The heat equation with power-exponential nonlinearities

Mohamed Majdoub

TUNISIA

August 2, 2018

International Congress of Mathematicians (ICM 2018)

Joint work with: S. Tayachi

Consider the following Cauchy problem

$$u_t - \Delta u = u|u|^{m-1}\left(\mathrm{e}^{|u|^q} - 1\right), \quad t > 0,\ x \in \mathbb{R}^N. \tag{1}$$

$$u(0) = u_0 \in \exp L^p(\mathbb{R}^N). \tag{2}$$

- $u = u(t,x) \in \mathbb{R}, \quad u_t = \frac{\partial u}{\partial t}$.
- $m \geq 1, \quad q > 0, \quad p > 1.$
- $u_0 \in \exp L^p(\mathbb{R}^N) \iff \int_{\mathbb{R}^N} \left(\mathrm{e}^{\frac{|u(x)|^p}{\lambda^p}} - 1\right) dx < \infty$ for some $\lambda > 0$.

The main goals are:

- Non-existence in $\exp L^p(\mathbb{R}^N)$ for $p < q$.
- Local well-posedness in $\exp L^p_0(\mathbb{R}^N)$ (some sub-space of $\exp L^p(\mathbb{R}^N)$) for $p \geq q$.
- Global existence for u_0 small in $\exp L^p(\mathbb{R}^N)$, $p \geq q$ and $m \geq 1 + \frac{2p}{N}$.
- Asymptotic estimates for global solutions.

The nonlinearity appearing in (1) can be replaced by a more general $f(u)$ verifying some hypotheses concerning the behaviors at infinity and near $u = 0$.

- The problem of studying nonlinear evolution equations with exponential nonlinearities has received intense attention in the last few years: Adam Azzam, Colliander, Ibrahim, Ioku, Masmoudi, Nakanishi, Ruf, Struwe, Tayachi, M., ...

- Exponential-type nonlinearities have been considered in several physical models on a model of self-trapped beams in plasma.

- The non-dimensional ignition model for a supercritical high activation energy thermal explosion of a solid fuel in a bounded container can be described by $\mathbf{u_t} - \mathbf{\Delta u} = \mathrm{e}^{\mathbf{u}}$. See J. W. Bebernes, Bebernes, Bressan & Eberly.

- For decreasing exponential nonlinearities: T. Cazenave [1979] proved global well-posedness together with scattering in the case of the nonlinear Schrödinger equation (NLS).

- For increasing exponential nonlinearities, the problem is much more difficult, since there is no a priori L^∞ control of the nonlinear term.

NLW

- Consider the following 2D nonlinear wave equation

$$\mathbf{u_{tt}} - \mathbf{\Delta u} = \mathbf{u}\left(\mathrm{e}^{\mathbf{4\pi u^2}} - \mathbf{1}\right).$$

- $\mathbf{4}\pi \longleftrightarrow$ Moser-Trudinger inequality

$$\|\mathrm{e}^{\alpha u^2} - 1\|_1 \leq c_\alpha \|u\|_2^2 \text{ if } \|\nabla u\|_2 \leq 1 \;\&\; \alpha < 4\pi\,.$$

- S. Ibrahim, N. Masmoudi & M. M.[2006]: Global well-posedness in the energy space for $E_0 := \|\nabla u_0\|_2^2 + \|u_1\|_2^2 + \frac{1}{4\pi}\int_{\mathbb{R}^2}\left(\mathrm{e}^{4\pi u_0^2} - 1\right) dx \leq 1$.
- S. Ibrahim, N. Masmoudi & M. M.[2011]: Ill-posedness in the supercritical regime i.e. $E_0 > 1$.
- M. Struwe[2013]: Global existence of classical solutions for arbitrary smooth initial data.
- S. Ibrahim, N. Masmoudi K. Nakanishi & M. M[2009]: Scattering in the energy space for $E_0 \leq 1$.

