INTEGRABLE COMBINATORICS

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- Jean Zinn-Justin
- Paul Ginsparg
- David Sénéchal

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STRUCTURES OF INTEGRABILITY

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# Structures of Integrability

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### Tools of Combinatorics

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# Tools of Combinatorics

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INTEGRABLE COMBINATORICS

PHILIPPE DI FRANCESCO

1. Lorentzian gravity: an archetype of quantum Integrable Trees
2. Planar Maps: an archetype of discrete Integrable Trees
3. Quantum integrability and the enumeration of ASM, DPP, TSSCPP.
4. Graded tensor product characters: paths, cluster Algebra and beyond

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INTEGRABLE COMBINATORICS

PHILIPPE DI FRANCESCO

1. Lorentzian gravity: an archetype of quantum Integrable Trees
2. Planar Maps: an archetype of discrete Integrable Trees
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1. 1+1-Dimensional Lorentzian Gravity

[PDF + Emmanuel Guitter + Charlotte Kristjansen '99]
[Ambjorn + Loll + Watabiki + Jurkiewicz + Kristjansen].

Triangulations that are
1. Random in space direction
2. Regular in time direction

[Diagram showing a grid with arrows indicating time and space directions]
1+1 Dimensional Lorentzian Gravity

Triangulations that are:
1. Random in space direction
2. Regular in time direction
1+1 Dimensional Lorentzian Gravity

\[ \text{Dual } = \text{trivalent graph} \]

→ squeeze ←
$1+1$ Dimensional Lorentzian Gravity

$\equiv$ TREE
(Basic Combinatorial object)
Triangulations that are:
1. Random in space direction
2. Regular in time direction

$\Rightarrow \text{TRANSFER MATRIX}$

\[
T_{ij} = \binom{i + \bar{d}}{\bar{i}}
\]

($i, \bar{d} \in \mathbb{Z}_+$)
Include curvature weight $a / 1$ or $1$

area weight $g / 1$ or $1$

Then

$$T_{ij}(g,a) = (ag)^{i+j} \sum_{k=0}^{\min(i,j)} \binom{i}{k} \binom{j}{k} a^{-2k}$$

or Generating Function

$$\sum_{i,j \geq 0} z^i w^j T_{ij}(g,a) = \frac{1}{1 - ga(z+w) - g^2(1-a^2)zw}$$
INTEGRABILITY \[\text{DF, Guiller, Kristjansen 99}\]

\[
\left[T(g,a) , T(g',a')\right] = 0
\]

\[\iff \varphi(g,a) = \varphi(g',a')\]

\[
\varphi(g,a) = \frac{1 - g^2(1-a^2)}{ag}
\]

- Diagonalization of \(T\)
- Correlations

INFINITE MATRICES!
2. PLANAR MAPS & GEODESICS

2D lattice model \[ \sigma(0) \]

2D quantum gravity
(Euclidean)
[90's]

2D random tessellation

[David, Disher-Kawai '90]

[Gross-Migdal, Douglas-Shenker, Brezin-Kazakov '90]
2. PLANAR MAPS & GEODESICS

- 2D lattice model
- 2D quantum Gravity (Euclidean) [90’s]
- 2D random tessellation

- Matrix models, exact solutions [90’s, 00’s]
- Integrability
- Correlations at fixed geodesic distance? [DF+ Guiller 02]
• Model for discrete random surfaces

→

Planar map
• trivalent
• 2 legs
• intrinsic geometry

Planar map
  • tetrahedral
  • 2 legs
  • geodesic distance between the legs = 2 here
MAP - TREE BIJECTION

$R_n = \text{generating function for maps with legs at distance } \leq n$

[Schaeffer 98]
MAP-TREE BIJECTION

\[ R_n = \text{generating function for maps with legs at distance } \leq n \]

\[ R_n = 1 + gR_n(R_{n+1} + R_n + R_{n-1}) \]
INTEGRABILITY & SOLUTION

\[ 0 = R_n - 1 - g R_n (R_{n+1} + R_n + R_{n-1}) \]

