The Teichmüller TQFT

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ICM, Rio de Janeiro, August 8, 2018

Motivation: quantum Chern–Simons theory

non-compact gauge groups

Given a non-compact Lie group G (e.g. $GL(2,R)$), a 3-manifold M.
Gauge fields $=$ G-connections $A \in \mathcal{A} := \Omega^1(M, Lie\ G)$.
Group of gauge transformations $\mathcal{G} := \mathcal{C}^\infty(M, G)$,

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A}, \quad (A, g) \mapsto A^g := g^{-1}Ag + g^{-1}\,\mathrm{d}g$$

Chern–Simons action functional

$$CS_M(A) := \int_M \mathrm{Tr}\,(3A \wedge \mathrm{d}A + 2A \wedge A \wedge A)$$

Phase space $=$ space of flat connections $= \hom(\pi_1(M), G)/G$.
Quantum partition function

$$Z_\hbar(M) = \int_{\mathcal{A}/\mathcal{G}} e^{\frac{i}{\hbar} CS_M(A)} \mathcal{D}A$$

expected to be a topological invariant of M.
Problem: give a mathematically rigorous definition and a calculational recipe for $Z_\hbar(M)$.

Some of the relevant works

1. Regge–Ponzano (1968)
2. Witten (1988/89, 1991, 2011), Bar-Natan–Witten (1991)
3. Verlinde (1990)
4. Reshetikhin–Turaev (1991), Turaev–Viro (1992)
5. K (1994, 1998)
6. Fock–Chekhov (1999)
7. Hikami (2001, 2007)
8. Baseilhac–Benedetti (2004)
9. Teschner (2007)
10. K–Reshetikhin (2007)
11. Dimofte–Gukov–Lenells–Zagier (2009)
12. Dijkgraaf–Fuji–Manabe (2011)
13. Geer–K–Turaev (2012)
14. Dimofte–Gaiotto–Gukov (2013)
15. Andersen–K (2014), Garoufalidis–K, K–Luo–Vartanov (2015)
16. Andersen–Marzioni, Andersen–Kratmann-Nissen (2017)
17. Mikhaylov (2018)

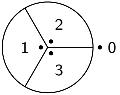
Teichmüller space

$S = S_{g,s}$ oriented surface of genus g punctured at s points, $(2g-2+s)s > 0$;
Chern–Simons theory with $G = PSL(2,\mathbb{R})$, $M = S \times \mathbb{R}$.
Teichmüller space $\mathcal{T}_S \subset \hom(\pi_1(S), G)/G$

- connected component corresponding to discrete faithful representations=set of hyperbolic structures on S;
- Weil–Petersson symplectic structure;
- mapping class group Γ_S acting by symplectomorphisms.

Useful fact: Γ_S freely acts on the set Δ_S of decorated ideal triangulations of S

- each triangle carries a marked corner;
- all triangles are linearly ordered (enumerated).

Example: $S = S_{0,4}$,  $\in \Delta_S$

To any group G freely acting on a set X, one associates a connected groupoid $\mathcal{G}_{G,X}$ as follows:

1. $\operatorname{Ob} \mathcal{G}_{G,X} = X/G$;
2. $\operatorname{Mor} \mathcal{G}_{G,X} = (X \times X)/G$ (diagonal action);
3. source $s([x,y]) = [x]$, target $t([x,y]) = [y]$;
4. $[x,y]$ and $[u,v]$ are composable iff $[y] = t([x,y]) = s([u,v]) = [u]$, i.e. $\exists!\ g \in G,\ y = gu$, $[x,y][u,v] = [x,y][gu,gv] = [x,y][y,gv] = [x,gv]$ (adopting the composition convention of fundamental groupoids).

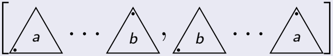
$\mathrm{id}_{[x]} = [x,x]$, $[x,y]^{-1} = [y,x]$, $\operatorname{Mor}([x],[x]) \simeq G$.

