Special geometry on Calabi-Yau moduli spaces and Q-invariant Frobenius rings.

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2018

Introduction.

Compactification of the Superstring theory on a Calabi–Yau (CY) threefold X leads to the effective quantum field theory defined in terms of so-called Special Kähler geometry of the CY moduli space.

It is known that the Kähler potential is given by the logarithm of the holomorphic volume of Calabi-Yau manifold X_ϕ:

$$G(\phi)_{a\bar{b}} = \partial_a \overline{\partial_b} K(\phi, \bar{\phi}),$$
$$e^{-K(\phi)} = \int_{X_\phi} \Omega \wedge \overline{\Omega},$$

This can be rewritten in terms of periods of Ω as:

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$
$$e^{-K} = \omega_\mu(\phi) C_{\mu\nu} \overline{\omega_\nu(\phi)},$$

where $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is an intersection matrix of 3-cycles.

New approach

In practice, the computation of all periods is a very complicated problem and was done explicitly only in few examples.

I will present a new method to easily compute the Kähler metric for the large class of CY defined as hypersurfaces in weighted projective spaces.

The method uses a correspondence between the Hodge structure on the middle cohomology of CY manifold and the structure on the invariant Frobenius ring associated with the CY manifold.

This correspondence is realized by Oscillatory integral presentation for the periods of the holomorphic Calabi-Yau 3-form.

Clarifying this correspondence we obtain the efficient method for computing Special geometry on the CY moduli space.

Correspondence of the Hodge structure of $H^3(X)$ and R^Q.

Let X be a CY manifold realized as the zero-set of a quasi-homogeneous polynomial $W(x)$ in weighted P^4. Cohomology $H^3(X)$ with Hodge decomposition $H^3(X) = H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X)$, the complex conjugation and Poincaré pairing is isomorphic to the invariant Milnor ring R^Q defined by $W(x)$ with its decomposition given by degree grading, an antiholomorphic involution M and the residue pairing $\eta_{\mu\lambda}$ on R^Q.

Using this correspondence we can transform the formula for Kähler potential $K(\phi)$ to

$$e^{-K(\phi)} = \sigma_\mu(\phi)\,\eta_{\mu\lambda}\,M_{\lambda\nu}\,\overline{\sigma_\nu(\phi)}.$$

$\sigma_\mu(\phi)$ are periods that are in correspondence with the elements of the basis in R^Q presented by oscillatory integrals,
$\eta_{\mu\nu}$ is the residue pairing in the Milnor ring,
$M_{\mu\nu}$ is the antiholomorphic involution of the ring R^Q that is in

Example. 101-d moduli space of Quintic threefold

Let a quintic CY manifold X be given as the solution of the equation

$$W(x,\phi)=\sum_{i=1}^{5}x_i^5+\sum_{\mathbf{s}=\mathbf{1}}^{101}\phi_{\mathbf{s}}\prod_i x_i^{s_i}=0$$

$\mathbf{s}=(s_1,s_2,s_3,s_4,s_5)$, $0\le s_i\le 3$, $\deg(\mathbf{s}):=\sum_{i=1}^{5}s_i=5$.
The complex structures Kähler potential in this case is

$$e^{-K(\phi)}=\sum_{\mu=0}^{203}(-1)^{\deg(\mu)/5}\prod\gamma\left(\frac{\mu_i+1}{5}\right)|\sigma_\mu(\phi)|^2,$$

$$\sigma_\mu(\phi)=\sum_{n_1,\ldots,n_5\ge 0}\prod_{i=1}^{5}\frac{\Gamma(\frac{\mu_i+1}{5}+n_i)}{\Gamma(\frac{\mu_i+1}{5})}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_s}}{m_s!},$$

$\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)$, $0\le\mu_i\le 3$, $\quad\sum_{i=1}^{5}\mu_i=0,5,10,15.$

$$\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)},\qquad \Sigma_n=\{m_{\mathbf{s}}\mid\sum_s m_{\mathbf{s}}s_i=5n_i+\mu_i\}$$

CY as the hypersurface in the weighted projective space

Let now $x_1, \ldots, x_5$ be homogeneous coordinates in the weighted projective space $\mathbb{P}^4_{(k_1,\ldots,k_5)}$, and let a Calabi-Yau X be defined as

