{p \in \Lambda^*_+} \widehat{V}(p/N) \left[a_p^* a_{-p}^* \sqrt{\frac{(N - \mathcal{N}_+)(N - 1 - \mathcal{N}_+)}{N^2}} + \text{h.c.} \right] \\
& + \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda^*_+ : p+q \neq 0} \widehat{V}(p/N) \left[\sqrt{\frac{N + 1 - \mathcal{N}_+}{N}}\, a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right] \\
& + \frac{\kappa}{2N} \sum_{p,q \in \Lambda^*_+, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}
\end{aligned}$$

Gain: conjugation with U generates new **constant** and **quadratic** contributions.

Problem: in contrast with mean-field regime, after conjugation with U there are still **large contributions** in higher order terms.

Reason: U removes factors of φ_0, **correlations** left in excitation vectors and carry large energy.

Natural idea: conjugate $\mathcal{L}_N$ with a **Bogoliubov** transformation of the form

$$\widetilde{T} = \exp\left[\frac{1}{2}\sum_{p\in\Lambda_+^*} \eta_p \left(a_p^* a_{-p}^* - a_p a_{-p}\right)\right]$$

to describe correlations.

Challenge: $\widetilde{T}$ does not preserve the excitation space $\mathcal{F}_+^{\leq N}$.

Modified operators: for $p \in \Lambda_+^*$, define

$$b_p^* = a_p^* \sqrt{\frac{N - \mathcal{N}_+}{N}} \qquad\qquad b_p = \sqrt{\frac{N - \mathcal{N}_+}{N}}\, a_p$$

Remark: for all $p \in \Lambda_+^*$,

$$U^* \, b_p^* \, U = a_p^* \frac{a_0}{\sqrt{N}}, \qquad U^* b_p U = \frac{a_0^*}{\sqrt{N}} a_p$$

Thus, b_p^* creates **excitation**, total number of particles **preserved**.

On states with $\mathcal{N}_+ \ll N$, we expect $b_p \simeq a_p$, $b_p^* \simeq a_p^*$.

Generalized Bogoliubov transformations: let $w = 1 - f$ and

$$\eta_p = -\frac{1}{N^2}\,\widehat{w}(p/N) \qquad \Rightarrow \qquad \eta_p \simeq \frac{C}{p^2}\,e^{-|p|/N}$$

We define

$$T = \exp \sum_{p \in \Lambda_+^*} \eta_p \left(b_p^* b_{-p}^* - b_p b_{-p} \right) : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Then

$$T^* b_p T = \cosh(\eta_p) b_p + \sinh(\eta_p) b_{-p}^* + d_p$$

where

$$\| d_p \xi \| \leq C N^{-1} \| (\mathcal{N}_+ + 1)^{3/2} \xi \|$$

Observation: let $\mathcal{K} = \sum p^2 a_p^* a_p$. Then

$$T^* \mathcal{N}_+ T \simeq \mathcal{N}_+ + \|\eta\|_2^2 \simeq \mathcal{N}_+ + C$$
$$T^* \mathcal{K} T \simeq \mathcal{K} + \|\eta\|_{H^1}^2 \simeq \mathcal{K} + CN$$

T generates **finitely many** excitations but **macroscopic** energy.

Renormalized excitation Hamiltonian: define

$$\mathcal{G}_N = T^* \mathcal{L}_N T = T^* U H_N U^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Theorem: we have

$$\mathcal{G}_N = 4\pi \mathfrak{a}_0 N + \mathcal{H}_N + \mathcal{E}_N$$

where $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$,

$$\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p,$$
$$\mathcal{V}_N = \frac{\kappa}{N} \sum_{p,q,r \in \Lambda_+^*} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

and where, for every $\delta > 0$, there exists $C > 0$ with

$$\pm \mathcal{E}_N \leq \delta \mathcal{H}_N + C\kappa(\mathcal{N}_+ + 1)$$

Mechanism: $\mathcal{G}_N = T^* \mathcal{L}_N T$ with

$$\mathcal{L}_N \simeq \frac{\widehat{V}(0)}{2} N + \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p$$
$$+ \sum_{p \in \Lambda_+^*} \kappa \widehat{V}(p/N) a_p^* a_p + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N) \left[a_p^* a_{-p}^* + a_p a_{-p} \right]$$
$$+ \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N) \left[a_{p+q}^* a_{-p}^* a_q + \text{h.c.} \right]$$
$$+ \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

Conjugation with T **generates new terms**, for example:

$$\frac{\kappa}{2N} \sum \widehat{V}(r/N)\, T a_{p+r}^* a_q^* a_p a_{q+r} T^* \to \frac{\kappa}{2N} \sum \widehat{V}(r/N) \eta_{q+r} a_{p+r}^* a_q^* a_p a_{-q-r}^*$$
$$\to \frac{\kappa}{2N} \sum \widehat{V}(r/N) \eta_{q+r} a_{-q}^* a_q^*$$

This term combines with other quadratic terms, reconstructing the **scattering equation**.