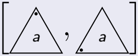
Groupoid of decorated ideal triangulations

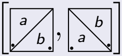
Theorem (Hyun Kyu Kim, 2016)

The groupoid $\mathcal{G}_{\Gamma_S,\Delta_S}$ admits the following presentation.

Generators:

1. *transpositions* $\tau_{a,b} :=$ 

;
2. *corner changes* $\rho_a :=$ 

;
3. *diagonal flips* $\omega_{a,b} :=$ 

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Relations:

1. $\tau_{a,b} = \tau_{b,a} = \tau_{a,b}^{-1}$, $\tau_{a,b}\tau_{b,c}\tau_{a,b} = \tau_{a,c}$;
2. $\rho_a\rho_a = \rho_a^{-1}$, $\omega_{a,b}\omega_{b,c} = \omega_{b,c}\omega_{a,c}\omega_{a,b}$, $\rho_a\rho_b\omega_{b,a}\rho_a^{-1}\omega_{a,b} = \tau_{a,b}$;
3. *trivial relations:* $\rho_a\rho_b = \rho_b\rho_a$, $\omega_{a,b}\rho_c = \rho_c\omega_{a,b}$, $\tau_{a,b}\rho_a = \rho_b\tau_{a,b}$, *etc.*

Basic algebraic systems

Let $\mathcal{C} = (\mathcal{C}, \otimes, \{P_{X,Y}\}_{X,Y \in \mathrm{Ob}\,\mathcal{C}})$ be a symmetric monoidal category. A *basic algebraic system* (BAS) in $\mathcal{C}$ consists of an object $V \in \mathrm{Ob}\,\mathcal{C}$ and two morphisms $R \in \mathrm{End}(V)$ and $W \in \mathrm{End}(V \otimes V)$ such that

1. $R^3 = \mathrm{id}_V$;
2. $W_{1,2} W_{2,3} = W_{2,3} W_{1,3} W_{1,2}$ in $\mathrm{End}(V^{\otimes 3})$;
3. $R_1 R_2 W_{2,1} R_1^{-1} W_{1,2} = P_{1,2}$ in $\mathrm{End}(V^{\otimes 2})$, $P_{1,2} := P_{V,V}$.

Theorem

Let (V, R, W) be a BAS in a symmetric monoidal category $\mathcal{C}$. Then there exists a canonical representation $\mathcal{F} : \mathcal{G}_{\Gamma_S, \Delta_S} \to \mathrm{Aut}\left(V^{\otimes 2(2g-2+s)}\right)$ such that $\tau_{a,b} \mapsto P_{a,b}$, $\rho_a \mapsto R_a$, $\omega_{a,b} \mapsto W_{a,b}$.

BAS from Teichmüller theory

By using ratios of Penner's λ-coordinates for $\mathcal{T}_S$, one obtains a Teichmüller BAS (V, R, W) in the symmetric monoidal category of sets $(\mathbf{Set}, \times)$, where $V = \mathbb{R}^2_{>0}$ and $R\colon V \to V$, $W\colon V^2 \to V^2$ are defined by

$$R(x_1, x_2) = \left(\frac{1}{x_2}, \frac{x_1}{x_2}\right), \quad W(\vec{x}, \vec{y}) = (\vec{x} \cdot \vec{y}, \vec{x} * \vec{y})$$

where

$$\vec{x} \cdot \vec{y} := (x_1 y_1, x_1 y_2 + x_2), \quad \vec{x} * \vec{y} := \left(\frac{x_2 y_1}{x_1 y_2 + x_2}, \frac{y_2}{x_1 y_2 + x_2}\right).$$