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} | \ W_0(x) = 0\}$$

for some quasi-homogeneous polynomial $W_0(x)$,

$$W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i)$$

and

$$\deg W_0(x) = d = \sum_{i=1}^{5} k_i.$$

The last relation ensures that X is a CY manifold.
The moduli space of complex structures is then given by homogeneous polynomial deformations of this singularity:

$$W(x, \phi) = W_0(x) + \sum_{s=0}^{h_{21}-1} \phi_{\mathbf{s}} e_s(x),$$

where $e_s(x)$ are monomials of x which have the same degree d.

Hodge structure on middle cohomology

The holomorphic everywhere non-vanishing 3-form Ω is defined as

$$\Omega = \frac{x_5 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}{\partial W(x)/\partial x_4}$$

Periods of Ω needed for our goal are integrals over cycles of $H_3(X, \mathbb{R})$

$$\omega_\mu(\phi) := \int_{q_\mu} \Omega, \quad q_\mu \in H_3(X, \mathbb{R}).$$

$H^3(X)$ possesses Hodge structure $H^3(X) = \oplus_{k=0}^3 H^{3-k,k}(X)$,

$\dim H^{3,0}(X) = \dim H^{0,3}(X) = 1,\ \dim H^{2,1}(X) = \dim H^{1,2}(X) = h^{2,1}.$

Poincaré pairing can be written through integrals over cycles q_μ as

$$< \chi_a, \chi_b >= \int_X \chi_a \wedge \chi_b = \int_{q_\mu} \chi_a \, C_{\mu\nu} \int_{q_\nu} \chi_b,$$

is invariant with respect to complex conjugation of (p, q)-forms. $C_{\mu\nu} = [q_\mu] \cap [q_\nu]$ is the intersection matrix of 3-cycles.

Q-invariant Milnor ring

On the other hand the polynomial $W_0(x)$ defines a Milnor ring R_0. We consider its subring R^Q invariant with respect to the symmetry group Q that acts on $\mathbb{C}^5$ diagonally and preserves $W(x, \phi)$

$$R^Q = \left(\frac{\mathbb{C}[x_1, \cdots, x_5]}{\mathrm{Jac}(W_0)}\right)^Q, \quad \mathrm{Jac}(W_0) = \langle \partial_i W_0 \rangle_{i=1}^5.$$

R^Q becomes Frobenius ring if it is endowed with pairing

$$\eta(e_\alpha, e_\beta) = \mathrm{Res}\, \frac{e_\alpha(x) e_\beta(x)\, \mathrm{d}^5 x}{\prod_{i=1}^N \partial_i W_0(x)}.$$

$\dim R^Q = \dim H^3(X)$ and R^Q has the Hodge structure arising from the grading with degrees $0, d, 2d, 3d$

$$R^Q = (R^Q)^0 \oplus (R^Q)^1 \oplus (R^Q)^2 \oplus (R^Q)^3$$

. $\dim (R^Q)^0 = \dim (R^Q)^3 = 1, \quad \dim (R^Q)^1 = \dim (R^Q)^2 = h^{2,1}$

$Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$

The next step is to introduce two differentials $D_\pm$

$$D_\pm = \mathrm{d} \pm \mathrm{d}W_0 \wedge, \qquad (D_\pm)^2 = 0$$

and two groups of $Q-$invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_Q$.
These groups inherit the grading from R^Q.
Choosing in the ring R^Q some basis $\{e_\mu(x)\}$
we take $\{e_\mu(x)\,\mathrm{d}^5x\}$ as a basis of $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

These cohomology groups are in one-to-one correspondence with the middle cohomology group $\in H^3(X)$(Candelas 1988).

This isomorphism, defined below, maps the components $H^{3-q,q}(X)$ to the Hodge decomposition components of $H^5_\pm(\mathbb{C}^5)_Q$ spanned by $e_\mu(x)\,\mathrm{d}^5x$ with $e_\mu(x) \in (R^Q)^q$.

It also maps the Poincare pairing of differential forms to X to the pairing $\eta(e_\alpha, e_\beta)$ on the invariant ring R^Q.