Theorem: we have

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N + \mathcal{H}_N + \mathcal{E}_N$$

with

$$\pm\mathcal{E}_N \leq \delta\mathcal{H}_N + C\kappa(\mathcal{N}_+ + 1)$$

Let us discuss some **consequences** of this theorem.

Upper bound: by definition

$$E_N \leq \langle U^* T\Omega, H_N U^* T\Omega\rangle = \langle \Omega, \mathcal{G}_N \Omega\rangle \leq 4\pi\mathfrak{a}_0 N + C$$

Lower bound: with $\delta = 1/2$, we find, for $\kappa > 0$ **small enough**,

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{2}\mathcal{H}_N - C\kappa(\mathcal{N}_+ + 1) \geq 4\pi\mathfrak{a}_0 N + \frac{1}{4}\mathcal{H}_N - C$$

Hence

$$E_N \geq 4\pi\mathfrak{a}_0 N - C$$

Condensation: from lower bound

$$\mathcal{G}_N \geq 4\pi\mathfrak{a}_0 N + \frac{1}{4}\mathcal{H}_N - C$$

we also obtain condensation with **optimal rate**.

Let $\psi_N \in L^2_s(\Lambda^N)$, with

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi\mathfrak{a}_0 N + \zeta$$

Let $\xi_N = T^* U \psi_N \in \mathcal{F}_+^{\leq N}$. We find, for $\kappa > 0$ small enough,

$$\begin{aligned} \langle \xi_N, \mathcal{N}_+ \xi_N \rangle &\leq C \langle \xi_N, \mathcal{H}_N \xi_N \rangle \\ &\leq C + C \langle \xi_N, (\mathcal{G}_N - 4\pi\mathfrak{a}_0 N) \xi_N \rangle \\ &\leq C + \langle \psi_N, (H_N - 4\pi\mathfrak{a}_0 N) \psi_N \rangle \leq C(1+\zeta) \end{aligned}$$

Number of excitations remains **bounded**. Hence

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(\zeta + 1)}{N}$$

Stronger condensation bound: if $\psi_N \in L^2(\Lambda)^{\otimes_s N}$ with

$$\psi_N = \chi(H_N \leq E_N + \zeta)\psi_N$$

Then $\psi_N = U_N^* T \xi_N$ with

$$\left\langle \xi_N, \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right] \xi_N \right\rangle \leq C(\zeta + 1)^3$$

Proposition: renormalized excitation Hamiltonian is such that

$$\mathcal{G}_N = C_N + \mathcal{Q}_N + \mathcal{C}_N + \mathcal{V}_N + \mathcal{E}_N$$

where C_N is a **constant**, Q_N is **quadratic**,

$$\mathcal{C}_N = \frac{\kappa}{\sqrt{N}} \sum_{\substack{p,q \in \Lambda_+^* \\ q \neq -p}} \widehat{V}(p/N) \left[b_{p+q}^* b_{-p}^* \Big(\gamma_q b_q + \sigma_q b_{-q}^* \Big) + \text{h.c.} \right]$$

and, where,

$$\pm \mathcal{E}_N \leq \frac{C}{\sqrt{N}} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Problem: $\mathcal{G}_N$ still contains non-negligible **cubic** and **quartic** terms! This is the main difference compared with case $\beta < 1$!

Not surprising: from [Erdős-S.-Yau, 08], [Napiorkowski-Reuvers-Solovej, '15] it is clear that Bogoliubov states can only approximate ground state energy up to an **error** $\mathcal{O}(1)$.

Cubic phase: we define

$$A = \frac{1}{\sqrt{N}} \sum_{r \in P_H, v \in P_L} \eta_r \Big[\sigma_v b^*_{r+v} b^*_{-r} b^*_{-v} + \gamma_v b^*_{r+v} b^*_{-r} b_v - \text{h.c.} \Big]$$

with $P_L = \{p \in \Lambda^*_+ : |p| \leq \sqrt{N}\}$, $P_H = \{p \in \Lambda^*_+ : |p| \geq \sqrt{N}\}$.

Set $S = e^A$ and introduce **new excitation Hamiltonian**

$$\mathcal{J}_N = S^* T^* U_N H_N U^*_N T S : \mathcal{F}^{\leq N}_+ \to \mathcal{F}^{\leq N}_+$$

Remark: a similar cubic conjugation was used in [Yau-Yin, 09].