NLS

- Consider the following 2D nonlinear Schrödinger equation

$$\mathbf{i u_t} + \mathbf{\Delta u} = \mathbf{u}\left(\mathrm{e}^{\mathbf{4}\pi|\mathbf{u}|^{\mathbf{2}}} - \mathbf{1}\right).$$

Define $E_0 := \|\nabla u_0\|_2^2 + \frac{1}{4\pi}\int_{\mathbb{R}^2}\left(\mathrm{e}^{4\pi|u_0|^2} - 1 - 4\pi|u_0|^2\right)dx.$

- J. Colliander, S. Ibrahim, N. Masmoudi & M. M.[2009]: GWP in the subcritical and critical regimes i.e. $E_0 \leq 1$. Ill-posedness in the supercritical regime i.e. $E_0 > 1$.
- S. Ibrahim, N. Masmoudi K. Nakanishi & M. M[2012]: H^1-scattering in the subcritical case.
- H. Bahouri, S. Ibrahim & G. Perelman[2014]: H^1-scattering in the radial framework in the critical regime.
- A. Adam Azzam[2017]: H^1-scattering in the critical regime.

Polynomial Growth

$$\mathbf{u_t - \Delta u = u|u|^{m-1}}$$

- The Lebesgue space $L^{q_c}(\mathbb{R}^N)$ with index $q_c := \frac{N(m-1)}{2}$ is the only one invariant under the scaling:

$$u(t,x) \Longrightarrow u_\lambda(t,x) := \lambda^{2/(m-1)}\, u(\lambda^2 t, \lambda x)$$

- **Supercritical case**: $q \geq q_c,\ q \geq 1$.
  - Weissler [1980]: Existence and uniqueness in $\mathcal{C}_T(L^q) \cap L^\infty_{loc}(L^\infty)$.
  - Brezis-Cazenave [1996]: Unconditional uniqueness.

Polynomial Growth

- **Critical case**: $q = q_c$ and $N \geq 3$. There are two sub-cases:
  - $q_c > m$, or equivalently $m > \frac{N}{N-2}$. The existence was proved by Weissler and the uniqueness by Brezis-Cazenave.
  - $q = q_c = m$, or equivalently $q = \frac{N}{N-2}$ and $m = \frac{N}{N-2}$ (`double critical`).
    - Weissler [1981]: Conditional wellposedness.
    - Ni-Sacks [1985]: Nonuniqueness for the unit ball.
    - Terraneo [2002]: Nonuniqueness for the whole space.
    - Matos-Terraneo [2003]: Nonuniqueness for general data.

Exponential Growth

$$\mathbf{u_t - \Delta u = u|u|^{m-1}\left(e^{|u|^2} - 1\right)}$$

- Ruf-Terraneo [2002]: Only local existence in Orlicz space $\exp L^2$ for small initial data.
- Ioku [2011]: Global existence in Orlicz space $\exp L^2$ for small initial data, $m = 1 + (N/4)$, $N \leq 4$.
- Ioku-Ruf-Terraneo [2015]:
  - Non-existence for large initial data in Orlicz space $\exp L^2$.
  - Local well-posedness in $\exp L^2_0$.
- Ibrahim-Jrad-Saanouni & M. M. [2014]: Local well-posedness in $W^{1,2}(\mathbb{R}^2)$.

Remark 1

If the nonlinearity f has a polynomial growth, then there exists a Lebesgue space L^p, $p < \infty$, for which there is LWP.

See R. Laister, J.C. Robinson , M. Sierzega, A. Vidal-Lopez, AIHP (2016)

Remark 2

It seems that this is not possible if f has an exponential growth. The standard Lebesgue spaces may be replaced with the so-called Orlicz spaces.

Orlicz spaces

Definition

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex increasing function such that $\phi(0) = 0$, $\lim\limits_{s\to\infty} \phi(s) = \infty$. Define

$$L^\phi(\mathbb{R}^N) = \left\{ u \text{ measurable on } \mathbb{R}^N; \text{s.t. } \exists\, \alpha > 0 \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\alpha}\right)\, dx < \infty \right\}$$

and

$$\|u\|_{L^\phi} = \inf \left\{ \alpha > 0, \quad \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\alpha}\right)\, dx \leq 1 \right\}.$$

- $\left(L^\phi(\mathbb{R}^N), \|\cdot\|_{L^\phi}\right)$ is a Banach space.
- $\phi(s) = s^p$, $1 \leq p < \infty \implies L^\phi = L^p$.
- $L^1 \cap L^\infty \subset L^\phi \subset L^1 + L^\infty$.

See Adams, Fournier (2003); Malingranda, Orlicz spaces and interpolation, 1989.