- This is a discrete integrable equation \( (n = \text{time}) \)

\[ \exists \, \mathcal{V}(x, y) = xy (1 - g (x+y)) - x - y \quad \text{such that} \]

\[ \mathcal{V}(R_n, R_{n+1}) - \mathcal{V}(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) \times \]

\[ \times \]

\[ (x) \]

\[ (R_{-1} = 0, R_{\infty} = R) \]
INTEGRABILITY & SOLUTION

\[ 0 = R_{n-1} - g R_n (R_{n+1} + R_n + R_{n-1}) \times \quad (\star) \quad (R_{-1} = 0, \ R_\infty = R) \]

- This is a discrete integrable equation \((n = \text{time})\)

\[ \exists \ \varphi(x,y) = xy (1 - g(x+y)) - x - y \quad \text{such that} \]
\[ \varphi(R_n, R_{n+1}) - \varphi(R_{n-1}, R_n) = (R_{n+1} - R_{n-1}) \times \ (\star) = 0 ! \]

- Exact soliton solution \([\text{DF, Boultier, Guiller 05-07}]\)

\[
R_n = R \frac{1 - x^{n+1}}{1 - x^{n+2}} \frac{1 - x^{n+4}}{1 - x^{n+3}} \\
R = 1 + 3g R^2 \\
x + \frac{1}{x} + 4 = \frac{1}{g R}, \ |x| < 1.
\]

- \(d_F = 4\)
- Scaling functions
- Continuum limit/proba

\([\text{LeGall, Miermont, DF, Boultier, Guiller}]\)
3. 6V, ASM

Alternating sign matrices (Robbins-Rumsey 82)
Six vertex model (Lieb 67)
6V, ASM, TSSCPP

Alternating sign matrices [Robbins-Rumsey 82]
Six vertex model [Lieb 67]

Totally symmetric self-complementary plane partition (70's)
6V, ASM, TSSCPP, DPP

Alternating sign matrices
[Robbins–Rumsey 82]
Six vertex model
[Lieb 67]

Descending Plane partitions
Andrews (70's)

Totally Symmetric Self-Complementary Plane Partition
\( A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \)

Zeilberger \quad Kuperberg \quad (96)

MRR \quad Conj. \quad (refinement) \quad Mills Robbins Rumsey 83

Alternating sign matrices \quad Robbins Rumsey 82 \quad Six vertex model \quad Lieb 67

Descending plane partitions \quad [Andrews 70, 75]

Totally symmetric self-complementary plane partition \quad [DF Behrend Zinn Justin 11-12]
GV, ASM, TSSCPP, DPP and LOOP MODEL

Alternating sign matrices

Six vertex model

Loop model

q KZ solution

Nienhuis, deGier, Razumov (00)
Stroganov
DF Zinn Justin (04)

Continisportelli [10]

Totally Symmetric Self-Complementary Plane Partition
GV, ASM, TSSCPP, DPP, Loop Model & The Nilpotent Variety

Alternating sign matrices
Six vertex model

\[ V = \{ M \in \mathbb{M}_{2n}(\mathbb{C}), M^2 = 0 \text{ and strictly upper triangular} \} \]

Totally Symmetric Self-Complementary Plane Partition

[DF-Zinn Justin 04-08]
[Knutson ZJ 10]
LOOP MODEL & THE qKZ equation

[quantum Knizhnik-Zamolodchikov]

R-matrix $R(z,w) := \frac{q^2 - q^1}{q^1 2 - q^2} \left( 1 + \frac{z - w}{q^1 + q^2 - q} \right) e_i$

generators of the Temperley-Lieb algebra

product of commuting transfer matrices

ground state $\Psi(z_1, z_{2n})$

$\Psi$ [diagram]

$\mathcal{T} \Psi(z_1, \ldots, z_{2n}) = \Psi(z_1, \ldots, z_{2n})$

← 1more!
LOOP MODEL & THE qKZ equation

[quantum Knizhnik-Zamolodchikov]

\[ R \text{-matrix } R(z,w) := \frac{z - q^{-1} w}{q^{-1} z - q w} e^{i} \]

uses Yang-Baxter equation

\[ R(z_i, z_{i+1}) R(t, z_i) R(t, z_{i+1}) \]

\[ R(t, z_{i+1}) R(t, z_i) R(z_i, z_{i+1}) \]
LOOP MODEL & THE qKZ equation

quantum Knizhnik-Zamolodchikov

\[ R(-) \]

