Long time existence results for solutions of water waves equations

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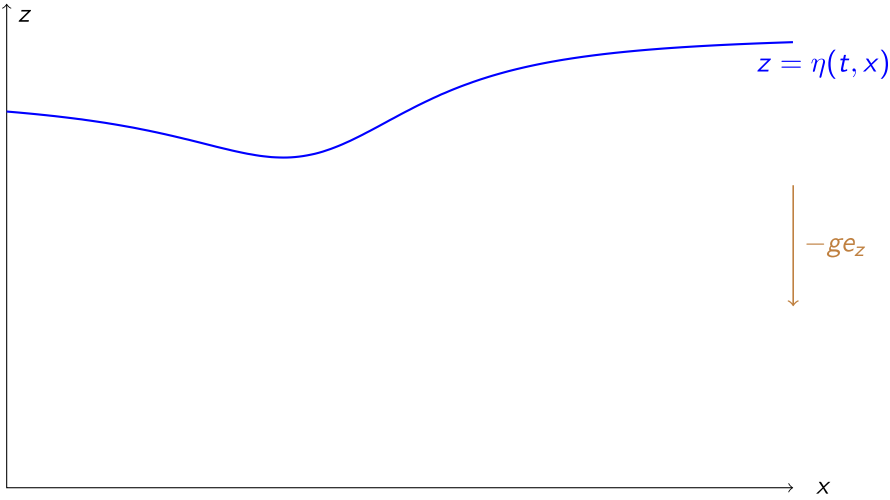
1. Water waves equations



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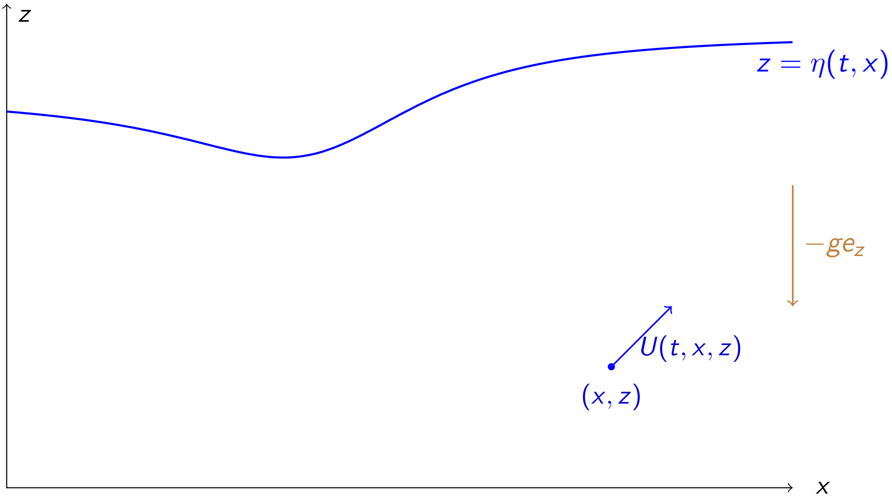

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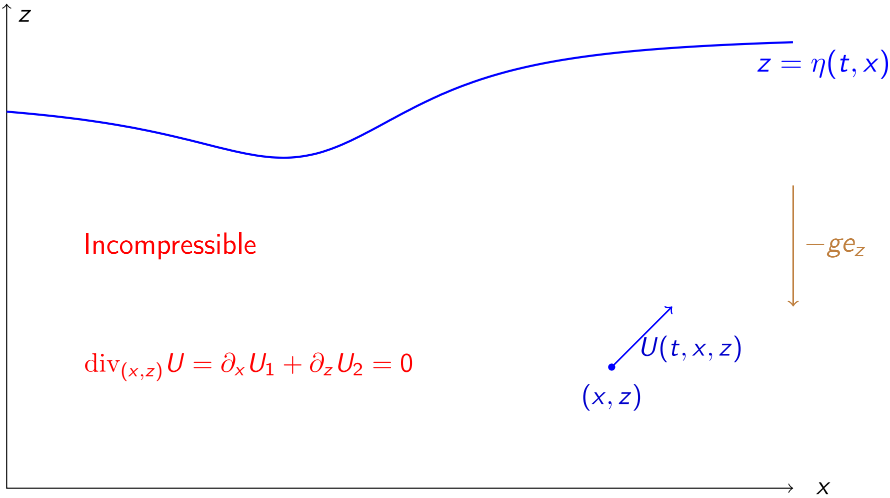

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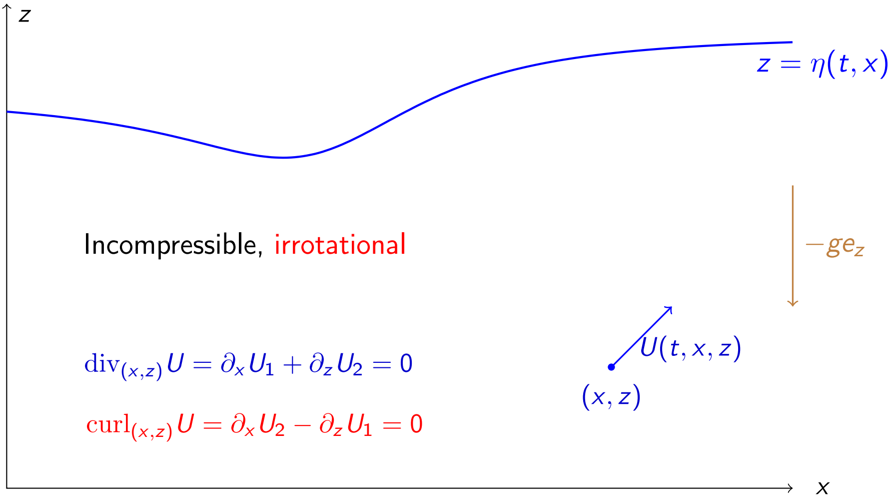