Orlicz spaces

- Clearly, $C_0^\infty(\mathbb{R}^N)$ is dense in $L^\phi(\mathbb{R}^N)$ for $\phi(s) = s^p$, $p < \infty$.
- This is not the case for any ϕ.
- This motivates the following definition.

Definition

$$L_0^\phi(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N)}^{L^\phi} = \text{the closure of } C_0^\infty(\mathbb{R}^N) \text{ in } L^\phi(\mathbb{R}^N).$$

$$L_0^\phi(\mathbb{R}^N) = \left\{ u, \text{ s.t. } \int_{\mathbb{R}^N} \phi\left(\frac{|u(x)|}{\alpha}\right) dx < \infty, \ \forall \ \alpha > 0 \right\}.$$

- $\phi(s) = \mathrm{e}^{s^p} - 1 \Longrightarrow L^\phi(\mathbb{R}^N) := \exp L^p(\mathbb{R}^N)$ and $L_0^\phi(\mathbb{R}^N) := \exp L_0^p(\mathbb{R}^N)$.

- $\exp L^p(\mathbb{R}^N) = \left\{ u, \text{ s.t. } \int_{\mathbb{R}^N} \left(e^{\frac{|u(x)|^p}{\alpha^p}} - 1 \right) dx < \infty, \text{ for some } \alpha > 0 \right\}.$
- $\|u\|_{\exp L^p(\mathbb{R}^N)} := \inf\left\{ \alpha > 0; \ \int_{\mathbb{R}^N} \left(e^{\frac{|u(x)|^p}{\alpha^p}} - 1 \right) dx \leq 1 \right\}.$
- $\exp L_0^p(\mathbb{R}^N) = \left\{ u, \text{ s.t. } \int_{\mathbb{R}^N} \left(e^{\frac{|u(x)|^p}{\alpha^p}} - 1 \right) dx < \infty, \text{ for every } \alpha > 0 \right\}.$

- $\exp L_0^p \subsetneq \exp L^p, \ p \geq 1$: $u(x) = |\log(|x|)|^{1/p} \chi_{\{|x|<1\}}.$
- $\exp L^p \not\hookrightarrow L^\infty, \ p \geq 1$: $u(x) = \left(\log\left(1 - \log|x|\right) \right)^{1/p} \chi_{\{|x|<1\}}.$
- $\exp L^p \not\hookrightarrow L^r$, for all $1 \leq r < p$: $u(x) = |x|^{-N/r} \chi_{\{|x|>1\}}.$
- $\|u\|_{\exp L^p} \leq \frac{1}{(\log 2)^{\frac{1}{p}}} \left(\|u\|_q + \|u\|_\infty \right), \ 1 \leq q \leq p.$

We have the embedding: $\exp L^p \hookrightarrow L^r$ for every $p \leq r < \infty$.

Lemma [Ruf-Terraneo]

$$\|u\|_r \leqslant \left(\Gamma\left(\frac{r}{p}+1\right)\right)^{\frac{1}{r}} \|u\|_{\exp L^p}.$$

The following lemma will be useful in the proofs.

Lemma

Let $\lambda > 0$, $1 \leq p,\ q < \infty$ and $K > 0$ such that $\lambda q K^p \leq 1$. Assume that

$$\|u\|_{\exp L^p} \leq K.$$

Then

$$\|\mathrm{e}^{\lambda |u|^p} - 1\|_q \leq (\lambda q K^p)^{\frac{1}{q}}\ .$$

Orlicz spaces

- Orlicz spaces used by Bahouri, Masmoudi & M. M. [2011], [2014]: to describe the lack of compactness of the 2D critical Sobolev embedding

$$W^{1,2}(\mathbb{R}^2) \hookrightarrow \exp L^2(\mathbb{R}^2).$$

- This embedding is non compact at least for two reasons: lack of compactness at infinity and a concentration-type phenomenon illustrated by

$$u_k(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{2k\pi}} & \text{if} \quad \mathrm{e}^{-k} \leq |x| \leq 1, \\ \sqrt{\frac{k}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-k}. \end{cases}$$

- $\|\nabla u_k\|_2 = 1, \;\; u_k \rightharpoonup 0$ in $W^{1,2}$ and $\|u_k\|_{\exp L^2} \to \frac{1}{\sqrt{4\pi}}$.