\[ \prod_{i=1}^{\infty} \text{commuting transfer matrices} \]

\[ \text{ground state } \Psi(z_1, \ldots, z_n) \]

\[ z_i^{w_i} = \prod_{i=1}^{\infty} z_i^{w_i} \]

\[ \text{generators of the Temperley-Lieb algebra} \]

\[ R_{ij} \Psi(z_1, \ldots, z_n) = \delta_{ij} \Psi(z_1, \ldots, z_n) \]

\[ R_{ij} \Psi(z_1, \ldots, z_n) = \delta_{ij} \Psi(z_1, \ldots, z_n) \]

\[ \text{permutation } z_i \leftrightarrow z_{i+1} \]
Quantum integrability has provided the missing link between these mysterious enumeration results.

- Solving qKZ eqns $\rightarrow$ Entries of groundstate 4 for the loop model-nilpotent variety equivariant cohomology

- lead to refined sum rules for ASM, TSSCPP, DPP.

[DF, P.Zinn-Justin 04, 05, 06] [DF 07, 08]
[DF, Behrend, P.Zinn-Justin 11, 12]
[Wheeler, Knutson+Zinn-Justin 16, 17]
4. GRADED TENSOR PRODUCT MULTIPLEITIES
CLUSTER ALGEBRAS & NONCOMMUTATIVE PATHS

Context: Combinatorics of Bethe Ansatz solution for the inhomogeneous generalized Heisenberg quantum spin chain for $\mathfrak{g}$

$R$-matrix $V_0 \otimes V_i \rightarrow V_i \otimes V_0$

Kirillov-Reshetikhin modules of Yangian ($\mathfrak{g}$) ($\mathfrak{g}$)

[Chari]

- Bethe [1931]: case $\mathfrak{g} = \mathfrak{sl}_2$, $V_i \cong \mathbb{C}^2$
- State space $V_1 \otimes \cdots \otimes V_N$ with natural grading by combinatorial linearized energy function \cite{KirillovReshetikhin87}

- Compute graded characters/multiplicities

\[ x_{V_1 \otimes \cdots \otimes V_N}(q, z) = \sum_\lambda \text{Mult}_q(V_1 \otimes \cdots \otimes V_N; V_\lambda) \, \text{ch}_{V_\lambda}(z) \]

~ Hilbert Polynomial

\[ \text{graded mult., irreducibles} \]

\[ \text{Fusion product} \]

\[ \text{irred. wt.-characters} \]
CLASSICAL CASE \((q=1)\)

Notations:

- \(V_i\) have the form \(KR_{\alpha,i} \supset V_i \omega_{\alpha} \)
- \(V_1 \otimes \ldots \otimes V_N = \bigotimes_{\alpha,i} KR_{\alpha,i}^{\otimes n_{\alpha,i}} \ (\alpha=1 \ldots r)\)
- \(X_{V_1 \otimes \ldots \otimes V_N}(q,z) = X_n(q,z)\)
$Q$-system for $\mathfrak{g}$

A non-linear recursion relation of the form:

$$Q_{\alpha_i, i+1} Q_{\alpha_i, i-1} = Q_{\alpha_i}^2 - \prod_{\beta : C_{\alpha \beta} = -1} Q_{\beta, i}$$

$C = $ Cartan matrix

($\mathfrak{g}$ simply-laced)
The Q-system is obeyed by KR characters

\[ \mathcal{Q}_{\alpha, i} = X_{KR_{\alpha_i}}(z) \]

\[ \left\{ \begin{array}{l}
\mathcal{Q}_{\alpha, 0} = 1 \\
\mathcal{Q}_{\alpha, 1} = f_{\text{character}}
\end{array} \right. \]

Example

- \( \omega_j = A_{N-1} \)

\[ \mathcal{Q}_{\alpha, 1} = S_{\{ \alpha \}} \]

\[ \mathcal{Q}_{\alpha, k} = S_{\{ \alpha \}} \]

\[ \mathcal{Q}_{\alpha, n+1} \mathcal{Q}_{\alpha, n-1} = \mathcal{Q}_{\alpha, n}^2 - \mathcal{Q}_{\alpha+h, n} \mathcal{Q}_{\alpha-l, n} \]
The Q-system is obeyed by KR characters

\[ Q_{\alpha_i} = X_{KR_{\alpha_i}}(z) \]

\[ \begin{cases} Q_{\alpha_{i0}} = 1 \\ Q_{\alpha_{i1}} = f_i \end{cases} \quad \text{(BC*)} \]

