On large time behavior of growth by birth and spread

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Contents

1. **Introduction: Models and Problems**
2. **The case of no curvature and spherical symmetric case**
3. **Existence of asymptotic speed for a Lipschitz source term**
4. **Estimate for asymptotic speed: flow with obstacle**
5. **A few open problems**

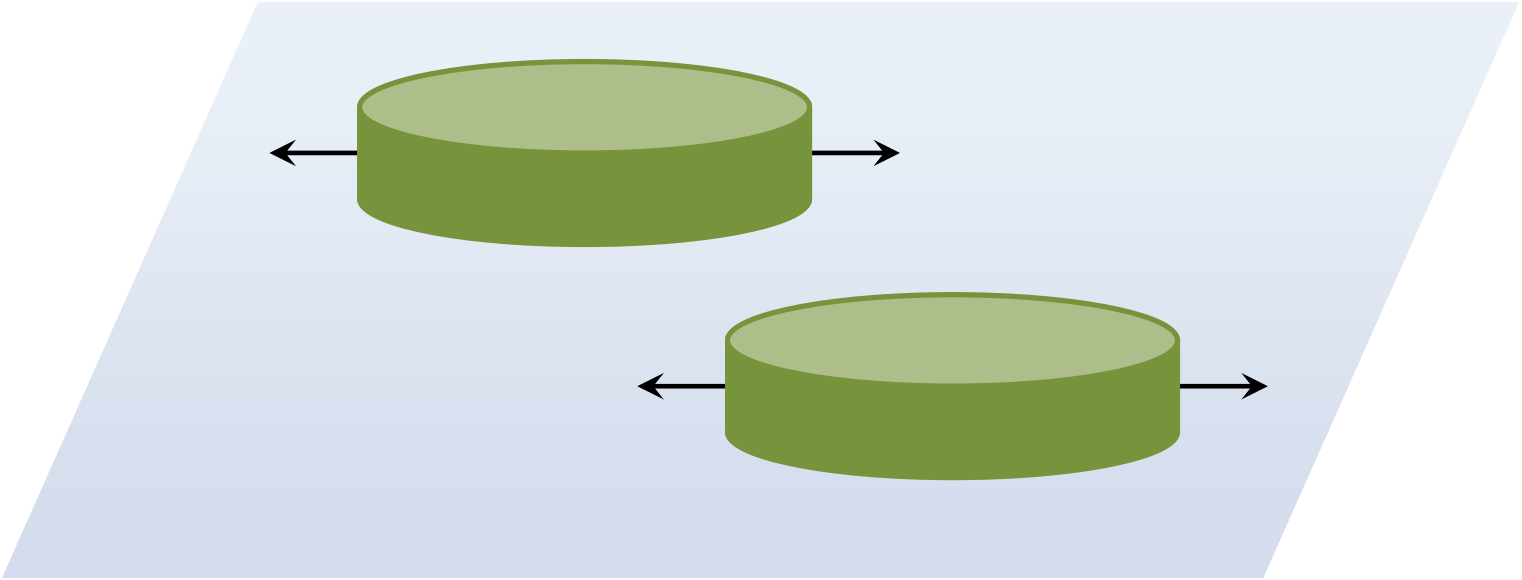
1. Introduction

1.1 Models (for growth of a crystal surface)

Two-dimensional nucleation: A flat crystal surface grows by adatoms over the surface. How should one model this phenomenon to measure the growth rate?

There are several models. A typical one is "Birth and Spread Model" (cf. M. Ohara – R. C. Reid (1973)).

Two-dimensional nucleation



crystal surface

Birth and Spread Model

1. Birth

Adatoms touch to the crystal surface on a set E.

The set E is a set of nucleation centers.

The height is assumed to be $h > 0$.

birth

Birth and Spread Model (continued)

2. Spread / propagation

Each layer (step) moves horizontally by catching adatoms with horizontal normal speed:

$$V = v_\infty \, (\, \rho_c H + 1).$$

Here H : the curvature of curve enclosing each step

$v_\infty > 0$: step velocity

$\rho_c > 0$: critical radius

spread

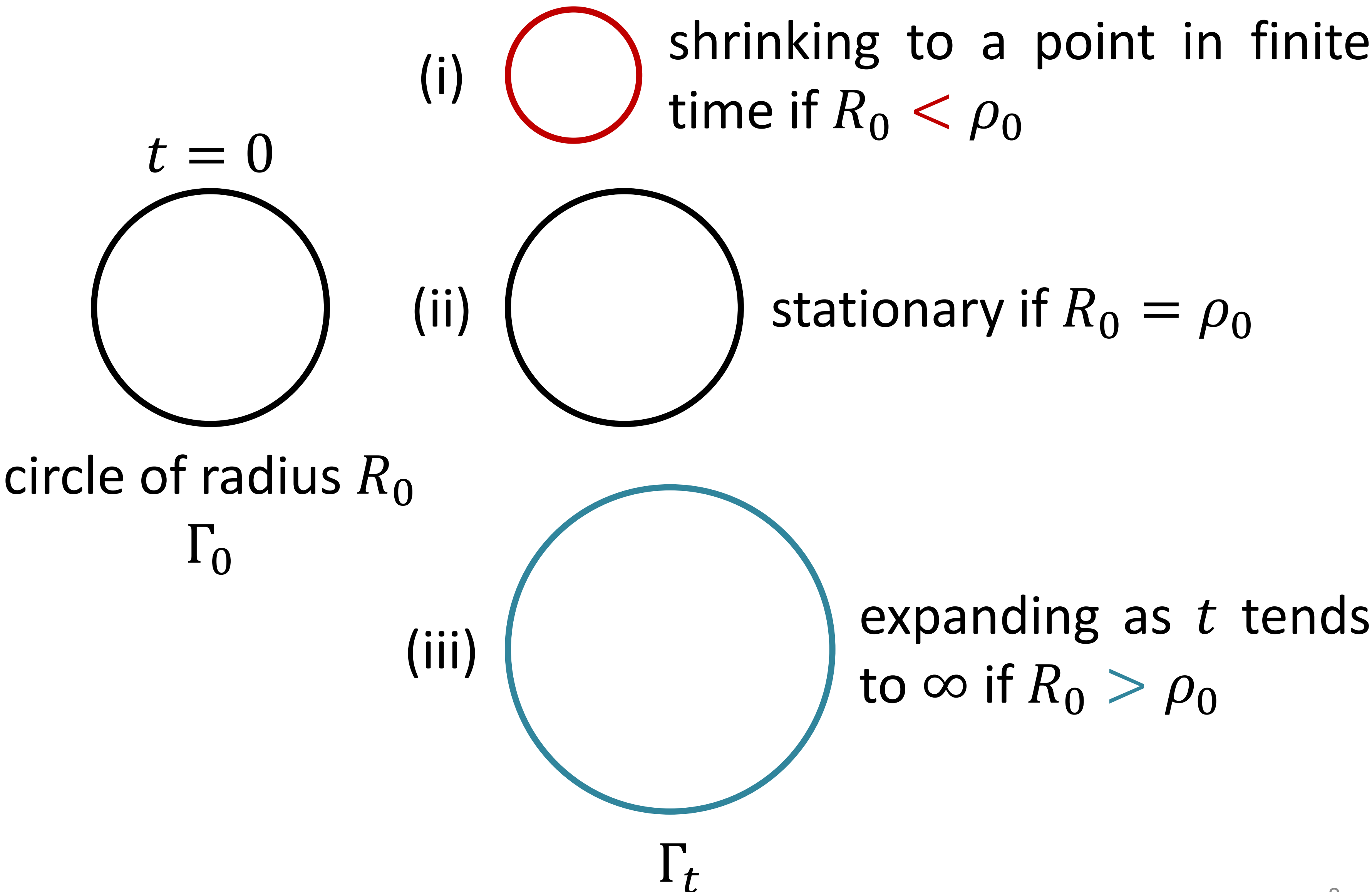
Eikonal-curvature flow equation

Equation

$$V = v_\infty(\rho_c H + 1)$$

for an evolving hypersurface Γ_t in $\mathbf{R}^N$ is called the **eikonal-curvature flow** equation. Here H is the ($N - 1$ times) mean curvature of Γ_t, i.e., the sum of the principal curvatures.

Eikonal-curvature flow in the plane



Birth and Spread Model (continued)

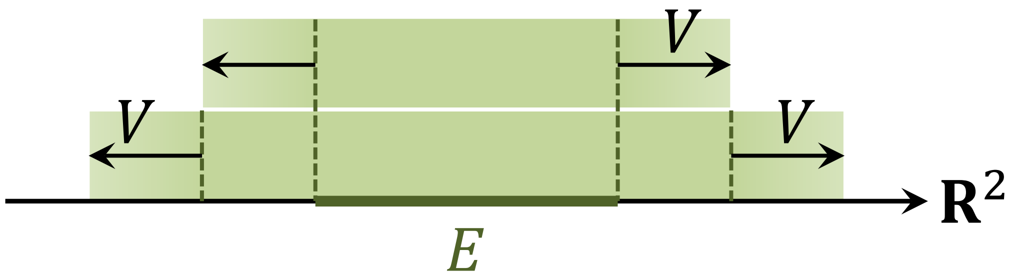
3. **Repeat the Step 1 (Forming the second layer)**
4. **Repeat the Step 2**

and repeat 3 and 4 successively.

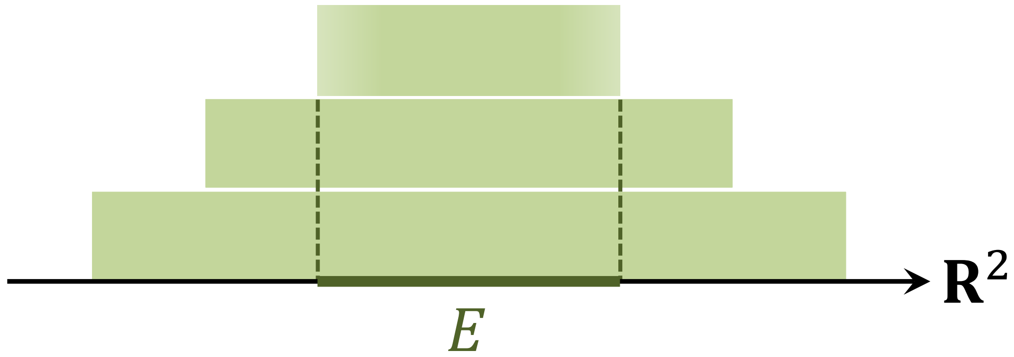
Step 3

Birth and Spread Model (continued)

Step 4



Step 5



Description of spreading law by height function

Consider each level-set of the height function u spreads by the eikonal-curvature flow equation

$$V = v_\infty\,(\,\rho_c H + 1)$$

in $\mathbf{R}^N$. Then $u = u(x,t)$ solves

$$\frac{u_t}{|Du|} = v_\infty\left(\rho_c\,\mathrm{div}\left(\frac{Du}{|Du|}\right) + 1\right).$$

This is the level-set equation of the eikonal curvature flow equation.

(Du : spatial gradient of u , i.e., $Du = (\partial u/\partial x_1\,, \ldots, \partial u/\partial x_N)$.)

($u_t \coloneqq \partial u/\partial t$, time derivative.)

Generalized solution

One is able to construct a generalized solution of eikonal-curvature flow equation **after** it develops singularities based on level-set equation.

A level-set method

- By adjusting the notion of viscosity solutions, we are able to solve the initial value problem for the level-set equation in $\mathbf{R}^N$ globally in time. Each level-set depends only on its initial level set.
- A viscosity solution is a weak notion of solution of PDEs based on order-preserving structure. (M. G. Crandall – P. L. Lions (1981), L. C. Evans, H. Ishii …)

A level set method (continued)

Numerically: S. Osher – A. Sethian (1989)
Analytically: Y.-G. Chen – Y. G. – S. Goto (1991)
L. C. Evans – J. Spruck (1991)
(mean curvature flow)

For singular anisotropy like crystalline curvature flow

M.-H. Giga – Y. G. (2001) $N = 2$
M.-H. Giga – Y. G. – N. Pozar (2014)
(total variation flow like equation) $N \geq 2$
A. Chambolle – M. Morini – P. Ponsiglione (2016)
+ M. Novaga (2017)
Y. G.-N. Pozar (2016, 2018)

Derivation of PDE model

Let $u = u(x,t)$ be the height function at the place $x \in \mathbf{R}^2$ and the time $t > 0$.