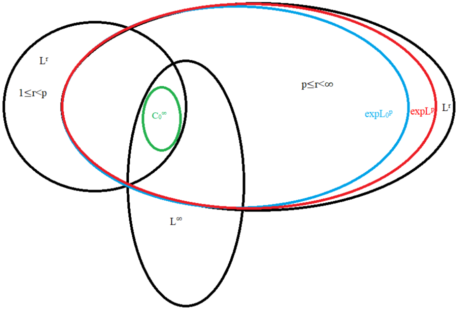

Figure: titre

Non-existence

- Assume that $f:\mathbb{R}\to\mathbb{R}$ is continuous, positive on $[0,\infty)$ and satisfies

$$\liminf_{s\to\infty}\left(f(s)\,\mathrm{e}^{-\lambda s^p}\right)>0,$$

  for some constants $p>1$ and $\lambda>0$.
- A typical example is $f(u)=u|u|^{m-1}\left(\mathrm{e}^{|u|^q}-1\right)$, $q>p$.
- Define $\Phi_\alpha(x)=\Big|\log|x|\Big|^{\frac{1}{p}}\chi_{\{|x|<1\}}$. Then $\Phi_\alpha\in\exp L^p\setminus\exp L^p_0$.

Theorem [S. Tayachi & M. M.]

There exists $\alpha_0>0$ such that for every $\alpha\geq\alpha_0$ and $T>0$ the Cauchy problem (1)-(2) with $u_0=\Phi_\alpha$ has no nonnegative $\exp L^p-$classical solution in $[0,T]$.

LWP

- Since $C_0^\infty(\mathbb{R}^N)$ is dense in $\exp L_0^p(\mathbb{R}^N)$, we are able to prove local existence and uniqueness in $\exp L_0^p(\mathbb{R}^N)$.
- We assume that the nonlinearity f satisfies

$$f(0)=0, \qquad |f(u)-f(v)| \leq C|u-v|(\mathrm{e}^{\lambda |u|^p}+\mathrm{e}^{\lambda |v|^p}),$$

  for some constants $C>0$, $p>1$ and $\lambda>0$.
- **Example**: $\mathbf{f(u)=u|u|^{m-1}\left(e^{|u|^q}-1\right),\ q\leq p}$.

Theorem [S. Tayachi & M. M.]

Given any $u_0 \in \exp L_0^p(\mathbb{R}^N)$ with $p>1$, there exist a time $T=T(u_0)>0$ and a unique weak solution $u \in C([0,T]; \exp L_0^p(\mathbb{R}^N))$ to (1)-(2) .

Global existence & Decay estimates

- This depends on the behavior of the nonlinearity $f(u)$ near $u = 0$.
- We suppose that the nonlinearity f satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \leq C\,|u - v|\left(|u|^{m-1}\mathrm{e}^{\lambda|u|^p} + |v|^{m-1}\mathrm{e}^{\lambda|v|^p}\right),$$

  for some positive constants C, λ and $p > 1$.
- Examples:
  - $f(u) = \pm u|u|^{m-1}$.
  - $f(u) = e^u - 1 - u$.
  - $f(u) = \pm u|u|^{m-1}e^{|u|^q},\ q \leq p$.
  - $f(u) = e^{|u|^q} - 1,\ q \leq p$.
- By the embedding: $\exp L^p \hookrightarrow L^q$ for $1 < p \leq q$, we need $N(m-1)/2 \geq p,\ m \geq p$.

Global existence & Decay estimates

Theorem [S. Tayachi & M. M.]

Assume that $\mathbf{N} > \frac{\mathbf{2p}}{\mathbf{p-1}}$. There exists a positive constant $\varepsilon > 0$ such that every initial data $u_0 \in \exp L^p(\mathbb{R}^N)$ with $\|u_0\|_{\exp L^p(\mathbb{R}^N)} \leqslant \varepsilon$, there exists a weak-mild solution $u \in L^\infty(0,\infty; \exp L^p(\mathbb{R}^N))$ of (1)-(2) satisfying

- $\lim_{t\longrightarrow 0} \|u(t) - \mathrm{e}^{t\Delta} u_0\|_{\exp L^p(\mathbb{R}^N)} = 0.$

- If $m > 3/2$ then there exists a constant $C > 0$ such that,

$$\|u(t)\|_a \leq C\, t^{-\sigma}, \quad \forall\, t > 0.$$

- Here $\frac{N(m-1)}{2} < a < \frac{N(m-1)}{2}\frac{1}{(2-m)_+}$, $a > N/2$, and $\sigma = \frac{1}{m-1} - \frac{N}{2a} > 0$.