Q-invariant relative homology and oscilatory integrals

Having $H^5_{D_\pm}(\mathbb{C}^5)_Q$ we define two Q-invariant relative homology groups $\mathcal{H}_5^{\pm,Q} := H_5(\mathbb{C}^5,\ \mathrm{Re}W_0 = L \to \pm\infty)_Q$ as a quotient of the relative homology group $H_5(\mathbb{C}^5,\ \mathrm{Re}W_0 = L \to \pm\infty)$.

For this purpose we define the pairing via oscillatory integrals

$$\langle Q^\pm_\mu, e_\nu(x)\,\mathrm{d}^5 x\rangle := \int_{Q^\pm_\mu} e_\nu(x)\, e^{\mp W(x)} \mathrm{d}^5 x.$$

Using this pairing we define the relative invariant homology groups $\mathcal{H}_5^{\pm,Q}$ to be the quotient of $H_5(\mathbb{C}^5, W_0 = L,\ \mathrm{Re}L \to \pm\infty)$ by its subspace whose elements are orthogonal to all elements of $H^5_{D_\pm}(\mathbb{C}^5)_Q$.

$H_3(X)$ versus R^Q correspondence

The crucial fact for what follows is that R^Q and $H^3(X)$ and all the additional structures on these rings are isomorhic to each other.

First of all there exists an isomorphism S

$$S(Q_\mu^+) = q_\mu, \quad Q_\mu^+ \in \mathcal{H}_5^{\pm,Q}, \;\; q_\mu \in H_3(X,\mathbb{Z})$$

characterized as follows:
Let $\{q_\mu\}$ be a basis of $H_3(X,\mathbb{Z})$, then we choose the basis $Q_\mu^\pm$ of $\mathcal{H}_5^{\pm,Q}$ such that the following integrals over the corresponding cycles of these two bases are equal

$$\int_{q_\mu} \Omega_\phi = \int_{Q_\mu^\pm} e^{\mp W(x,\phi)}\, \mathrm{d}^5 x.$$

$H^3(X)$ versus $H^5_{D_\pm}(\mathbb{C}^5)_Q$ correspondence

Having isomorphism between $H_3(X)$ and $\mathcal{H}_5^{\pm,Q}$ we define the isomorphism between the two cohomology groups $H^3(X)$ and $H^5_{D_\pm}(\mathbb{C}^5)_Q$ also with help of oscilatory integrals.

Namely, take a basis of cycles $q_\mu \in H_3(X)$ and the corresponding to them basis of cycles $Q_\mu^\pm \in \mathcal{H}_5^{\pm,Q}$ at $\phi = 0$, then the form $\chi_\alpha \in H^3(X)$ corresponds to the form $e_\alpha(x)\,\mathrm{d}^5 x \in H^5_{D_\pm}(\mathbb{C}^5)_Q$ iff

$$\int_{q_\mu} \chi_\alpha = \int_{Q_\mu^\pm} e_\alpha(x)\, e^{\mp W(x,\phi)}\,\mathrm{d}^5 x$$

for all pairs $\{q_\mu, Q_\mu\}$.
So these two forms are isomorphic if they have the equal coordinates (that is, periods) in some isomorphic bases.

This isomorphism preserves Hodge decomposition (Candelas).
The pairing of the form $\in H^3(X)$ and the pairing of the corresponding elements $\in R^Q$ coincide.

Coincidence of two pairings and the intersection matrix

We can rewrite the Poincaré pairing of $\chi_a, \chi_b \in H^3(X)$

$$< \chi_a, \chi_b >:= \int_X \chi_a \wedge \chi_b$$

as the bilinear expression of periods

$$< \chi_a, \chi_b >= \int_{q_\mu} \chi_a \; C_{\mu\nu} \int_{q_\nu} \chi_b,$$

where $C_{\mu\nu} = q_\mu \cap q_\nu$.
On the other hand the residue pairing $\eta(e_a, e_b)$ in the ring R^Q can be written through the periods and $\hat{C}_{\mu\nu} = Q^+_\mu \cap Q^-_\nu$ as

$$\eta(e_a, e_b) = \int_{Q^+_\mu} e_a e^{-W(x,\phi)} \mathrm{d}^5 x \; \hat{C}_{\mu\nu} \int_{Q^-_\mu} e_b e^{W(x,\phi)} \mathrm{d}^5 x,$$