These maps are consistent with the symplectic structure

$$\omega_V = \frac{\mathrm{d}x_2 \wedge \mathrm{d}x_1}{x_2 x_1}.$$

By quantizing the Teichmüller BAS, for any $\hbar \in \mathbb{R}_{>0}$, one associates a (projective) quantum Teichmüller BAS $(\hat{V}, \hat{R}, \hat{W})$ in the symmetric monoidal category of (separable) complex Hilbert spaces $(\mathbf{Hilb}, \hat{\otimes})$, where $\hat{V} = L^2(\mathbb{R})$ and unitary operators

$$\hat{R} := e^{-\frac{i\pi}{3}} e^{i3\pi\hat{q}^2} e^{i\pi(\hat{p}+\hat{q})^2}, \quad \hat{W} := \Phi_\hbar(\hat{q}_1 + \hat{p}_2 - \hat{q}_2) e^{-2\pi i \hat{p}_1 \hat{q}_2}$$

with (normalised self-adjoint) Heisenberg's operators in $L^2(\mathbb{R})$

$$\hat{p}f(x) = (2\pi i)^{-1} f'(x), \quad \hat{q}f(x) = xf(x)$$

and Faddeev's quantum dilogarithm function

$$\Phi_\hbar(x) := \exp\left(\int_{\mathbb{R}+i\epsilon} \frac{e^{-i2xz}}{4\sinh(zb)\sinh(zb^{-1})z}\,\mathrm{d}z\right), \quad b+b^{-1} = 1/\sqrt{\hbar}.$$

Quantized Teichmüller BAS (continued)

This is a projective BAS

1. $\hat{R}^3 = \mathrm{id}_{\hat{V}}$;
2. $\hat{W}_{1,2}\hat{W}_{2,3} = \hat{W}_{2,3}\hat{W}_{1,3}\hat{W}_{1,2}$;
3. $\hat{R}_1\hat{R}_2\hat{W}_{2,1}\hat{R}_1^{-1}\hat{W}_{1,2} = e^{\frac{i\pi}{12\hbar}}\hat{P}_{1,2}$.

The corresponding properties of $\Phi_\hbar(x)$

$$\Phi_\hbar(x)\Phi_\hbar(-x) = \Phi_\hbar(0)^2 e^{i\pi x^2}, \quad \Phi_\hbar(0) = e^{\frac{i\pi}{24}\left(\frac{1}{\hbar}-2\right)},$$

$$\Phi_\hbar(\hat{p})\Phi_\hbar(\hat{q}) = \Phi_\hbar(\hat{q})\Phi_\hbar(\hat{p}+\hat{q})\Phi_\hbar(\hat{p}),$$

or equivalently

$$\Phi_\hbar(x)\Phi_\hbar(y) = \int_{\mathbb{R}^3} Q_{x,y}^{u,v,w}\Phi_\hbar(u)\Phi_\hbar(v)\Phi_\hbar(w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w,$$

$$Q_{x,y}^{u,v,w} := e^{i\pi\left(\frac{1}{4}+2(u-x)(w-y)-(u-v+w)^2\right)}$$

Further remarks on $\Phi_\hbar(x)$

Special value $\hbar = 1/4$,

$$e^{-\frac{i\pi}{12}}\Phi_{1/4}(x) = e^{\frac{1}{2\pi i}\operatorname{Li}_2\left(1-e^{2\pi x}\right)} = e^{i\pi x^2/2}F(ix)$$

Hölder's double sine function (1886)

$$F(x) := e^x \prod_{n\geq 1}\left(\left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^n e^{2x}\right)$$

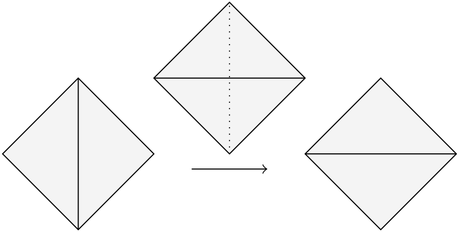
$F_\hbar(x) := e^{i\pi x^2/2}\Phi_\hbar(-ix)/\Phi_\hbar(0)$ [Shintani (1977), Ruijsenaars (1997)]

Relation to the Liouville central charge $c_L = 1 + \frac{6}{\hbar}$
Quasi-classical behaviour

