The Teichmüller TQFT

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Motivation: quantum Chern–Simons theory
non-compact gauge groups

Given a non-compact Lie group $G$ (e.g. $GL(2, R)$), a 3-manifold $M$. Gauge fields $= G$-connections $A \in \mathcal{A} := \Omega^1(M, \text{Lie } G)$. Group of gauge transformations $\mathcal{G} := C^\infty(M, G)$,

$$\mathcal{A} \times \mathcal{G} \to \mathcal{A}, \quad (A, g) \mapsto A^g := g^{-1}Ag + g^{-1}dg$$

Chern–Simons action functional

$$CS_M(A) := \int_M \text{Tr} \left( 3A \wedge dA + 2A \wedge A \wedge A \right)$$

Phase space $= \text{space of flat connections} = \text{hom}(\pi_1(M), G)/G$. Quantum partition function

$$Z_{\hbar}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{i\hbar CS_M(A)} DA$$

expected to be a topological invariant of $M$. **Problem**: give a mathematically rigorous definition and a calculational recipe for $Z_{\hbar}(M)$.
Some of the relevant works

1. Regge–Ponzano (1968)
3. Verlinde (1990)
17. Mikhaylov (2018)
Teichmüller space

$S = S_{g,s}$ oriented surface of genus $g$ punctured at $s$ points, $(2g - 2 + s)s > 0$;

Chern–Simons theory with $G = PSL(2, \mathbb{R})$, $M = S \times \mathbb{R}$.

Teichmüller space $T_S \subset \text{hom}(\pi_1(S), G)/G$

- connected component corresponding to discrete faithful representations $=$ set of hyperbolic structures on $S$;
- Weil–Petersson symplectic structure;
- mapping class group $\Gamma_S$ acting by symplectomorphisms.

Useful fact: $\Gamma_S$ freely acts on the set $\Delta_S$ of decorated ideal triangulations of $S$

- each triangle carries a marked corner;
- all triangles are linearly ordered (enumerated).

Example: $S = S_{0,4}$, $\begin{array}{c}
\bullet & \bullet & \circ \\
1 & 2 & 0 \\
\circ & \circ & \bullet \\
3 & \bullet & \bullet
\end{array} \in \Delta_S$
To any group \( G \) freely acting on a set \( X \), one associates a connected groupoid \( \mathcal{G}_{G,X} \) as follows:

1. \( \text{Ob} \mathcal{G}_{G,X} = X/G \);
2. \( \text{Mor} \mathcal{G}_{G,X} = (X \times X)/G \) (diagonal action);
3. source \( s([x, y]) = [x] \), target \( t([x, y]) = [y] \);
4. \([x, y]\) and \([u, v]\) are composable iff
   \([y] = t([x, y]) = s([u, v]) = [u] \), i.e. \( \exists! g \in G, y = gu \),
   \([x, y][u, v] = [x, y][gu, gv] = [x, y][y, gv] = [x, gv] \) (adopting the composition convention of fundamental groupoids).

\( \text{id}_{[x]} = [x, x], [x, y]^{-1} = [y, x], \text{Mor}([x], [x]) \simeq G. \)
The groupoid $\mathcal{G}_{\Gamma_S, \Delta_S}$ admits the following presentation.

**Generators:**

1. **Transpositions** $\tau_{a,b} := \begin{bmatrix} a & \ldots & b \cr b & \ldots & a \end{bmatrix}$;
2. **Corner changes** $\rho_a := \begin{bmatrix} a \cr a \end{bmatrix}$;
3. **Diagonal flips** $\omega_{a,b} := \begin{bmatrix} a & b \cr a & b \end{bmatrix}$.

**Relations:**

1. $\tau_{a,b} = \tau_{b,a} = \tau_{a,b}^{-1}, \tau_{a,b} \tau_{b,c} \tau_{a,b} = \tau_{a,c}$;
2. $\rho_a \rho_a = \rho_a^{-1}, \omega_{a,b} \omega_{b,c} = \omega_{b,c} \omega_{a,b}, \rho_a \rho_b \omega_{b,c} \rho_a^{-1} \omega_{a,b} = \tau_{a,b}$;
3. Trivial relations: $\rho_a \rho_b = \rho_b \rho_a, \omega_{a,b} \rho_c = \rho_c \omega_{a,b}, \tau_{a,b} \rho_a = \rho_b \tau_{a,b},$ etc.
Let $\mathcal{C} = (\mathcal{C}, \otimes, \{P_{X,Y}\}_{X,Y \in \text{Ob} \mathcal{C}})$ be a symmetric monoidal category. A basic algebraic system (BAS) in $\mathcal{C}$ consists of an object $V \in \text{Ob} \mathcal{C}$ and two morphisms $R \in \text{End}(V)$ and $W \in \text{End}(V \otimes V)$ such that