Since $C_{\mu\nu} = \hat{C}_{\mu\nu}$ and $\int_{q_\mu} \chi_a = \int_{Q^\pm_\mu} e_a e^{\mp W(x,\phi)} \mathrm{d}^5 x$
we obtain the equality $< \chi_a, \chi_b >= \eta(e_a, e_b)$, which gives the useful formula for $C_{\mu\nu}$ in terms the pairing η.

Anti-Involution $*$ on R^Q

The same isomorphism allows to define a complex conjugation $*$ on the Q-invariant cohomology $H^5_{D_\pm}(\mathbb{C}^5)_{inv}$.

Let the form $\phi_\mu \in H^3(X)$ correspond to $\{e_\mu(x)\} \in R^Q$ and let

$$*\phi_\mu = M_{\nu\mu}\phi_\nu.$$

The R^Q inherits this involution, and for the basis $\{e_\mu(x)\}$ the antiholomorphic operation $*$ reads as

$$* \, e_\mu(x) = M_{\nu\mu} e_\nu(x).$$

From this definition and since $(*)^2 = I$, it follows that $\bar{M}M = I$. We introduce the convenient basis $\Gamma^\pm_\mu \in \mathcal{H}_5^{\pm,Q}$ dual to the basis $\{e_\mu(x)\}$ such that:

$$\langle \Gamma^\pm_\mu, e_\nu(x)\, \mathrm{d}^5 x\rangle = \int_{\Gamma^\pm_\mu} e_\nu(x)\, e^{\mp W_0(x)} \mathrm{d}^5 x = \delta_{\mu\nu}.$$

This definition induces the antiholomorphic operation $*$ on $\Gamma^\pm_\mu$

$$*\Gamma^\pm_\mu = \bar{M}_{\mu\nu}\Gamma^\pm_\nu$$

How to compute $M_{\mu\nu}$

We define T as the transition matrix from cycles $\Gamma^{\pm}_{\mu}$ to any real cycles, say, Lefschetz thimbles $L^{\pm}_{\mu} = *L^{\pm}_{\mu}$

$$L^{\pm}_{\mu} = T_{\mu\nu}\Gamma^{\pm}_{\nu}.$$

Then we have

$$L^{\pm}_{\mu} = \overline{T}_{\mu\nu}\,(*\Gamma^{\pm}_{\nu}).$$

Comparing this relation with

$$*\Gamma^{\pm}_{\mu} = \bar{M}_{\mu\nu}\,\Gamma^{\pm}_{\nu},$$

we obtain for M the useful expression in terms T

$$M = T^{-1}\,\bar{T}.$$

Obviously M does not depend on the choice of real cycles. Using the definition $\langle\Gamma^{\pm}_{\mu}, e_{\nu}(x)\,\mathrm{d}^5x\rangle = \delta_{\mu\nu}$ we obtain the useful for computing $T_{\mu\nu}$ (and $M_{\mu\nu}$) relation

$$T_{\mu\nu} = \int_{L^{\pm}_{\mu}} e_{\nu}(x)\,e^{\mp W_0(x)}\mathrm{d}^5x.$$

Deriving the main formula for Kähler potential

Now use the CY/R^Q correspondence to transform the expression

$$e^{-K} = \omega_b^+(\phi)\ C_{ab}\ \overline{\omega_b^-(\phi)},$$

periods $\omega_a^\pm(\phi)$ are given by oscilatory integrals over cycles $L_a^\pm$:

$$\omega_a^\pm(\phi) = \int_{L_a^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x = T_{a\mu}\sigma_\mu^\pm(\phi),$$

and periods $\sigma_\mu^\pm(\phi)$ are integrals over cycles $\Gamma_\mu^\pm$

$$\sigma_\mu^\pm(\phi) = \int_{\Gamma_\mu^\pm} e^{\mp W(x,\phi)} \mathrm{d}^5 x.$$