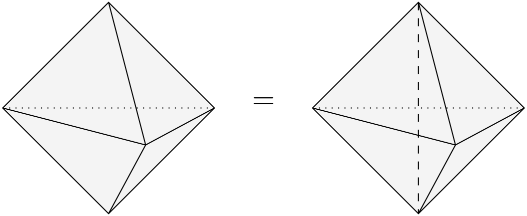
$$\Phi_\hbar\left(\frac{x}{2\pi\sqrt{\hbar}}\right) \overset{\hbar\to 0}{\sim} e^{\frac{1}{2\pi i\hbar}\operatorname{Li}_2(-e^x)}$$

Lifting to three dimensions

Diagonal flips can be realized in 3d terms by gluing tetrahedra



The pentagon (5-term) identity and the $2 - 3$ Pachner moves



Triangulations

X is "triangulated" $\Leftrightarrow$ an ordered Δ-complex (Hatcher) is a CW-complex where all cells are standard simplexes $\Delta^n = \{t\colon \{0,\dots,n\} \to [0,1] \mid \sum_i t_i = 1\}$ with face inclusion maps

$$f_i\colon \Delta^{n-1} \to \Delta^n, \quad f_i(t)_j = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

and if $\alpha\colon \Delta^n \to X$ is a characteristic map of a n-cell then $\partial_i \alpha := \alpha \circ f_i$ is a characteristic map of a $(n-1)$-cell.

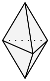
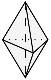
Notation:

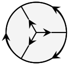
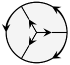
- $X_i :=$ the set of i-dimensional simplexes of X
- $X_{i,j} := X_i \times (\Delta^i)_j$

Shape structures

Let X be a closed ($\partial X = \emptyset$) oriented triangulated pseudo 3-manifold.

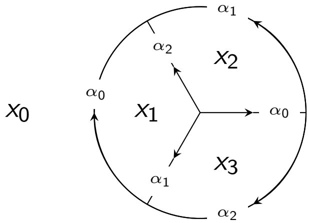
- A shape structure $\alpha\colon X_{3,1} \to]0,\pi[$ provides each tetrahedron with dihedral angles of an ideal hyperbolic tetrahedron.
- Gauge group action in the space of shape structures is generated by the total dihedral angles around edges through the Neumann–Zagier Poisson bracket.
- An edge is balanced if the associated total dihedral angle is 2π. A shape structure with all edges balanced is known as angle structure (Casson, Lackenby, Rivin).

The gauge equivalence class of a shape structure is invariant under angled 2-3 Pachner moves  $=$  (balanced internal edge).

In a triangulated oriented pseudo 3-manifold there are two types of tetrahedra: positive  and negative 

Two types of labellings:

- real numbers on faces $x\colon T_2 \to \mathbb{R}$ (quantum states)
- dihedral angles of ideal hyperbolic tetrahedra on edges $\alpha\colon T_1 \to \mathbb{R}_{>0}$, $\sum_i \alpha(\partial_i \partial_j T) = \pi$



$$x_i := x(\partial_i T)$$
$$\alpha_i := \alpha(\partial_i \partial_0 T)$$
$$\alpha_0 + \alpha_1 + \alpha_2 = \pi$$

Neumann–Zagier symplectic structure: $\omega_{NZ} = d\alpha_0 \wedge d\alpha_2$

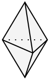
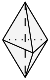
Tetrahedral weight functions

Define $W_\hbar(s,t,x,y,u,v) := \delta(x+u-y)\phi_{s,t}(v-u)e^{-2\pi i x(v-u)}$ where $\phi_{s,t}(z) := \Phi_\hbar\left(z + \frac{\pi - s}{2\pi i\sqrt{\hbar}}\right) e^{tz/\sqrt{\hbar}}$.
To an oriented tetrahedron T with labellings x and α, associate the weight function

$$Z_\hbar(T,x,\alpha) = W_\hbar(\alpha_0,\alpha_2,x_0,x_1,x_2,x_3)$$

if T is negative and complex conjugate otherwise.

Properties

- angled pentagon identity  $=$  (balanced internal edge);
- tetrahedral symmetry.