1. Birth (with speed $c > 0$)

$$u(x,t) = c1_E t, \quad 1_E(x) = \begin{cases} 1, x \in E \\ 0, x \notin E \end{cases}$$

$$\text{or} \quad u_t = c\,1_E$$

2. Spread

$$u_t = v_\infty \left(\rho_c \operatorname{div}\left(\frac{Du}{|Du|} \right) + 1 \right) |Du|$$

Repeating these processes alternatively with the time grid τ and sending $\tau \to 0$ to get the equation

$$u_t - v_\infty \left(\rho_c \operatorname{div}\left(\frac{Du}{|Du|} \right) + 1 \right) |Du| = c1_E.$$

(Trotter-Kato product formula)

Problem

Consider

$$\begin{cases} u_t - v_\infty \left(\rho_c \operatorname{div}\left(\dfrac{Du}{|Du|} \right) + 1 \right) |Du| = c 1_E \\ u\Big|_{t=0} = 0. \end{cases}$$

What is the large time behavior of u? For example, investigate the asymptotic speed (growth rate)

$$\lim_{t \to \infty} \frac{u(x,t)}{t} = ?$$

(The value may not be c.)

2. The case of no curvature and spherical symmetric case

2.1 The case $\rho_c = 0$

Equation becomes

$$\begin{cases} u_t - v_\infty |Du| = c 1_E \quad (\, E : \text{bounded closed set} \,) \\ u\Big|_{t=0} = 0. \end{cases}$$

The unique "envelope" solution is

$$u(x,t) = c(t - \mathrm{dist}(x,E)/v_\infty)_+$$

so that $u/_t \to c$ as $t \to \infty$. The set E can be a point or a discrete set, so we need a notion of an envelope solution (Y. G. – N. Hamamuki (2013)).

More examples in the case $\rho_c = 0$

$$\begin{cases} u_t - v_\infty |Du| = \sum_{i=1}^{m} c_i \, 1_{\{a_i\}} & c_i \to 0 \\ u\Big|_{t=0} = 0 \end{cases}$$

The unique envelope solution is given

$$u(x,t) = \max_{1 \le i \le m} c_i (t - |x - a_i|/v_\infty)_+$$

(cf. T. P. Schulze – R. V. Kohn (1999)).

The problem is **coercive** so general growth rate can be obtained. (N. Hamamuki (2013))

Large-time asymptotics for non-coercive Hamiltonians (e.g. Y. G. – Q. Liu – H. Mitake (2012, 2014). E. Yokoyama – Y. G. – P. Rybka (2008))

Asymptotic speed and profile in the case $\rho_c = 0$

Consider

$$\begin{cases} u_t - v_\infty |Du| = r(x) \geq 0 \ \text{ in } \ \mathbf{R}^N \\ u\big|_{t=0} = u_0 \end{cases}$$

$r(x), u_0$ compactly supported, $u_0 \in C(\mathbf{R}^N)$

Theorem 2.1 (N. Hamamuki, 2013). Let $R = \max r$. Then $\lim_{t\to\infty} u^\lambda(x,t) = R(t - |x|/v_\infty)_+$ as $\lambda \to \infty$. In particular, $\lim_{t\to\infty} u(x,t)/t = R$.

Here $u^\lambda(x,t) = u(\lambda x, \lambda t)/\lambda$.

Unscaled asymptotic profile

Problem. Find R and w such that

$$\lim_{t\to\infty} \sup_{x\in B} |u(x,t) - Rt - w(x)| = 0,$$

where B is an arbitrary big ball.

R: asymptotic speed
w: unscaled asymptotic profile

Formal argument

Formally, R and w solves

$$R - v_\infty |Dw| = r(x).$$

Cell problem. Find R and w.

For periodic case, such R and w is known to exist.
(G. Namah – J.-M. Roquejoffre (1999))
(A. Fathi (1998))

Feature of the problem

Weak KAM theory
(Kolmogorov – Arnold – Moser)

- R is uniquely determined provided that a weak comparison hold.
- w is not unique even up to additive constant.

Example

$$E = \left\{\frac{1}{4} + m, \frac{3}{4} + m\right\}, \quad m \in \mathbf{Z}$$

Consider

$$u_t - |Du| = 1_E.$$

The asymptotic speed R must be 1.