1. $R^3 = \text{id}_V$;
2. $W_{1,2} W_{2,3} = W_{2,3} W_{1,3} W_{1,2}$ in $\text{End}(V \otimes^3)$;
3. $R_1 R_2 W_{2,1} R_1^{-1} W_{1,2} = P_{1,2}$ in $\text{End}(V \otimes^2)$, $P_{1,2} := P_{V,V}$.

**Theorem**

Let $(V, R, W)$ be a BAS in a symmetric monoidal category $\mathcal{C}$. Then there exists a canonical representation $\mathcal{F} : \mathcal{G}_{\Gamma, \Delta} \to \text{Aut}(V \otimes^2(2g-2+s))$ such that $\tau_{a,b} \mapsto P_{a,b}$, $\rho_a \mapsto R_a$, $\omega_{a,b} \mapsto W_{a,b}$. 

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BAS from Teichmüller theory

By using ratios of Penner’s $\lambda$-coordinates for $T_S$, one obtains a Teichmüller BAS $(V, R, W)$ in the symmetric monoidal category of sets $(\text{Set}, \times)$, where $V = \mathbb{R}^2_{>0}$ and $R: V \to V$, $W: V^2 \to V^2$ are defined by

$$R(x_1, x_2) = \left( \frac{1}{x_2}, \frac{x_1}{x_2} \right), \quad W(\vec{x}, \vec{y}) = (\vec{x} \cdot \vec{y}, \vec{x} \ast \vec{y})$$

where

$$\vec{x} \cdot \vec{y} := (x_1y_1, x_1y_2 + x_2), \quad \vec{x} \ast \vec{y} := \left( \frac{x_2y_1}{x_1y_2 + x_2}, \frac{y_2}{x_1y_2 + x_2} \right).$$

These maps are consistent with the symplectic structure

$$\omega_V = \frac{dx_2 \wedge dx_1}{x_2x_1}.$$
By quantizing the Teichmüller BAS, for any $\hbar \in \mathbb{R}_{>0}$, one associates a (projective) quantum Teichmüller BAS $(\hat{V}, \hat{R}, \hat{W})$ in the symmetric monoidal category of (separable) complex Hilbert spaces $(\text{Hilb}, \hat{\otimes})$, where $\hat{V} = L^2(\mathbb{R})$ and unitary operators

$$\hat{R} := e^{-i\pi/3} e^{i3\pi \hat{q}^2} e^{i\pi(\hat{p} + \hat{q})^2}, \quad \hat{W} := \Phi_{\hbar}(\hat{q}_1 + \hat{p}_2 - \hat{q}_2) e^{-2\pi i \hat{p}_1 \hat{q}_2}$$

with (normalised self-adjoint) Heisenberg’s operators in $L^2(\mathbb{R})$

$$\hat{p}f(x) = (2\pi i)^{-1} f'(x), \quad \hat{q}f(x) = xf(x)$$

and Faddeev’s quantum dilogarithm function

$$\Phi_{\hbar}(x) := \exp \left( \int_{\mathbb{R}+i\epsilon} \frac{e^{-iz^2}}{4 \sinh(zb) \sinh(zb^{-1})z} \, dz \right), \quad b + b^{-1} = 1/\sqrt{\hbar}.$$
Quantized Teichmüller BAS (continued)

This is a projective BAS

1. \( \hat{R}^3 = \text{id}_\hat{\mathcal{V}} \);

2. \( \hat{W}_{1,2} \hat{W}_{2,3} = \hat{W}_{2,3} \hat{W}_{1,3} \hat{W}_{1,2} \);

3. \( \hat{R}_1 \hat{R}_2 \hat{W}_{2,1} \hat{R}_1^{-1} \hat{W}_{1,2} = e^{i\pi \frac{12}{\hbar}} \hat{P}_{1,2} \).