In the (degenerate) case of a negative flat tetrahedron with dihedral angles $\alpha_0 = \alpha_2 = 0$, $\alpha_1 = \pi$, the weight function is given by the operator kernel (in the coordinate representation) of the operator $\hat{W}$ of the quantum Teichmüller BAS $Z_\hbar(T,x,\alpha) = \langle x_1, x_3 | \hat{W} | x_0, x_2 \rangle$.

The partition function

For a closed oriented triangulated pseudo 3-manifold X with shape structure α, associate the partition function

$$Z_\hbar(X,\alpha) := \int_{x\in\mathbb{R}^{X_2}} \prod_{T\in X_3} Z_\hbar\left(T, x, \alpha|_T\right) \mathrm{d}x$$

Theorem (Andersen–K)

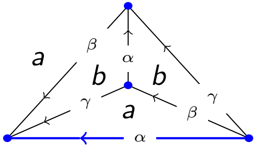
If $H_2(X\setminus X_0,\mathbb{Z})=0$, then $Z_\hbar(X,\alpha)$ is well defined (i.e. the integral converges), and its absolute value

- *depends on only the gauge equivalence class of α;*
- *is invariant under angled $2-3$ and $3-2$ Pachner moves.*

The construction extends to manifolds with boundary eventually giving rise to a generalised 3d TQFT.

Example of calculation

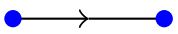
Hyperbolic S^3 with conical singularities along an embedded wedge of two circles

$(X, \alpha, \beta, \gamma) =$ 

$\simeq S^3,$

total dihedral angles

$w($  $) = \alpha$

$w($  $) = 2\pi - \alpha$

$$|Z_\hbar(X, \alpha, \beta, \gamma)| = |\Phi_\hbar\left(\frac{\alpha - \pi}{2\pi i\sqrt{\hbar}}\right)| = F_\hbar\left(\frac{\alpha - \pi}{2\pi\sqrt{\hbar}}\right) \overset{\hbar\to 0}{\sim} e^{-\frac{\mathrm{vol}(X_\alpha)}{2\pi\hbar}}$$

where

$$\mathrm{vol}(X_\alpha) := \sup\{\mathrm{vol}(\Delta^3, \alpha, \beta, \gamma) \mid \alpha \text{ is fixed}\} = 2\Lambda(\alpha/2),$$

with Milnor's formula

$$\mathrm{vol}(\Delta^3, \alpha, \beta, \gamma) := \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

and Lobachevsky's function

$$\Lambda(x) := -\int_0^x \log|2\sin t|\,\mathrm{d}t$$

The volume conjecture for the Teichmüller TQFT

Let $[\alpha]$ be the gauge equivalence class of a shape structure $\alpha\colon X_{3,1} \to]0,\pi[$. Define

$$\mathrm{vol}(X_{[\alpha]}) := \sup_{\beta\in[\alpha]} \sum_{T\in X_3} \mathrm{vol}(\Delta^3, \beta|_T)$$

Conjecture (Volume Conjecture for $Z_\hbar$)

$$\lim_{\hbar\to 0} 2\pi\hbar \log |Z_\hbar(X,\alpha)| = -\mathrm{vol}(X_{[\alpha]})$$

The VC for $Z_\hbar$ holds for some knot complements

- in S^3: 4_1 and 5_2 [Andersen–K, 2014], 6_1 [Andersen–Kratmann-Nissen, 2017], twist knots [Ben-Aribi–Piguet-Nakazawa, in progress];
- in $\mathbb{R}P^3$: [Piguet-Nakazawa, in progress].

Summary

- By using the combinatorial framework of triangulations, the quantum Teichmüller theory gives rise to a generalised 3d TQFT.
- The shape structures given by dihedral angles of ideal hyperbolic tetrahedra ensure the topological invariance and convergence.
- It is expected that the associated invariant exponentially decays for $\hbar \to 0$, the decay rate being given by the hyperbolic volume.