Consider

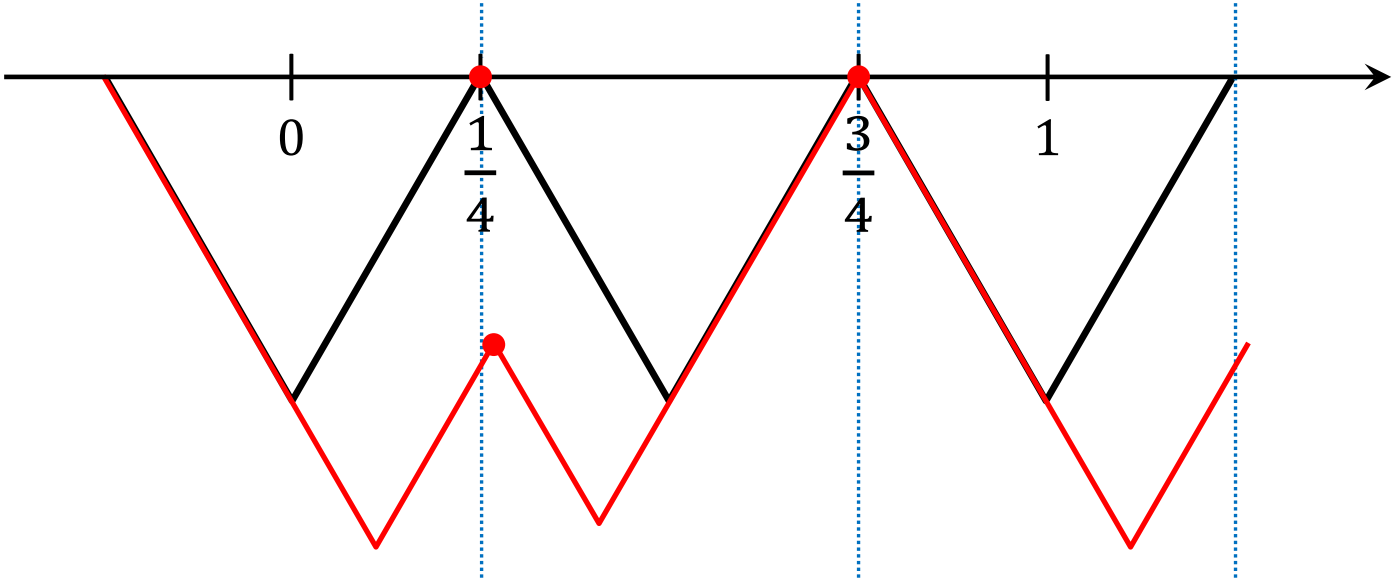
$$1 - 1_E - |Dw| = 0.$$

$$1 - 1_E = 0 \quad \text{iff} \quad x = \frac{1}{4} + m \quad \text{or} \quad \frac{3}{4} + m.$$

(This set is what is called the Aubry set.)

Many solutions in $\mathbf{R}/\boldsymbol{Z}$

- Viscosity solutions are not unique even up to constant.



- The asymptotic profile is chosen by initial data.
- w is determined by the value at the Aubry set.

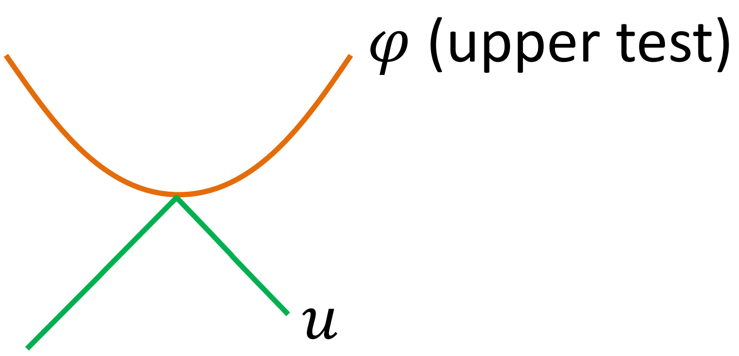
Definition of a viscosity solution

Definition. A function $u \in C(\mathbf{R})$ is a viscosity subsolution of $f(x) - |Dw(x)| = 0$ if

$$f_*(\hat{x}) - |D\varphi(\hat{x})| \leq 0$$

whenever $(\varphi, \hat{x}) \in C^1(\mathbf{R}) \times \mathbf{R}$ satisfies $\max(u - \varphi) = (u - \varphi)(\hat{x})$. Here f_* denotes the lower semi-continuous envelope (a largest lower semi-continuous function less than or equal to f). A supersolution is defined in a symmetric way. Viscosity solution = viscosity sub and supersolution.

In the case of $f(x) = 1 - 1_E$, local maximum cannot be taken except at E to be a subsolution.



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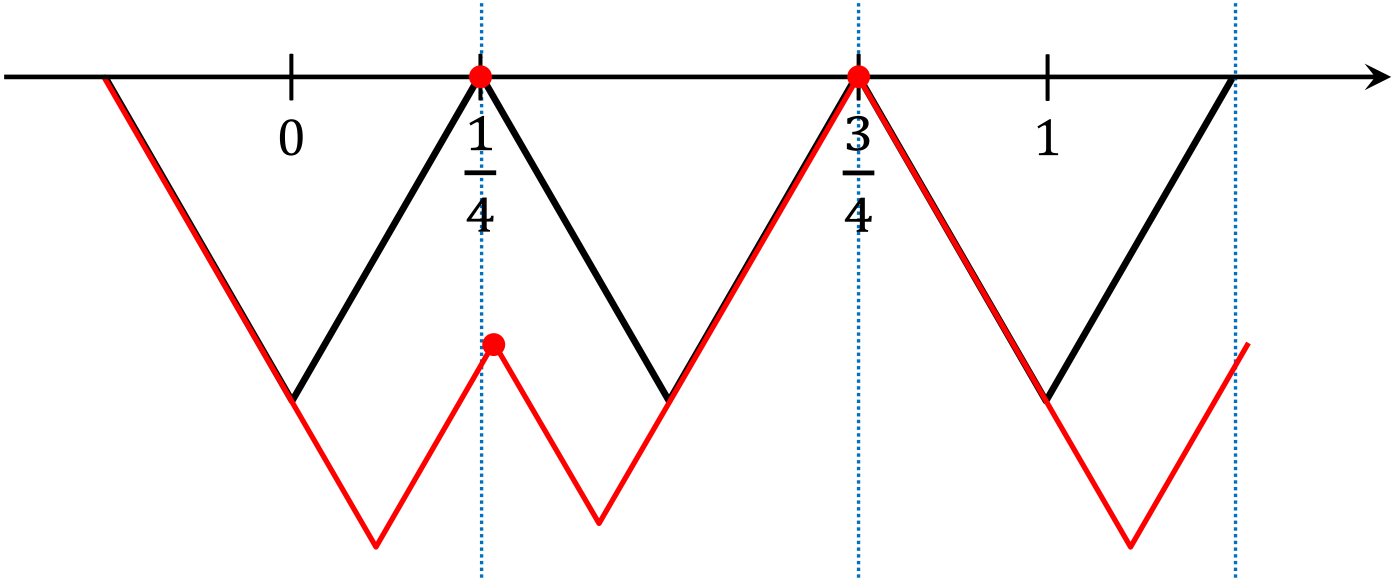


- The asymptotic profile is chosen by initial data.
- w is determined by the value at the Aubry set.