In the previous Theorem, if $u_0 \in \exp L^p_0$ with $\|u_0\|_{\exp L^p}$ sufficiently small, then the unique maximal solution u of (1)-(2) given by Theorem of LWP, satisfies :

1. u is global.
2. $u \in C([0,\infty); \exp L^p_0) \cap L^\infty(0,\infty; \exp L^p)$.
3. $\|u(t)\|_a \leq \; C\, t^{-\left(\frac{1}{m-1}-\frac{N}{2a}\right)}, \quad \forall\, t>0.$

Linear estimates

Proposition

For all $1 \leq r \leq \rho \leq \infty$, we have

$$\|e^{t\Delta}\varphi\|_{\rho} \leqslant t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{\rho})}\|\varphi\|_{r}.$$

Proposition

Let $1 \leqslant q \leqslant p$, $1 \leqslant r \leqslant \infty$. Then the following estimates hold:

(i) $\|e^{t\Delta}\varphi\|_{\exp L^p} \leqslant \|\varphi\|_{\exp L^p}, \ \forall\, t > 0, \ \forall\, \varphi \in \exp L^p.$

(ii) $\|e^{t\Delta}\varphi\|_{\exp L^p} \leqslant \ t^{-\frac{N}{2q}}\left(\log(t^{-\frac{N}{2}}+1)\right)^{-\frac{1}{p}}\|\varphi\|_{q}, \ \forall\, t > 0, \ \forall\, \varphi \in L^q.$

(iii) $\|e^{t\Delta}\varphi\|_{\exp L^p} \leqslant \dfrac{1}{(\log 2)^{\frac{1}{p}}}\left[t^{-\frac{N}{2r}}\|\varphi\|_{r} + \|\varphi\|_{q}\right], \ \forall\, t > 0, \forall\, \varphi \in L^r \cap L^q.$

Proof: (i) For any $\alpha > 0$, expanding the exponential function leads to

$$\int_{\mathbb{R}^N} \left(\exp \left| \frac{\mathrm{e}^{t\Delta}\varphi}{\alpha} \right|^p - 1 \right) dx \quad = \quad \sum_{k=1}^{\infty} \frac{\|\mathrm{e}^{t\Delta}\varphi\|_{pk}^{pk}}{k!\alpha^{pk}}.$$

Then by the $L^{pk} - L^{pk}$ estimate of the heat semi-group, we obtain

$$\int_{\mathbb{R}^N} \left(\exp \left| \frac{\mathrm{e}^{t\Delta}\varphi}{\alpha} \right|^p - 1 \right) dx \quad \leqslant \quad \sum_{k=1}^{\infty} \frac{\|\varphi\|_{pk}^{pk}}{k!\alpha^{pk}} = \int_{\mathbb{R}^N} \left(\exp \left| \frac{\varphi}{\alpha} \right|^p - 1 \right) dx.$$

Therefore we obtain

$$\|\mathrm{e}^{t\Delta}\varphi\|_{\exp L^p} \quad = \quad \inf \left\{ \alpha > 0, \int_{\mathbb{R}^N} \left(\exp \left| \frac{\mathrm{e}^{t\Delta}\varphi}{\alpha} \right|^p - 1 \right) dx \leq 1 \right\} \leq \|\varphi\|_{\exp L^p}.$$

(ii) Using the smoothing effect estimate estimate with $q \leq p$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\left(\exp\left|\frac{e^{t\Delta}\varphi}{\alpha}\right|^p-1\right)dx &= \sum_{k=1}^{\infty}\frac{\|e^{t\Delta}\varphi\|_{pk}^{pk}}{k!\alpha^{pk}} \\ &\leqslant \sum_{k=1}^{\infty}\frac{t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{pk})pk}\|\varphi\|_q^{pk}}{k!\alpha^{pk}} \\ &= t^{\frac{N}{2}}\left(\exp\left(\frac{t^{-\frac{N}{2q}}\|\varphi\|_q}{\alpha}\right)^p-1\right).\end{aligned}$$

It follows that

$$\|e^{t\Delta}\varphi\|_{\exp L^p} \leq t^{-\frac{N}{2q}}\left(\log(t^{-\frac{N}{2}}+1)\right)^{-\frac{1}{p}}\|\varphi\|_q.$$

This proves (ii).