The corresponding properties of \( \Phi_\hbar(x) \)

\[
\Phi_\hbar(x)\Phi_\hbar(-x) = \Phi_\hbar(0)^2 e^{i\pi x^2}, \quad \Phi_\hbar(0) = e^{i\pi \frac{1}{24} (\frac{1}{\hbar} - 2)},
\]

\[
\Phi_\hbar(\hat{p})\Phi_\hbar(\hat{q}) = \Phi_\hbar(\hat{q})\Phi_\hbar(\hat{p} + \hat{q})\Phi_\hbar(\hat{p}),
\]

or equivalently

\[
\Phi_\hbar(x)\Phi_\hbar(y) = \int_{\mathbb{R}^3} Q_{x,y}^{u,v,w} \Phi_\hbar(u)\Phi_\hbar(v)\Phi_\hbar(w) \, du \, dv \, dw,
\]

\[
Q_{x,y}^{u,v,w} := e^{i\pi \left(\frac{1}{4} + 2(u-x)(w-y)-(u-v+w)^2\right)}
\]
Further remarks on $\Phi_\hbar(x)$

Special value $\hbar = 1/4$,

$$e^{-\frac{i\pi}{12}} \Phi_{1/4}(x) = e^{\frac{1}{2\pi i}} \text{Li}_2(1 - e^{2\pi x}) = e^{i\pi x^2/2} F(ix)$$

Hölder’s double sine function (1886)

$$F(x) := e^x \prod_{n \geq 1} \left( \left( \frac{1 - \frac{x}{n}}{1 + \frac{x}{n}} \right)^n e^{2x} \right)$$

$$F_\hbar(x) := e^{i\pi x^2/2} \Phi_\hbar(-ix)/\Phi_\hbar(0) \quad [\text{Shintani (1977), Ruijsenaars (1997)}]$$

Relation to the Liouville central charge $c_L = 1 + \frac{6}{\hbar}$

Quasi-classical behaviour

$$\Phi_\hbar \left( \frac{x}{2\pi \sqrt{\hbar}} \right) \xrightarrow{\hbar \to 0} e^{\frac{1}{2\pi i\hbar}} \text{Li}_2(-e^x)$$
Lifting to three dimensions

Diagonal flips can be realized in 3d terms by gluing tetrahedra

The pentagon (5-term) identity and the 2 − 3 Pachner moves
Triangulations

$X$ is “triangulated” $\iff$ an ordered $\Delta$-complex (Hatcher) is a CW-complex where all cells are standard simplexes

$\Delta^n = \{ t: \{0, \ldots, n\} \rightarrow [0,1] \mid \sum_i t_i = 1 \}$ with face inclusion maps

$$f_i: \Delta^{n-1} \rightarrow \Delta^n, \quad f_i(t)_j = \begin{cases} t_j & \text{if } j < i \\ 0 & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases}$$

and if $\alpha: \Delta^n \rightarrow X$ is a characteristic map of a $n$-cell then

$\partial_i \alpha := \alpha \circ f_i$ is a characteristic map of a $(n-1)$-cell.

Notation:

- $X_i :=$ the set of $i$-dimensional simplexes of $X$
- $X_{i,j} := X_i \times (\Delta^i)_j$
Let $X$ be a closed ($\partial X = \emptyset$) oriented triangulated pseudo 3-manifold.

- A **shape structure** $\alpha : X_{3,1} \rightarrow ]0, \pi[$ provides each tetrahedron with dihedral angles of an ideal hyperbolic tetrahedron.
- **Gauge group** action in the space of shape structures is generated by the total dihedral angles around edges through the Neumann–Zagier Poisson bracket.
- An edge is **balanced** if the associated total dihedral angle is $2\pi$. A shape structure with all edges balanced is known as an **angle structure** (Casson, Lackenby, Rivin).

The gauge equivalence class of a shape structure is invariant under **angled 2-3 Pachner moves** $\begin{array}{c|c}
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\end{array} = \begin{array}{c|c}
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\end{array}$ (balanced internal edge).
Labelled oriented tetrahedra

In a triangulated oriented pseudo 3-manifold there are two types of tetrahedra: positive and negative.