2.2 Spherical symmetric case

From now on, we assume $\rho_c = 1, v_\infty = 1$.

Consider the level-set equation of the eikonal-curvature flow $V = H + 1$ with source term r :

(P)

$$\begin{cases} u_t - \left(\operatorname{div}\left(\frac{Du}{|Du|}\right) + 1\right)|Du| = r(x) & \text{in } \mathbf{R}^N \times (0,\infty), \\ u\Big|_{t=0} = u_0 & \text{in } \mathbf{R}^N. \end{cases}$$

Assumptions on r and u_0

Here r is bounded and $r \geq 0$, $\operatorname{supp} r$ is compact, u_0 is continuous, $\operatorname{supp} u_0$ is compact.

Basic properties

- Even if r is discontinuous, there exists a global-in-time viscosity solution, which may not be unique (Y. G. – H. Mitake – H. V. Tran, SIMA (2016)).
- Weak comparison principle holds.

Spherical symmetric case

A non-coercive equation :

$$u_t - \left(1 - \frac{N-1}{r}\right) u_r = c1_{B(0,R_0)}.$$

Assume that $E = B(0, R_0)$ a closed ball of radius R_0 centered at zero. Let u be the maximum solution of (P) with $r = c1_E, u_0 = 0$. The solution u can be obtained by an explicit calculation.

- $R_0 < N - 1 \Rightarrow$ the growth is completed in finite time.
- $R_0 > N - 1 \Rightarrow$ the growth rate equals c.
- $R_0 = N - 1 \Rightarrow u(x,t) = tc1_{B(0,R_0)}$.

  It grows with speed c in $B(0, R_0)$ but never grows outside $B(0, R_0)$

This is first order problem but not coercive because it is linear equation. $u_t + F(x, Du) = 0$ is **coercive** if $F(x,p) \to -\infty$ as $|p| \to \infty$ locally uniformly in x.

Spherical symmetric case

Theorem 2.2. (i) If $R_0 < N-1$, then

$$u(x,t) = \min(ct, \varphi(r))$$

φ: stationary solution

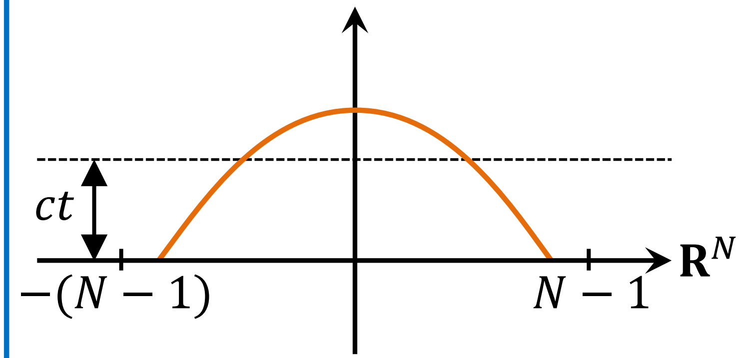
(ii) If $R_0 > N-1$, then

$$\lim_{t\to\infty} \frac{u(x,t)}{t} = c \quad \text{(locally uniformly)}$$

(There is an explicit formula for u)

(iii) If $R_0 = N-1$, $u(x,t) = t\, c 1_{B(0,R_0)}$

Here u is the maximal viscosity solution.



graph of φ

Note that even if $E = \partial B(0, R_0)$ with $R_0 > N-1$, we get $\frac{u(x,t)}{t} \to c$.

3. Existence of asymptotic speed for a Lipschitz source term

(Y. G. - H. Mitake - H. V. Tran - T. Ohtsuka, in preparation)

Consider

$$(P)\quad \begin{cases} u_t - \left(\operatorname{div}\left(\dfrac{Du}{|Du|}\right) + 1\right)|Du| = r(x) & \text{in } \mathbf{R}^N \times (0,\infty) \\ u\big|_{t=0} = u_0 & \text{in } \mathbf{R}^N. \end{cases}$$

r : bounded, $r \not\equiv 0$, $\operatorname{supp} r$: compact

u_0 : continuous, $\operatorname{supp} u_0$: compact

(There is a unique viscosity solution if r is Lipschitz.)

Theorem 3.1. Assume that r and u_0 are Lipschitz. Let u be the viscosity solution of (P). Then $\lim_{t\to\infty} u(x,t)/t = a$ exists and the convergence is locally uniform.

What is the growth rate a?

Let $\phi(t)$ be the maximum value of u at time t, i.e.,

$$\phi(t) := \max_{x \in \mathbf{R}^N} u(x,t)\,.$$

Theorem 3.2 (Subadditivity of ϕ). $\phi(t+s) \leq \phi(t) + \phi(s)$ for $t, s > 0$.

Proof. Set $v(x,t) := u(x,t+s) - \phi(s)$ so that $v(x,0) \leq 0$. By the comparison principle, $v(x,t) \leq u(x,t)$, which implies

$$u(x,t+s) \leq u(x,t) + \phi(s)$$

for all $x \in \mathbf{R}^N$. □

What is the growth rate a? (continued)

Lemma 3.3.

$$a = \lim_{t \to \infty} \frac{\phi(t)}{t} = \inf_{t > 0} \frac{\phi(t)}{t}$$

Proof. Fekete's lemma on subadditivity says that $\phi(t)/t$ is nonincreasing. Thus $\lim_{t \to \infty} \phi(t)/t$ exists. $\square$

The growth rate in Theorem 3.1 must be $\inf \phi(t)/t$.

Idea of the proof of Theorem 3.1

Lemma 3.4 (Lipschitz bound). There exists $C > 0$ depending only on r and u_0 such that

$$\|u_t\|_{L^\infty(\mathbf{R}^N\times[0,\infty))} + \|Du\|_{L^\infty(\mathbf{R}^N\times[0,\infty))} \le C.$$

Proof of Theorem 3.1. We note that the max point x_t of $u(\cdot, t)$ is always inside the convex hull of $\mathrm{supp}\, r$. Thus Lemma 3.3 and Lemma 3.4 give the desired result.