(iii) By the embedding $L^q \cap L^\infty \hookrightarrow \exp L^p$, we have

$$\|e^{t\Delta}\varphi\|_{\exp L^p} \leqslant \frac{1}{(\log 2)^{1/p}} \left[\|e^{t\Delta}\varphi\|_\infty + \|e^{t\Delta}\varphi\|_q\right].$$

Using the $L^r - L^\infty$ estimate for the heat semi-group, we get

$$\|e^{t\Delta}\varphi\|_{\exp L^p} \leqslant \frac{1}{(\log 2)^{1/p}} \left[t^{-\frac{N}{2r}}\|\varphi\|_r + \|\varphi\|_q\right].$$

This proves (iii).

As a consequence we have the following.

Corollary

Let $p > 1$, $N > \frac{2p}{p-1}$, $r > \frac{N}{2}$. Then, for every $g \in L^1 \cap L^r$, we have

$$\|e^{t\Delta}g\|_{\exp L^p} \leq \kappa(t)\, \|g\|_{L^1 \cap L^r}, \ \forall\, t > 0,$$

where $\kappa \in L^1(0,\infty)$.

Proof: We have, by (ii) with $q = 1$,

$$\|e^{t\Delta}g\|_{\exp L^p} \leq t^{-\frac{N}{2}}\left(\log(t^{-\frac{N}{2}}+1)\right)^{-\frac{1}{p}}\|g\|_1.$$

Using (iii) with $q = 1$, we get

$$\|e^{t\Delta}g\|_{\exp L^p} \leq \frac{1}{(\log 2)^{\frac{1}{p}}}\left(t^{-\frac{N}{2r}}+1\right)\left[\|g\|_r + \|g\|_1\right].$$

Combining these inequalities, we obtain

$$\|e^{t\Delta}g\|_{\exp L^p} \leq \kappa(t)\left(\|g\|_1 + \|g\|_r\right).$$

By the assumption $N > \frac{2p}{p-1},\ r > \frac{N}{2}$, we can see that $\kappa \in L^1(0, \infty)$.

Non-existence

The following lemma is the key of the proof.

Lemma

There exists $\alpha_0 > 0$ such that for any $\alpha \geq \alpha_0$, $\varepsilon > 0$ and $r > 0$, we have

$$\int_0^{\varepsilon} \int_{|x|<r} \exp\left(\lambda \left(\mathrm{e}^{t\Delta}\Phi_\alpha\right)^p\right) dx\, dt = \infty\,.$$

Proof: Let $B(a, \rho)$ denotes the open ball centered at $a \in \mathbb{R}^N$ and with radius $\rho > 0$. Fix ε, $r > 0$. For $\rho = \min\left(r, \frac{1}{4}\right)$, we have $B(3x, |x|) \subset B(0, 1)$ for any $|x| < \rho$.

Non-existence

It follows that

$$\begin{aligned}\left(\mathrm{e}^{t\Delta}\Phi_\alpha\right)(x) &\geq \frac{\alpha}{(4\pi t)^{N/2}}\int_{|y-3x|<|x|}\mathrm{e}^{-\frac{|x-y|^2}{4t}}\Big(-\log|y|\Big)^{\frac{1}{p}}\,dy\\ &\geq C\alpha\left(\frac{|x|^2}{t}\right)^{N/2}\mathrm{e}^{-\frac{9}{4}\frac{|x|^2}{t}}\Big(-\log 4|x|\Big)^{1/p}.\end{aligned}$$

Let $\eta=\min\left(\varepsilon,\rho^2\right)$. Then, for any $0<t<\eta$, we have $B(0,\sqrt{t})\subset B(0,\rho)$. Hence

$$\begin{aligned}\int_0^\varepsilon\int_{|x|<r}\exp\left(\lambda\left(\mathrm{e}^{t\Delta}\Phi_\alpha\right)^p\right)\,dx\,dt &\geq \int_0^\eta\int_{\frac{\sqrt{t}}{2}<|x|<\sqrt{t}}\exp\left(-C\lambda\alpha^p\log(4|x|)\right)\,dx\,dt\\ &\geq C_\alpha\int_0^\eta t^{\frac{N}{2}-\frac{C\lambda\alpha^p}{2}}\,dt=\infty,\end{aligned}$$

for $\alpha\geq\alpha_0:=\left(\frac{N+2}{C\lambda}\right)^{1/p}$.