Also we take the expression for pairing on R^Q at $\phi = 0$

$$\eta_{\mu\nu} = \int_{L_a^+} e_\mu e^{-W_0(x)} \mathrm{d}^5 x\ C_{ab} \int_{L_b^-} e_\nu e^{W_0(x)} \mathrm{d}^5 x = T_{a\mu} C_{ab} T_{b\nu}$$

Eliminating the matrix C_{ab} from these relations we obtain

$$e^{-K(\phi)} = \sum_{\mu,\nu,\lambda} \sigma_\mu^+(\phi)\ \eta_{\mu\lambda}\ M_{\lambda\nu}\ \overline{\sigma_\nu^-(\phi)}.$$

Example. Fermat threefolds

In this case CY manifold X is given by the equation

$$X = \{x_1, \ldots, x_5 \ \in \mathbb{P}^4_{(k_1,\ldots,k_5)} | \ W(x, \phi) = 0\}$$

$$W(x,\phi) = \sum_{i=1}^{5} x_i^{\frac{d}{k_i}} + \sum_{\mathbf{s}=1}^{h_{21}} \phi_{\mathbf{s}} e_s(x), \quad d = \sum_{i=1}^{5} k_i,$$

and $\frac{d}{k_i}$ are positive integers.

The monomials $e_s(x) = e_{(s_1,\cdots,s_5)} := \prod_i x_i^{s_i}$ correspond to the deformation of the complex structure of X.
Their weights are equal to $\sum_{i=1}^{5} k_i s_i = d$ and each variable x_i has a non-negative integer power $s_i \leq \frac{d}{k_i} - 2$.
The number of such monomials is equal to the Hodge number h_{21}.

Q-invariant Ring

Considering $W_0(x)$ as an isolated singularity in $\mathbb{C}^5$ we have an associated Milnor ring

$$R_0 = \frac{\mathbb{C}[x_1, \cdots, x_5]}{\langle \partial_i W_0 \rangle}.$$

The bases of Milnor rings R_0 consist of monomials $e_\mu(x) = \prod_i x_i^{\mu_i}$. Each non-negative integer $\mu_i \leq \frac{d}{k_i} - 2$ and $\dim R_0 = \prod_i (\frac{d}{k_i} - 1)$.

Define the subring $R^Q \in R_0$ generated by the monomials $e_s(x)$ of weight d.

The basis of R^Q are elements $e_\mu(x) = e_{(\mu_1, \cdots, \mu_5)} := \prod_i x_i^{\mu_i}$ whose weights $\sum_{i=1}^5 k_i \mu_i$ are equal $0, d, 2d$ and $3d$.

The basis includes $e_\rho(x) := \prod_i x_i^{\frac{d}{k_i} - 2}$, $\rho = (\frac{d}{k_1} - 2, \cdots, \frac{d}{k_5} - 2)$. The dimensions of the subspaces of degrees $0, d, 2d$ and $3d$ are $1, h_{21}, h_{21}$ and 1.

This grading defines a Hodge structure on R^Q which is isomorphic to the Hodge structure on $H^3(X)$.

Q-invariant ring

Fermat polynomials have a nice property that there is a symmetry group $\prod_i \mathbb{Z}_{d/k_i}$ that diagonally acts on $\mathbb{C}^5$:
$\alpha \cdot (x_1, \cdots, x_5) = (\alpha_1^{k_1} x_1, \cdots, \alpha_5^{k_5} x_5),\ \alpha_i^d = 1.$
This action preserves $W_0 = \sum_{i=1}^5 x_i^{\frac{d}{k_i}}$.

The so–called quantum symmetry Q is the subgroup of $\mathbb{Z}_d^5$ defined as follows.
Polynomial $W(x)$ is quasihomogeneous, therefore, in particular,
$W(\alpha^{k_1} x_1, \ldots, \alpha^{k_5} x_5) = W(x_1, \ldots, x_5)$, if $\alpha^d = 1$.
This acts trivially on the weighted projective space
as well as on X.
Thus in Fermat case $Q \simeq \mathbb{Z}_d$ and the subring R^Q is the
Q-invariant part of the Milnor ring.