Proposition: we can decompose

$$\mathcal{J}_N = \widetilde{C}_N + \widetilde{Q}_N + \mathcal{V}_N + \widetilde{\mathcal{E}}_N$$

where $\widetilde{C}_N$ is a **constant**, $\widetilde{Q}_N$ is **quadratic** in creation and annihilation operators, and where

$$\pm\widetilde{\mathcal{E}}_N \leq \frac{C}{N^{1/4}}\left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3\right]$$

Mechanism: we have

$$\mathcal{J}_N = e^{-A}\mathcal{G}_N e^{A} \simeq \mathcal{G}_N + [\mathcal{G}_N, A] + \frac{1}{2}[[\mathcal{G}_N, A], A] + \dots$$

where

$$\mathcal{G}_N \simeq C_N + Q_N + \mathcal{C}_N + \mathcal{V}_N$$

We combine terms arising from $[Q_N, A], [\mathcal{V}_N, A]$ with $\mathcal{C}_N$ (use again **scattering equation**).

At same time $[\mathcal{C}_N, A]$ renormalizes constant and quadratic terms in $\mathcal{G}_N$.

Diagonalization: quadratic term given by

$$\widetilde{\mathcal{Q}}_N = \sum_{p\in\Lambda_+^*} F_p\, b_p^* b_p + \frac{G_p}{2}\left[b_p\, b_{-p} + b_p^*\, b_{-p}^*\right]$$

with

$$F_p = p^2(\sigma_p^2 + \gamma_p^2) + \kappa\Big(\widehat{V}(\cdot/N) * \widehat{f}_N\Big)_p (\gamma_p + \sigma_p)^2;$$
$$G_p = 2p^2\sigma_p\gamma_p + \kappa\Big(\widehat{V}(\cdot/N) * \widehat{f}_N\Big)_p (\gamma_p + \sigma_p)^2$$

We define new **Bogoliubov transformation**

$$\tanh(2\tau_p) = -\frac{G_p}{F_p} \quad \Rightarrow \quad R = \exp\Big[\frac{1}{2}\sum_{p\in\Lambda_+^*} \tau_p(b_p^* b_{-p}^* - b_p b_{-p})\Big]$$

Then

$$R^*\widetilde{Q}_N R = \frac{1}{2}\sum_{p\in\Lambda_+^*}\left[-F_p + \sqrt{F_p^2 - G_p^2}\right] + \sum_{p\in\Lambda_+^*}\sqrt{F_p^2 - G_p^2}\, a_p^* a_p + \delta_N$$

with

$$\pm\delta_N \le C N^{-1}(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)$$

Final excitation Hamiltonian: we define

$$\mathcal{M}_N = R^* \mathcal{J}_N R = R^* S^* T^* U_N H_N U_N^* T S R : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

Then

$$\begin{aligned} \mathcal{M}_N = \; & 4\pi \mathfrak{a}_N (N-1) \\ & + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[-p^2 - 8\pi \mathfrak{a}_0 + \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2} + \frac{(8\pi \mathfrak{a}_0)^2}{2p^2} \right] \\ & + \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 16\pi \mathfrak{a}_0 p^2} \, a_p^* a_p + \mathcal{V}_N + \mathcal{E}_N' \end{aligned}$$

where

$$\pm \mathcal{E}_N' \leq C N^{-1/4} \left[(\mathcal{H}_N + 1)(\mathcal{N}_+ + 1) + (\mathcal{N}_+ + 1)^3 \right]$$

Main theorem follows from **min-max principle**, because on low-energy states of diagonal quadratic Hamiltonian, we find

$$\mathcal{V}_N \leq C N^{-1} (\zeta + 1)^{7/2}$$

Open problems

Lee-Huang-Yang formula: consider N bosons in $\Lambda = [0;L]^{\times 3}$, with fixed density $\rho = N/L^3$. As $\rho \to 0$ (**dilute limit**), expect

$$\lim_{N,L\to\infty : N/L^3=\rho} \frac{E_N}{N} = 4\pi\mathfrak{a}_0\rho\left[1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}_0^3} + \text{smaller order}\right]$$

So far, only leading order is known from [**Lieb-Yngvason, '98**].

Upper bound shown to second order in [**Yau-Yin, '09**].

Lee-Huang-Yang proven for class of potentials scaling with ρ in [**Giuliani-Seiringer, '09**], [**Brietzke-Solovej, '18**].

Bose-Einstein condensation: long term goal is proof of BEC in thermodynamic limit, where $N, L \to \infty$ and fixed $\rho = N/L^3$. Ongoing work of **Ballaban-Feldman-Knörrer-Trubowitz**.