(cf. G. Barles (2013) lecture notes) □

Bernstein's argument to get a Lipschitz bound (formal proof of Lemma 3.4)

We recall that our equation can be written as

$$u_t - \sum_{i,j} a_{ij}(Du)\, u_{x_i x_j} - |Du| - r = 0$$

with $a_{ij}(p) = \delta_{ij} - p_i p_j/|p|^2$. We set $U = |Du|^2/2$ and differentiate in x_k the above equation and multiply u_{x_k} to get

$$2U_t - \sum_{i,j,k,\ell} a_{ij} \left(U_{x_i x_j} - u_{x_i x_k} u_{x_j x_k} \right) - r_{x_k} u_{x_k}$$
$$- \left\{ \left(a_{ij}\right)_{p_\ell} u_{x_i x_j} + \frac{2u_{x_\ell}}{|Du|} \right\} U_{x_\ell} = 0.$$

Formal proof continued 1

Take max point $(x_0, t_0) \in \mathbf{R}^N \times (0, T]$ of U, i.e.,

$$U(x_0, t_0) = \max_{\mathbf{R}^N \times [0,T]} U\,.$$

(We may assume that $t_0 > 0$.) At this point $U_t \le 0$, $DU = 0$, $D^2 U \le 0$. Thus

$$\sum_{i,j,k} a_{ij}\, u_{x_i x_k} u_{x_j x_k} - f_{x_k} u_{x_k} \le 0. \qquad (*)$$

Note that $0 \le A \le I$ for $A = (a_{ij})$.

A linear algebra inequality $(\operatorname{tr} AB)^2 \le \operatorname{tr} A \operatorname{tr} AB^2$ for $A \ge 0$ implies

$$\left(\sum a_{ij}\, u_{x_i x_j} \right)^2 \le N \sum a_{ij}\, u_{x_i x_k} u_{x_j x_k}.$$

Formal proof continued 2

Note that

$$\left(\sum a_{ij}\, u_{x_i x_j}\right)^2 = (u_t - |Du| - f)^2 \geq \frac{1}{2}|Du|^2 - {}^{\exists}M_0$$

provided that $|u_t| \leq M_1$ (since f is Lipschitz).

Thus (*) implies

$$\frac{1}{2n}|Du|^2 - Df \cdot Du \leq M_2 \quad \text{at} \quad (x_0, t_0).$$

This implies a bound for $|Du|$ (on U). (The bound for $|u_t| \leq M_1$ is easier.) $\square$

Actual proof needs approximation of the equation so that the equation is parabolic e.g. $|Du|$ is approximated by $(|Du|^2 + \varepsilon^2)^{1/2}$.

4. Estimate for asymptotic speed
(Y. G. – H. Mitake – H. V. Tran, SIMA (2016))

Problem. If $r(x) = c1_E$, in what E

$$\limsup_{t\to\infty} \frac{u(x,t)}{t} < c \quad \text{and} \quad \liminf_{t\to\infty} \frac{u(x,t)}{t} > 0?$$

We have studied this problem when E is a ball. In this case

$$\frac{u(x,t)}{t} \to 0 \quad \text{or} \quad \frac{u(x,t)}{t} \to c.$$

Are there any intermediate situation?

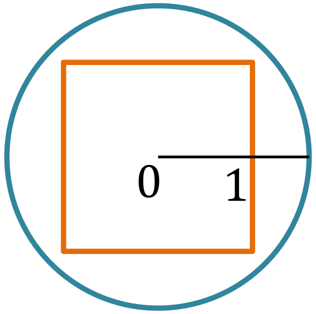
4.1 In the case of square

Assume that $E = \{(x_1, x_2) \,|\, |x_i| \le d, i = 1, 2\}$. Let u be the maximal solution of

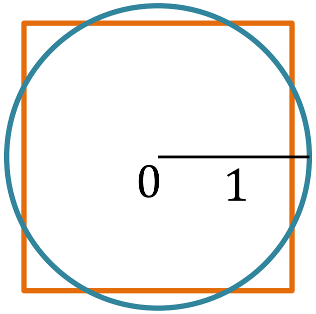
$$u_t - \left(\operatorname{div}\frac{Du}{|Du|} + 1\right)|Du| = c1_E,$$

$$u\Big|_{t=0} = 0.$$

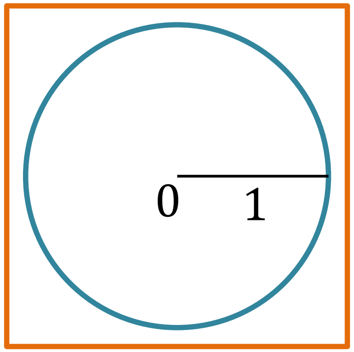
$d < 1/\sqrt{2}$



$1/\sqrt{2} < d < 1$



$d > 1$



Intermediate situation

Theorem 4.1 (Y. G. – H. Mitake – H. V. Tran, SIMA (2016)). Assume that $1/\sqrt{2} < d < 1$. Then there exist α and β such that $0 < \alpha < \beta < c$ and

$$\alpha \le \liminf_{t\to\infty} \frac{u(x,t)}{t} \le \limsup_{t\to\infty} \frac{u(x,t)}{t} \le \beta$$

locally uniformly for $x \in \mathbf{R}^2$.

Growth speed seriously depends on the shape of E.

4.2 Motion of the top – flow with obstacle

We consider

$$u_t - \left(\operatorname{div}\frac{Du}{|Du|} + 1\right)|Du| = c1_E, \quad u\Big|_{t=0} = 0$$

for a general compact set E in $\mathbf{R}^N$.

By comparison, $u^*(x,t) \leq ct$ in $\mathbf{R}^N \times (0,\infty)$. Here u^* is the upper semi-continuous envelope. In general, u may not be continuous.

Notation:

$$A_{\max}(t) = \{x \in \mathbf{R}^N | u^*(x,t) = ct\}.$$

Curvature flow with obstacle

Lemma 4.2. The set $A_{\max}(t)$ is a set theoretic solution of $V = H + 1$ (i.e., $\psi(x,t) = 1_{A_{\max}(t)}(x)$ is a viscosity subsolution of

(L) $$\psi_t - \left(\operatorname{div}\frac{D\psi}{|D\psi|} + 1\right)|D\psi| = 0.)$$

Moreover, $A_{\max}(t) \subset E$.