Phase symmetry, complex conjugation and pairing

The phase symmetry respects the Hodge decomposition.
The monomial basis $\{e_\mu(x) = e_{(\mu_1,\cdots,\mu_5)}(x) = \prod_i x^{\mu_i}\}$ of R^Q is an eigenbasis of the phase symmetry $\mathbb{Z}_d^5$, and each $e_\mu(x)$ has a unique weight. We can extend the phase symmetry action to the parameter space $\{\phi_s\}_{s=1}^{h_{2,1}}$ such that $W(x,\phi)$ is invariant under this action.
The complex conjugation acts on $H^3(X)$ as $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Due to the isomorphism between R^Q and $H^3(X)$ the complex conjugation acts also on the elements of the ring R^Q as $*e_\mu(x) = p_\mu e_{\rho-\mu}(x)$, where p_μ is a constant.
On the invariant ring R^Q there exists the pairing turning it into a Frobenius algebra:

$$\eta_{\mu\nu} = \mathrm{Res}\,\frac{e_\mu(x)\,e_\nu(x)}{\prod_i \partial_i W_0(x)}.$$

For our monomial basis it is $\eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$.

Computation of periods $\sigma_\mu(\phi)$

To explicitly compute $\sigma^{\pm}_\mu(\phi)$, first we expand the exponent in the integral in ϕ representing $W(x,\phi) = W_0(x) + \sum_s \phi_s e_s(x)$

$$\sigma^{\pm}_\mu(\phi) = \sum_m \int_{\Gamma^{\pm}_\mu} \prod_r e_r(x)^{m_r}\, e^{\mp W_0(x)}\, \mathrm{d}^5x \, \left(\prod_s \frac{(\pm\phi_s)^{m_s}}{m_s!} \right),$$

where $m := \{m_s\}_s,\ m_s \geq 0$ denotes a multi-index of powers of ϕ_s in the expansion above.
$\sigma^{-}_\mu(\phi) = (-1)^{|\mu|}\sigma^{+}_\mu(\phi)$, so we focus on $\sigma_\mu(\phi) := \sigma^{+}_\mu(\phi)$.

Each differential form $\prod_s e_s(x)^{m_s}\,\mathrm{d}^5x$ belongs to $H^5_{D_\pm}(\mathbb{C}^5)_Q$.
It follows that it is equal to a linear combination of
$e_\mu d^5x \in H^5_{D_\pm}(\mathbb{C}^5)_Q$ modulo D_+ - exact terms.
We use this fact for computing oscilatory integrals taking into account that they vanish for D_+-exact terms and

$$\int_{\Gamma^{+}_\mu} e^{-W_0(x)} P(x)\,\mathrm{d}^5x = \int_{\Gamma^{+}_\mu} e^{-W_0(x)}(P(x)\,\mathrm{d}^5x + D_+U)$$

for any polynomial $P(x)$ and any polynomial $4-$form U.

Computation of periods $\sigma_\mu(\phi)$

Let us denote $\sum_s m_s s_i = \nu_i + n_i \frac{d}{k_i},\ \nu_i < \frac{d}{k_i}$ (for later convenience). To compute

$$\int_{\Gamma_\mu^+} e^{-W_0(x)} \prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x,$$

we use the above property with

$$\prod_i x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x =$$

$$= (-1) \left(n_1 - 1 + \frac{k_1(\nu_1 + 1)}{d} \right) x^{\nu_1 + (n_1 - 1)\frac{d}{k_1}} \prod_{i>1} x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}^5 x + D_+ U.$$

where

$$U = \frac{k_1}{d} x_1^{\nu_1 + 1 + (n_1 - 1)\frac{d}{k_1}} \prod_{i>1} x_i^{\nu_i + n_i \frac{d}{k_i}} \, \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_5.$$

Repeating this 4 times we obtain (modulo an exact term)

$$\prod_i x_i^{\nu_i+n_i\frac{d}{k_i}}\,\mathrm{d}^5x = (-1)\prod_i\left(n_i-1+\frac{k_i(\nu_i+1)}{d}\right)\prod_i x_i^{\nu_i+(n_i-1)\frac{d}{k_i}}\,\mathrm{d}^5x.$$