**THM** There exists a linear functional \( \phi \) from the algebra of solutions to the Q-system with (BC*) to symmetric polynomials, such that:

\[ \phi \left( \prod Q_{\alpha_i}^{n_{\alpha_i}} \right) = X_n(1, z) \quad \text{(not the obvious)} \]
CLUSTER ALGEBRA connection

[Fomin-Zelevinsky 00]

- Q-systems are naturally associated to cluster algebras
  [Kedem 07, DF-Kedem 08]

- quiver = given by adjacency matrix
- Cartan matrix

\[
C = \begin{pmatrix}
C^t & C & C \\
-C^t & 0 & \ldots \\
\ldots & \ldots & \ldots
\end{pmatrix}_{2r \times 2r}
\]

Ex: \[
A_5 = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\circ & \circ & \circ & \circ & \circ
\end{array}
\]

\[
\begin{array}{cccccc}
Q_1, n & Q_2, n & Q_3, n & Q_4, n & Q_5, n \\
\circ & \circ & \circ & \circ & \circ
\end{array}
\]

- cluster variables \( \{ Q_{a,i}, Q_{a,i+1} \}_{i=1}^{2r} \) (2r variables in each cluster)
- with mutation rules that produce only Laurent polynomials of the initial variables
- of SU(N) Yang-Mills theories
- Seiberg-Witten curve A case
- \( \mathfrak{e}N=2 \) susy gauge theories
- \( 2r \) variables
CLUSTER ALGEBRA connection

[Fomin-Zelevinsky 00]

- Q-systems are naturally associated to cluster algebras
  [Kedem 07, DF-Kedem 08]

- quiver = given by adjacency matrix  \( B = \begin{pmatrix} C^t & C & C \\ -C^t & 0 \end{pmatrix} \) \(_{2r}^{2r+1} \) vertices

\[ C = \text{Cartan matrix} \]

Ex:

\[ D_4 \]

\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \begin{array}{c}
| \quad | \\
| \quad | \\
| \quad | \\
| \quad | \\
| \quad | \\
\end{array} \]

- cluster variables \( \{ Q_{x_1}, Q_{x_{i+1}} \} \) \(_{2r}^{2r+1} \) variables on each cluster

with mutation rules that produce only Laurent polynomials of

the initial variables

[cf SU(N) Yang-Mills theories
Seiberg-Witten curve A case
\( \text{er} = 2 \) susy gauge theories]
QUANTIZATION

- natural procedure using inverse of $B$-matrix; $\Lambda = \det C \cdot C^{-1}$
  - leads to quantum $Q$-system

\[
\kappa \lambda_{x} Q_{\alpha,i} Q_{\alpha,i-1} = Q_{\alpha,i}^2 - \prod_{\beta \neq \alpha} Q_{\beta,i}
\]

\[
Q_{\alpha,i} Q_{\beta,i+1} = \kappa^{\lambda_{\alpha\beta}} Q_{\beta,i+1} Q_{\alpha,i}
\]

$Q$ = non-commuting variable!

Define \[ q = \kappa^{-\det(C)} \rightarrow \text{same as previous $q$-grading} \]
QUANTUM CASE

There exists a linear functional $\phi$ from the algebra of solutions to the quantum Q-system to symmetric polynomials, such that:

$$\phi \left( \prod_{i=k}^{1} \prod_{\alpha=1}^{r} (Q_{\alpha,i})^{n_{\alpha,i}} \right) = \chi_n (q, z)$$

[DF, Redem, 16-18]

The quantum cluster algebra $\sim$ quantum deformation by KR energy $\sim$ deformed Grothendieck ring
Q-systems, discrete integrability and Paths: THE SL₂ EXAMPLE

SL₂ Q-system: \[ Q_{n+1} Q_{n-1} = Q_n^2 - 1 \]

Cluster algebra:

\[ \frac{Q_n^2 - 1}{Q_1} = Q_{-1} \]

\[ Q_2 = \frac{Q_1^2 - 1}{Q_0} \]

(\(\ast\)) is used as a forward mutation \( Q_{n+1} = \frac{(Q_n^2 - 1)}{Q_{n-1}} \)

backward mutation \( Q_{n-1} = \frac{(Q_n^2 - 1)}{Q_{n+1}} \)
Discrete Integrability \( (n = \text{discrete time}) \).