Actually, ψ is a subsolution of the obstacle problem

$$\max\left\{\psi_t - \left(\operatorname{div}\frac{D\psi}{|D\psi|} + 1\right)|D\psi|, \psi - 1_E\right\} = 0 \text{ in } \mathbf{R}^N \times (0,\infty).$$

Curvature flow with an obstacle: G. Mercier……

Idea of proof

Note that

$$u_c(x,t) := u(x,t) - ct$$

is a viscosity subsolution of (L) and $u_c \leq 0$. Moreover, $A_{\max}(t) = \{x \in \mathbf{R}^N \,|\, u_c^*(x,t) = 0\}$. Thus $A_{\max}$ is a set theoretic subsolution (cf. Y. G., Surface Evolution Equations, 2006).

Proof for $\boldsymbol{A_{max}} \subset \boldsymbol{E}$. If not, ${}^{\exists}x_0 \in A_{\max}(t_0) \cap E^c$ with some $t_0 > 0$. Then $\varphi(x,t) = ct$ is a test function of u from above. This is a contradiction

$$c = \varphi_t - \left(\mathrm{div}\frac{D\varphi}{|D\varphi|} + 1\right)|D\varphi|\Big|_{(x_0,t_0)} < c1_E(x_0) = 0. \quad \square$$

4.3 Upper estimate

Lemma 4.3. Assume that a flow $V = H + 1$ with obstacle E starting from E vanishes at $t = t_0$. Then there exists $b \in (0, c)$ such that $\max_x u(x, t_0) \le bt_0$.

Proof. Since $A_{\max}(t_0) = \phi$, we have

$$\max_x u(x, t_0) < ct_0.$$

We set

$$b = \max_x \frac{u(x, t_0)}{t_0}$$

to get the desired result. $\square$

Global upper estimate

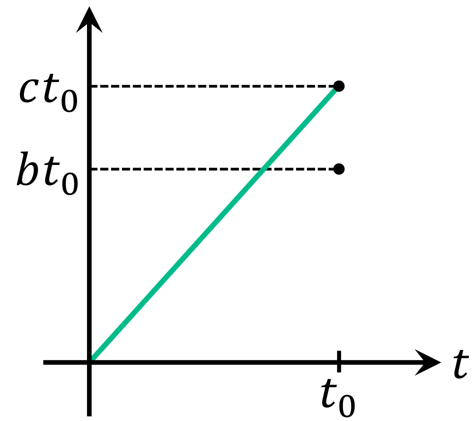
Theorem 4.4. Under the assumption of Lemma 4.3 with t_0. There exists $b \in (0, c)$ such that

$$u(x,t) \le bt + (c-b)t_0, \quad (x,t) \in \mathbf{R}^N \times (0,\infty).$$

In particular,

$$\limsup_{t\to\infty} \frac{u(x,t)}{t} \le b.$$

Global upper estimate (continued)



Since

$$\max_x u(x, t_0) \leq bt_0,$$

by induction we have

$$u(x, mt_0 + t) \leq u(x, t) + mbt_0 \text{ on } \mathbf{R}^N \times (0, \infty).$$

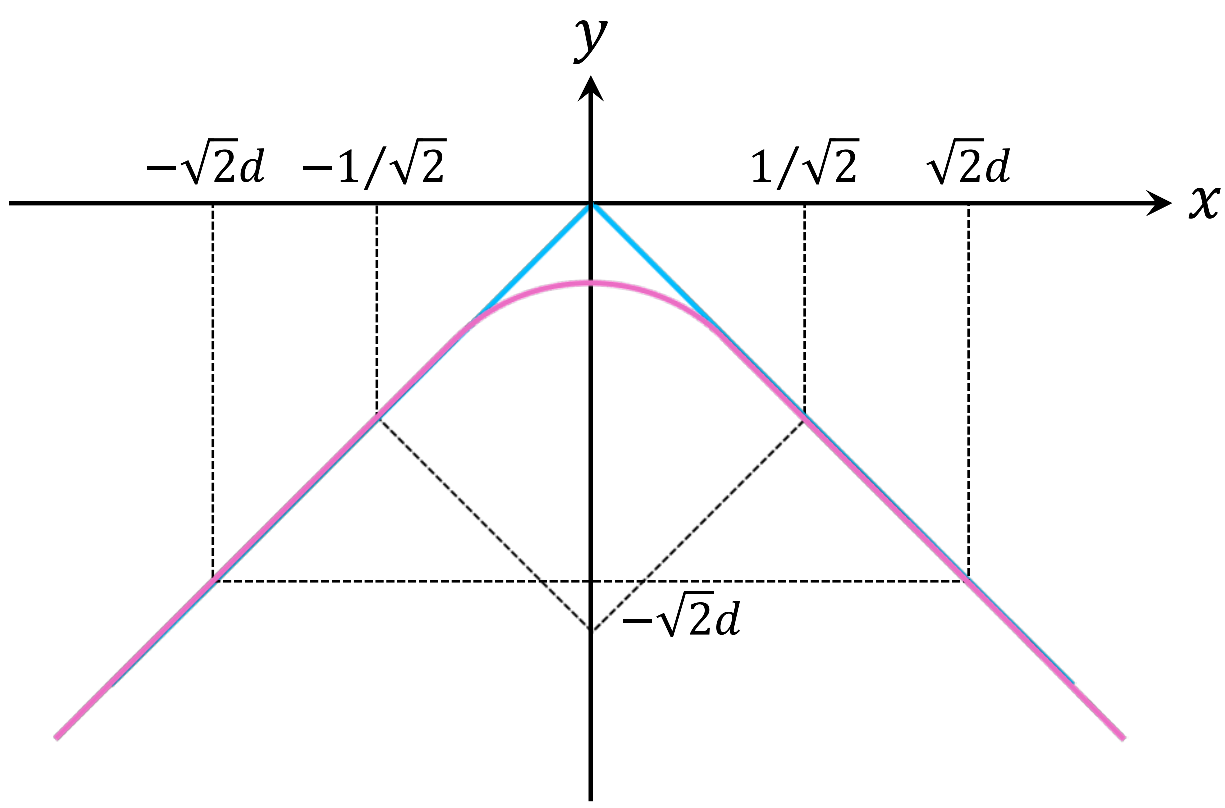
In particular, $u(x, mt_0) \leq mbt_0$. Thus for $t \in (mt_0, (m+1)t_0)$, $m \in \mathbf{N}$, we observe that

$$u(x,t) \leq u(x, mt_0) + c(t - mt_0) \leq bmt_0 + c(t - mt_0)$$
$$= bt + (c - b)(t - mt_0) \leq bt + (c - b)t_0.$$

Estimate from below is similar. Theorem 4.1 now follows.