It follows that

$$\prod_i x_i^{\nu_i+n_i\frac{d}{k_i}}\,\mathrm{d}^5x = (-1)^{\sum_i n_i}\prod_i\frac{\Gamma(\frac{k_i(\nu_i+1)}{d}+n_i)}{\Gamma(\frac{k_i(\nu_i+1)}{d})}\prod_i x_i^{\nu_i}\,\mathrm{d}^5x,\ \nu_i<\frac{d}{k_i}.$$

That is, if any $\nu_i=\frac{d}{k_i}-1$, the form is exact, and the integral is zero. Otherwise, the rhs is proportional to $e_\nu(x)\mathrm{d}^5x$.
Using the definition of Γ^+_μ cycles $\int_{\Gamma^+_\mu} e_\nu(x)\,e^{-W_0(x)}\,\mathrm{d}^5x=\delta_{\mu\nu}$ we perform integrating over Γ^+_μ and obtain that the period

$$\sigma_\mu(\phi)=\sum_{n_i\geq 0}\prod_i\frac{\Gamma\left(n_i+\frac{k_i(\mu_i+1)}{d}\right)}{\Gamma\left(\frac{k_i(\mu_i+1)}{d}\right)}\sum_{m\in\Sigma_n}\prod_s\frac{\phi_s^{m_s}}{m_s!},$$

where $\Sigma_n=\{m_s\mid\sum_s m_s s_i=\mu_i+\frac{d}{k_i}n_i\}$.

Formula for Kähler potential

Pick Lefschetz thimbles $L_\mu^\pm$ as a basis of cycles with real coefficients.
Define T as the transition matrix from cycles $\Gamma_\mu^\pm$ to Lefschetz thimbles $L_\mu^\pm$

$$\Gamma_\mu^\pm = (T^{-1})_{\mu\nu}\, L_\nu^\pm .$$

and compute the transition matrix $T_{\mu\nu}$ using the relation

$$T_{\mu\nu} = \int_{L_\mu^\pm} e_\nu(x)\, e^{\mp W_0(x)} \mathrm{d}^5 x.$$

Then we obtain

$$M = T^{-1}\, \bar{T}$$

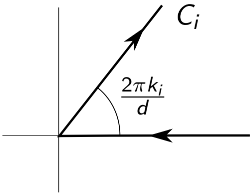
which we need to insert to the expression for Kähler potential together with $\eta_{\mu\nu} = \delta_{\mu,\rho-\nu}$.

Lefschetz thimbles

Lefschetz thimbles $L_\mu^\pm$ are products of one-dimensional cycles C_{μ_i}

$$L_\mu^+ = \prod_{i=1}^{5} C_{\mu_i},$$

and $C_{\mu_i} = \hat{\rho}_i^{\mu_i} \cdot C_i$ with $\rho_i = e^{\frac{2\pi i k_i}{d}}$.
This definition of one-dimensional cycle C_{α_i} means that this cycle is the path in x_i-plane obtained by rotating counter clockwise through angle $\frac{2\pi k_i \mu_i}{d}$ from the basic path C_i depicted on the figure



By construction $L_\mu^\pm$ are steepest descent/acsent cycles for $\mathrm{Re} W_0$.

We now compute $T_{\alpha\mu}$ explicitly

$$T_{\alpha\mu} = \int_{L_\alpha^+} e_\mu \, e^{-W_0} \, \mathrm{d}^5 x = \rho^{(\bar{\alpha},\bar{\mu})} A(\mu),$$

where A_μ is a product of five gamma integrals,

$$A_\mu = \prod_i \left(\frac{k_i}{d}\right) \Gamma\left(\frac{k_i(\mu_i + 1)}{d}\right).$$

Then

$$T^{-1}_{\bar{\mu}\bar{\alpha}} = B(\mu)[\bar{\rho}^{(\bar{\mu}+1,\bar{\alpha})} - 1],$$

$$B(\mu) = \prod_i \frac{1}{\Gamma\left(\frac{k_i(\mu_i+1)}{d}\right)},$$

$$M_{\mu\nu} = (T^{-1}\bar{T})_{\mu\nu} = \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) \delta_{\mu,\rho-\nu},$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$