GE & DE

- The proof uses a fixed point argument on the associated integral equation

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}(f(u))(s)ds.$$

- The nonlinearity satisfies

$$|f(u) - f(v)| \leq C|u - v| \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left(|u|^{pk+m-1} + |v|^{pk+m-1} \right).$$

- We will perform a fixed point argument on the space

$$Y_M := \left\{ u \in L^\infty(0, \infty, \exp L^p);\ \sup_{t>0} t^\sigma \|u(t)\|_a + \|u\|_{L^\infty(0,\infty;\exp L^p)} \leq M \right\},$$

where $a > \frac{N(m-1)}{2} \geq p$ and $\sigma = \frac{1}{m-1} - \frac{N}{2a} = \frac{N}{2}\left(\frac{2}{N(m-1)} - \frac{1}{a}\right) > 0$.

- Endowed with the metric $d(u, v) = \sup_{t>0} \left(t^\sigma \|u(t) - v(t)\|_r \right)$, Y_M is complete.

GE & DE

- For $u \in Y_M$, we define $\Phi(u)$ by

$$\Phi(u)(t) := \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}(f(u(s)))ds.$$

- We have

$$t^\sigma \|\mathrm{e}^{t\Delta} u_0\|_a \leq t^\sigma t^{-\frac{N}{2}\left(\frac{2}{N(m-1)} - \frac{1}{a}\right)} \|u_0\|_{\frac{N(m-1)}{2}} = \|u_0\|_{\frac{N(m-1)}{2}} \leq C\|u_0\|_{\exp L^p},$$

where we have used $1 \leq p \leq \frac{N(m-1)}{2} < a$.

GE & DE

- For $q > \frac{N}{2}$, we have

$$\begin{aligned}
\|\Phi(u)(t)\|_{\exp L^p} &\leq \|\mathrm{e}^{t\Delta}u_0\|_{\exp L^p} + \int_0^t \left\|\mathrm{e}^{(t-s)\Delta}(f(u(s)))\right\|_{\exp L^p}\, ds \\
&\leq \|\mathrm{e}^{t\Delta}u_0\|_{\exp L^p} + \int_0^t \kappa(t-s)\Big(\|f(u(s))\|_{L^1\cap L^q}\Big)\, ds \\
&\leq \|\mathrm{e}^{t\Delta}u_0\|_{\exp L^p} + \|f(u)\|_{L^\infty(0,\infty;(L^1\cap L^q))}\int_0^\infty \kappa(s)\, ds \\
&\leq \|\mathrm{e}^{t\Delta}u_0\|_{\exp L^p} + C\|f(u)\|_{L^\infty(0,\infty;(L^1\cap L^q))}.
\end{aligned}$$

- It follows that

$$\|\Phi(u)\|_{L^\infty(0,\infty;\exp L^p)} \leq \|u_0\|_{\exp L^p} + C\|f(u)\|_{L^\infty(0,\infty;\, L^1\cap L^q)}.$$

GE & DE

- Since $|f(u)| \leq C|u|^m \left(\mathrm{e}^{\lambda|u|^p} - 1\right) + C|u|^m$, we get

$$\|\Phi(u)\|_{L^\infty(0,\infty,\exp L^p)} \leq \|u_0\|_{\exp L^p} + CM^m \leq \varepsilon + CM^m.$$

- Let $u, v \in Y_M$. Using the Taylor expansion of f, linear estimates and Hölder's inequality, we obtain

$$d\Big(\Phi(u), \Phi(v)\Big) \leq Cd(u,v) \sum_{k=0}^{\infty} (C\lambda)^k M^{pk+m-1} \leq CM^{m-1} d(u,v).$$

- The conclusion follows via a fixed point argument.
- We stress that the decay estimate is included in the fixed point argument.

Perspectives

1. Lower dimensions $\mathbf{N} \leq \frac{\mathbf{2p}}{\mathbf{p-1}}$: work in progress.

2. Global existence for arbitrary (not necessary small) initial data.

3. Well-posedness in more general Orlicz spaces L^{ϕ}.

4. Qualitative behavior of global and blowing-up solutions.

Thank You