Conservation law:

\[
\frac{Q_{n+1} + Q_{n-1}}{Q_n} = \text{const \ modulo \ Q-system} = C
\]

- 2D phase space (initial data \((Q_0, Q_1)\)) \(\Rightarrow\) Integrable!
- 1 conserved quantity
General Solution

Define: \( y_1 = \frac{Q_1}{Q_0}, \quad y_2 = -\frac{1}{Q_0 Q_1}, \quad y_3 = \frac{Q_0}{Q_1} \)

Then: \( C = y_1 + y_2 + y_3 \quad 1 = y_1 y_3 \)

\[ F(t) := \sum_{n=0}^{\infty} t^n Q_n = \frac{Q_0}{1 - t y_1} \]

\[ \frac{1 - t y_2}{1 - t y_3} \]

partition function for weighted paths

continued fraction that stops...
MUTATIONS as CONTINUED FRACTION rearrangements

Lemma \[ \frac{a}{1 - b \frac{1 - c - u}{1 - c - u}} = a' + \frac{b'}{1 - c' \frac{1 - u}{1 - u}} \]
\[ \Rightarrow \begin{cases} a' = \frac{ac}{b+c} \\ b' = \frac{ab}{b+c} \\ c' = b+c \end{cases} \]

Example:

\[ y_1 = \frac{y_1 y_3}{y_2 + y_3} = \frac{Q_0}{Q_{-1}} \]
\[ y_2 = \frac{y_1 y_2}{y_2 + y_3} = -\frac{1}{Q_0 Q_{-1}} \]
\[ y_3 = \frac{y_2 + y_3}{Q_0 - Q_0^{-1}} = Q_{-1} / Q_0 \]

\[ \frac{Q_0}{1 - ty_1} \]
\[ \frac{1 - ty_2}{1 - ty_2} \]
\[ \frac{1 - ty_3}{1 - ty_3} \]

\[ F(t) = \sum_{n=0}^{\infty} t^n Q_n = \frac{Q_0}{1 - ty_1 - ty_2 - ty_3} \]

invariant under mutation \( n \rightarrow n-1 \)
QUANTIZATION

Quantum $SL_2$ $Q$-system:

\[
\begin{align*}
Q_{n+1}Q_{n-1} &= Q_n^2 - 1 \\
Q_nQ_{n+1} &= kQ_{n+1}Q_n
\end{align*}
\]

Still integrable: $y_1 = Q_1 Q_0^{-1}$, $y_2 = -Q_1^{-1} Q_0^{-1}$, $y_3 = Q_1^{-1} Q_0$

\[
Q_{n+1} + kQ_{n-1} = C Q_n \quad C = y_1 y_2 y_3, \quad k = y_3 y_1
\]

Same solution with "quantum paths"!

→ Extends to all $SL_N$ cases

→ Theory of Non-Commutative paths/continued fractions

[DF+Kedem 10] [Gelfand Retakh Wilson]
Question: formalize the operation that extends the spin chain by 1 site:

\[ \phi \left( Q_{\beta,k} \cdot \prod_{i} Q_{\alpha_i} \cdot n_{\alpha_i} \right) = \Omega_{\beta,k} \chi_{n_{\beta}} (q,z) = \chi_{n + \varepsilon_{\beta,k}} (q,z) \]
Answer: $\Theta$ is the following difference operator

\[ \Theta_{\beta, k} = \sum_{I \in \mathcal{P}[1, N]} \left( \prod_{i \in I} z_i \right)^k \left( \prod_{i \in I} \frac{z_i}{z_i - z_j} \right) \left( \prod_{i \in I} \Gamma_i \right) \]

where $\Gamma_i \uparrow f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, z_i, \uparrow, \ldots, z_N)$