Substituting the explicit expressions for the periods σ_μ, the pairing $\eta_{\mu\nu}$, and the anti-involution M in the above expression for the Kähler potential on the moduli space, we obtain

$$e^{-K(\phi)} = \sum_{\mu} (-1)^{\deg(\mu)/d} \prod_i \gamma\left(\frac{k_i(\mu_i+1)}{d}\right) |\sigma_\mu(\phi)|^2,$$

where

$$\sigma_\mu(\phi) = \sum_{n_1,\ldots,n_5 \geq 0} \prod_{i=1}^{5} \frac{\Gamma(\frac{k_i(\mu_i+1)}{d} + n_i)}{\Gamma(\frac{k_i(\mu_i+1)}{d})} \sum_{m \in \Sigma_n} \prod_s \frac{\phi_s^{m_s}}{m_s!},$$

$0 \leq \mu_i \leq \frac{d}{k_i} - 2, \quad \sum_{i=1}^{5} \mu_i = 0, d, 2d, 3d,$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}, \qquad \Sigma_n = \{m_s \mid \sum_s m_s s_i = \mu_i + \frac{d}{k_i} n_i\}.$$

THANK YOU!

Appendix A. Coinsidence of two pairings. Proof.

Following Chiodo et al we prove the second equality

$$\eta_{ab} = \mathrm{Res}\,\frac{e_a \cdot e_b \,\mathrm{d}^n x}{\partial_1 W \cdots \partial_n W} = \int_{L^+_\mu} e_a\, e^{-W} \mathrm{d}^n x\; C_{\mu\nu} \int_{L^-_\nu} e_b\, e^{W} \mathrm{d}^n x$$

Consider a small relevant perturbation of the isolated singularity $W(x,t) = W(x) + \sum e_i x_i$. Then the isolated singularity of W located in $x_i = 0$ transforms to a set of Morse critical points $\{p_\mu\}$. Consider instead of η_{ab} the bilinear form

$$\eta_{ab}(t,z) := \int_{L^+_\mu} e_a\, e^{-W(x,t)/z} \mathrm{d}^n x\; C_{\mu\nu} \int_{L^-_\nu} e_b\, e^{W(x,t)/z} \mathrm{d}^n x$$

First of all we notice, that if $t = 0$, then

$$\eta_{ab}(t=0,z) = z^n \cdot \eta_{ab}(t=0,1).$$

We take as basis of cycles $L^\pm_\mu$ the so-called of Lefschetz thimbles. They start from Morse points p_μ and goes along the gradient of $\mathrm{Re}W(x,t)$ in the direction of the steepest descent/ascent. With the proper orientation their intersections are $L^+_\mu \cap L^-_\nu = \delta_{\mu\nu}$.

Appendix A. Coincidence of two pairings. Proof.

Then rhs of the equality becomes in this basis:

$$\sum_{\mu} \int_{L_\mu^+} e_a \, e^{-W(x,t)/z} \mathrm{d}^n x \int_{L_\mu^-} e_b \, e^{W(x,t)/z} \mathrm{d}^n x$$

From stationary phase expansion as $z \to 0$ we obtain for a period:

$$\int_{L_\mu^+} e_a(x) \, e^{-W(x,t)/z} \mathrm{d}^n x = \frac{(2\pi z)^{n/2}}{\sqrt{\mathrm{Hess}\, W(p_\mu, t)}} (e_a(p_\mu) + O(z))$$

Using this we get

$$\eta_{ab}(t,z) = \sum_{\mu} (2\pi i z)^n \frac{e_a(p_\mu) \cdot e_b(p_\mu)}{\mathrm{Hess}(W(p_\mu, t))} (1 + O(z)) =$$

$$(2\pi i z)^n \left(\mathrm{Res} \, \frac{e_a \cdot e_b \, \mathrm{d}^n x}{\partial_1 W \cdots \partial_n W} + O(z) \right)$$

It holds for $t = 0$. Taking into account $\eta_{ab}(0,z) = z^n \cdot \eta_{ab}(0,1)$ we obtain the equality $< \chi_a, \chi_b >= \eta(e_a, e_b)$.