Two types of labellings:
- real numbers on faces \( x: T_2 \to \mathbb{R} \) (quantum states)
- dihedral angles of ideal hyperbolic tetrahedra on edges \( \alpha: T_1 \to \mathbb{R}_{>0}, \sum_i \alpha(\partial_i \partial_j T) = \pi \)

Neumann–Zagier symplectic structure: \( \omega_{NZ} = d\alpha_0 \wedge d\alpha_2 \)
Tetrahedral weight functions

Define \( W_\hbar(s, t, x, y, u, v) := \delta(x + u - y)\phi_{s, t}(v - u)e^{-2\pi i x(v-u)} \)
where \( \phi_{s, t}(z) := \Phi_\hbar \left(z + \frac{\pi - s}{2\pi i \sqrt{\hbar}}\right)e^{tz/\sqrt{\hbar}}. \)

To an oriented tetrahedron \( T \) with labellings \( x \) and \( \alpha \), associate the weight function

\[
Z_\hbar(T, x, \alpha) = W_\hbar(\alpha_0, \alpha_2, x_0, x_1, x_2, x_3)
\]

if \( T \) is negative and complex conjugate otherwise.

Properties

- angled pentagon identity \( \Diamond = \Diamond \) (balanced internal edge);
- tetrahedral symmetry.

In the (degenerate) case of a negative flat tetrahedron with dihedral angles \( \alpha_0 = \alpha_2 = 0, \alpha_1 = \pi \), the weight function is given by the operator kernel (in the coordinate representation) of the operator \( \hat{W} \) of the quantum Teichmüller BAS

\[
Z_\hbar(T, x, \alpha) = \langle x_1, x_3 | \hat{W} | x_0, x_2 \rangle.
\]
The partition function

For a closed oriented triangulated pseudo 3-manifold $X$ with shape structure $\alpha$, associate the partition function

$$Z_\hbar(X, \alpha) := \int_{x \in \mathbb{R}^2} \prod_{T \in X_3} Z_\hbar(T, x, \alpha|_T) \, dx$$

**Theorem (Andersen–K)**

If $H_2(X \setminus X_0, \mathbb{Z}) = 0$, then $Z_\hbar(X, \alpha)$ is well defined (i.e. the integral converges), and its absolute value

- depends on only the gauge equivalence class of $\alpha$;
- is invariant under angled $2 - 3$ and $3 - 2$ Pachner moves.

The construction extends to manifolds with boundary eventually giving rise to a generalised 3d TQFT.
Example of calculation
Hyperbolic $S^3$ with conical singularities along an embedded wedge of two circles

$$(X, \alpha, \beta, \gamma) = \begin{array}{c}
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\alpha
\end{array}
\end{array} \sim S^3, \quad \text{total dihedral angles}
$$

$$w(\rightarrow) = \alpha, \quad w(\rightarrow) = 2\pi - \alpha$$

$$|Z_{\hbar}(X, \alpha, \beta, \gamma)| = |\Phi_{\hbar} \left( \frac{\alpha - \pi}{2\pi i \sqrt{\hbar}} \right) | = F_{\hbar} \left( \frac{\alpha - \pi}{2\pi \sqrt{\hbar}} \right) \sim_{\hbar \to 0} e^{-\frac{\text{vol}(X_{\alpha})}{2\pi \hbar}}$$

where

$$\text{vol}(X_{\alpha}) := \sup \{ \text{vol}(\Delta^3, \alpha, \beta, \gamma) \mid \alpha \text{ is fixed} \} = 2\Lambda(\alpha/2),$$

with Milnor’s formula

$$\text{vol}(\Delta^3, \alpha, \beta, \gamma) := \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

and Lobachevsky’s function

$$\Lambda(x) := -\int_0^x \log |2 \sin t| \, dt$$
Let $[\alpha]$ be the gauge equivalence class of a shape structure $\alpha : X_{3,1} \to ]0, \pi[. Define

$$\text{vol}(X[\alpha]) := \sup_{\beta \in [\alpha]} \sum_{T \in X_3} \text{vol}(\Delta^3, \beta|_T)$$

Conjecture (Volume Conjecture for $Z_\hbar$)

$$\lim_{\hbar \to 0} 2\pi \hbar \log |Z_\hbar(X, \alpha)| = -\text{vol}(X[\alpha])$$

The VC for $Z_\hbar$ holds for some knot complements

- in $\mathbb{R}P^3$: [Piguet-Nakazawa, in progress].
By using the combinatorial framework of triangulations, the quantum Teichmüller theory gives rise to a generalised 3d TQFT.

The shape structures given by dihedral angles of ideal hyperbolic tetrahedra ensure the topological invariance and convergence.

It is expected that the associated invariant exponentially decays for \( \hbar \to 0 \), the decay rate being given by the hyperbolic volume.