Of Macdonald theory!
Theorem \( (\text{DF} + \text{Ko}) \) \( \mathcal{O}_{\beta, k} \) form a representation of the quantum Q-system for \( SL_N \). \( (\mathcal{O}_{N+1,k = 0}, \forall k) \)

\[
q^\alpha \mathcal{O}_{\alpha, k+1} \mathcal{O}_{\alpha, k-1} = \mathcal{O}_{\alpha, k}^2 - \mathcal{O}_{\alpha+1, k} \mathcal{O}_{\alpha-1, k}
\]

\[
\mathcal{O}_{\alpha, k} \mathcal{O}_{\beta, k+1} = q^{\min(\lambda, \beta)} \mathcal{O}_{\beta, k+1} \mathcal{O}_{\alpha, k}
\]

and

\[
\chi_n(q,z) = q^\# \prod \mathcal{O}_{\alpha_i, k}^{n_{\alpha_i}} \cdot 1
\]

Algebra of \( \mathcal{O}_{\alpha, k} \) is quotient of \( \mathcal{U}_q(\widehat{\text{Is}}^2) \) by \( \mathcal{O}_{N+1,k = 0} \)
A NATURAL $t$-DEFORMATION

Rem: Macdonald operators (for which Macdonald polynomials are common eigenfunctions) are a $t$-deformation of $\mathfrak{g}_0$.

\[ M_{\alpha,0} := \sum_{|I|=\alpha} \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in \mathfrak{g}} I_i \]
Rem: Macdonald operators (for which Macdonald polynomials are common eigenfunctions) are a $t$-deformation of $\Theta_{\alpha_0}$. [Macdonald]

\[
M_{\alpha_0} := \sum_{|I|=\alpha} \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} I_i
\]

Suggests to consider

\[
M_{\alpha,k} := \sum_{|I|=\alpha} \left( \prod_{i \in I} x_i \right)^k \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} I_i
\]
\( M_{\alpha,k} := \sum_{|I|=\alpha} \left( \prod_{i \in I} x_i \right)^k \prod_{i \in I} \prod_{j \in J} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I} I_i \)

- \( \lim_{t \to \infty} t^{-\alpha(N-\alpha)} M_{\alpha,k} = \mathcal{D}_{\alpha,k} \) (our operators)

- Question: what is the algebra of M's?
\[ M_{\alpha,k} := \sum_{|I|=\alpha} \left( \prod_{i \in I} x_i \right)^k \prod_{i \in I} \frac{t x_i - x_j}{x_i - x_j} \prod_{j \in J} I_i \]

- \( \lim_{t \to \infty} t^{-\alpha(N-\alpha)} M_{\alpha,k} = D_{\alpha,k} \) (our operators)

- **Question:** what is the algebra of \( M \)'s?

**Answers:**

1. Double Affine Hecke Algebra [Cherednik]

   \[ M_{\alpha,k} = T^k_+ (M_{\alpha,0}) \quad T_+ = \text{automorphism of } SL_2(\mathbb{Z}) \text{ action} \]

2. Quotient of Elliptic Hall Algebra [Schiffman-Vasserot]

   EHA \sim t\text{-deformation of } qQ\text{ system cluster algebra}
5. SUMMARY

- Integrability / Path-Tree formulations
  "integrable combinatorics" / continuous / discrete

- Cluster Algebra / Paths
  also dimer models / cluster integrable systems
  [Goncharov-Kenyon 13]

- Combinatorics survive beyond quantization
  → because of path formulations
  → quantum cluster algebra
    quantum determinant
HANKEL DETERMINANT and $A_{n-1}$ Q-system

Desnanot–Jacobi identity on minors of any matrix

\[
\begin{pmatrix}
\alpha + 1 & \alpha - 1 \\
\alpha & \alpha & \alpha & \alpha
\end{pmatrix}
\]

\[
Q_{\alpha + 1, n} \times Q_{\alpha - 1, n} = Q_{\alpha, n-1} Q_{\alpha, n+1} - \left( Q_{\alpha, n} \right)^2
\]

\[
Q_{\alpha, n} = \begin{vmatrix}
Q_{1, n-\alpha+1} & Q_{1, n-\alpha+2} & \cdots & Q_{1, n} \\
\vdots & \ddots & \ddots & \vdots \\
Q_{1, n} & \cdots & Q_{1, n+\alpha-1}
\end{vmatrix}
\]

is a Hankel determinant of the Q$_{1,n}$'s
QUANTUM DETERMINANT

\[ Q_{\alpha-1,n} \cdot Q_{\alpha-1,n} = q^\alpha Q_{\alpha,n+1} Q_{\alpha,n-1} - Q_{\alpha,n}^2 \]