4.4 In the case of square of medium size

Lemma 4.5. If $d < 1$, then there exists $t_0 > 0$ such that $A_{\max}(t_0) = \emptyset$.



Obstacle problem

We shall construct a supersolution of the obstacle problem

$$\max\left\{y_t - \frac{y_{xx}}{1+y_x^2} - (1+y_x^2)^{1/2}, y - g(x)\right\} = 0$$

$$\text{in } \left(-\sqrt{2}d, \sqrt{2}d\right) \times (0, \infty)$$

where $g(x) = -|x|$.

To simplify the work, we seek a self-similar solution of form

$$y(x,t) = \lambda(t) Y\left(\frac{x}{\lambda(t)}\right), \quad \lambda'(t) = \frac{1}{\lambda(t)} - 1.$$

Idea of the proof of Theorem 4.1

[$1/\sqrt{2} < d < 1$ yields an intermediate speed.]

Lemma 4.5 together with Theorem 4.4 yields an upper bound for u.

We construct a supersolution for the obstacle problem outside E for $V = H + 1$ which leads the estimate for u/t from below.

Summary

- Even second order birth and spread model we prove the existence of asymptotic speed.
- We prove that the asymptotic speed may not be equal to maximum of the source term which is quite different from first order model.

5. A few open problems

Full convergence problem

Problem A. Show the **full** convergence

$$\lim_{t\to\infty} \sup_{x\in B} |u(x,t) - Rt - w(x)| = 0$$

for

$$u_t - \left(\operatorname{div} \frac{Du}{|Du|} + 1\right)|Du| = r \geq 0,$$

$$u\Big|_{t=0} = 0.$$

Here R is the asymptotic speed and w is an unscaled asymptotic profile, i.e.,

$$R - \left(\operatorname{div} \frac{Dw}{|Dw|} + 1\right)|Dw| = r.$$

Here r is assumed to be Lipschitz.

Full convergence problem (continued)

- We know the convergence in Problem 1 if we take a subsequence $t_j \to \infty$. The unscaled asymptotic profile w may depend on the subsequence. Study **uniqueness set** and clarify the dependence of w with respect to initial data u_0.
- In periodic setting this problem is well studied for the first order equation.

  G. Namah – J. M. Roquejoffre (1999), A. Fathi (1998), G. Barles – P. E. Souganidis (2000) …, H. Ishii (2008) for $\mathbf{R}^N$.
- The second order degenerate parabolic problem is difficult to handle.

  D. Cagnetti – D. Gomes – H. Mitake – H. V. Tran (2015) based on adjoint method by L. C. Evans (2010). It applies to $u_t + H(x, Du) = a(x)\Delta u$, $a(x) \geq 0$.

Asymptotic speed

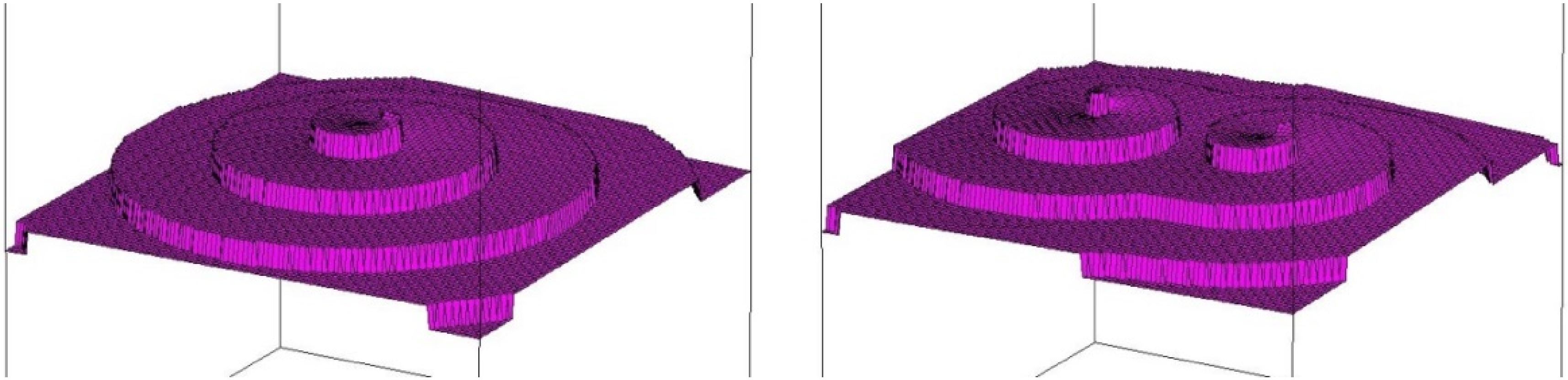
Problem B. Show the existence of R where r is discontinuous even in the case $r = 1_E$. Study how R depend on r. For example, does it depend on r continuously?

So far, we know that R depends on r monotonically if it exists.

Numerical study for spiral growth: T. Ohtsuka – Y.-H. R. Tsai – Y. G. (2015), (2018)

Y. G. – H. Mitake – T. Ohtsuka – H. V. Tran (in preparation)

Thank you for your attention.



T. Ohtsuka – Y.-H. R. Tsai – Y. G. (2015)
Computation by a modified level set method
for spiral growth