= quantum Desnanot–Jacobi relation

THM [DF+Kedem 16]

\[ Q_{\alpha,k} = C \left( \prod_{i<j} (u_i - u_j)^{\frac{k}{k-i}} (1-q \frac{u_i}{u_j}) \right) Q(u_1) Q(u_2) \cdots Q(u_\alpha) \]

constant term

\[ q\text{-Vandermonde} \quad Q(u) = \sum_{n \in \mathbb{Z}} u^n Q_{\alpha,n} \]

Homogeneous polynomial of degree \( \alpha \) in \( Q_n \)'s
QUANTUM DETERMINANT

\[ Q_{\alpha+1,n} Q_{\alpha-1,n} = q^\alpha Q_{\alpha,n+1} Q_{\alpha,n-1} - Q_{\alpha,n}^2 \]

= quantum Desnanot-Jacobi relation

\[ \text{Ex: } \alpha = 3 \]

\[ Q_{3,n} = \begin{vmatrix} Q_{n-2} & Q_{n-1} & Q_n \\ Q_{n-1} & Q_n & Q_{n+1} \\ Q_n & Q_{n+1} & Q_{n+2} \end{vmatrix} q \]

\[ = Q_n^3 - q Q_{n+1} Q_{n-1} Q_n - q(1-q)Q_{n+1} Q_n Q_{n-1} - q Q_n Q_{n+1} Q_{n-1} + q^2 Q_{n+1}^2 Q_{n-1} + q^2 Q_{n+2} Q_{n-1}^2 - q^3 Q_{n+2} Q_n Q_{n-2} \]

\( \lambda \)-determinant [MRR] = sum over Alternating Sign Matrices

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\]
NEW DIRECTIONS

- $(q,t)$-determinant? [known $\alpha=2,3$ DF+Kedem 18]
- direct relation between cluster algebra and $t=\infty$ DAHA (nil-DAHA)? in progress
- other types (B,C,D in progress; [DF+Kedem 18])
- Path formulations and Macdonald polynomials Whittaker functions
- Beyond quantization: Non-commutative geometry? (Paths/Integrability/etc...)
  [DF+Kedem 09,10] [Berenstein, Retakh 15]
THE NON-COMMUTATIVE $SL_2$ $Q$-system

$q_{n+1} q^{-1}_{n} q^{-1}_{n-1} = q_{n} - q^{-1}_{n} \quad (*)$

Integrability?

NB (*) is equivalent to the NC determinant formula:

$\frac{(q_{n} - q_{n+1} q^{-1}_{n} q_{n})}{q_{n}} = 1$

Gelfand-Relikh quasi-determinant of the matrix

$\begin{vmatrix}
q_{n} & q_{n-1} \\
q_{n+1} & q_{n}
\end{vmatrix}$
THE NON-COMMUTATIVE $SL_2$ $Q$-system

$Q_{n+1} Q_n^{-1} Q_{n-1} = Q_n - Q_n^{-1}$ \quad (n \in \mathbb{Z})

2 conserved qties:

$Q_n^{-1} Q_{n+1} Q_n Q_{n+1}^{-1} = K$

$(Q_{n+1} + K Q_{n-1}) Q_n^{-1} = C$

continued fraction solutions:

$y_1 = Q_1 Q_0^{-1}$, $y_2 = Q_1 Q_0$

$y_2 = Q_1 Q_0^{-1}$, $K = y_3 y_1$, $C = y_4 y_2 y_3$

$F(t) = \sum_0^\infty t^n Q_n = Q_0 (1 - t (1 - t (1 - t y_3 y_2 y_1)^{-1})^{-1})^{-1}$

NC mutations by rearrangements $\rightarrow$ NC cluster algebra

[DF+Kedem 10]