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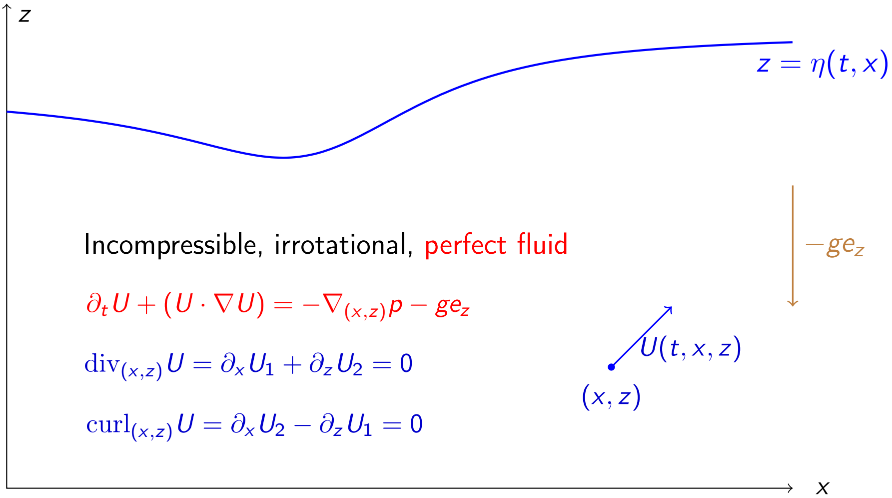

Incompressible, irrotational, perfect fluid

$$\partial_t U + (U \cdot \nabla U) = -\nabla_{(x,z)} p - g e_z$$

$$\mathrm{div}_{(x,z)} U = \partial_x U_1 + \partial_z U_2 = 0$$

$$\mathrm{curl}_{(x,z)} U = \partial_x U_2 - \partial_z U_1 = 0$$

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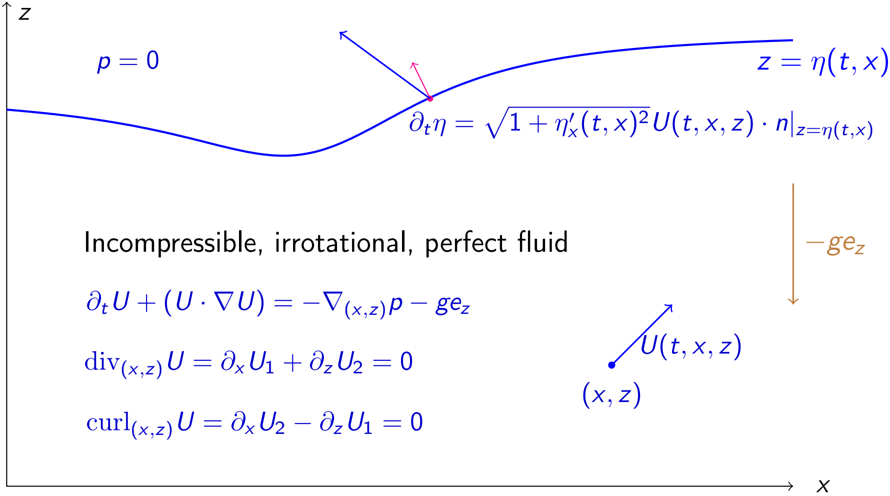

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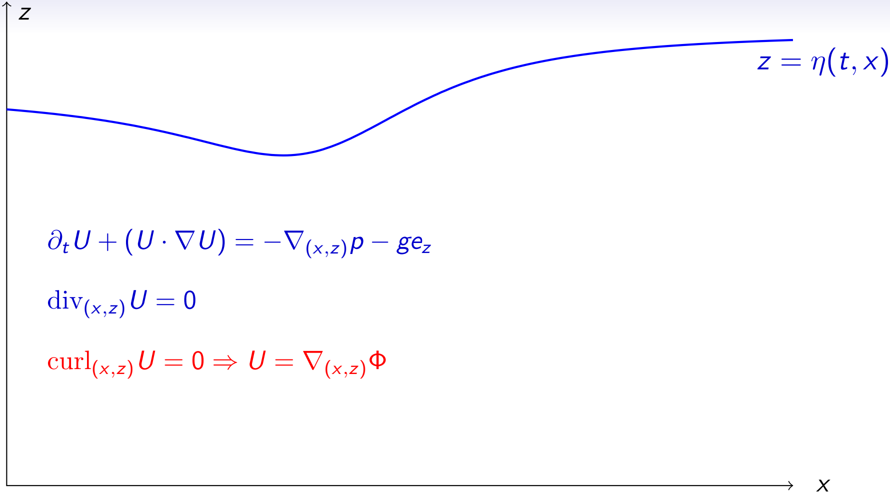

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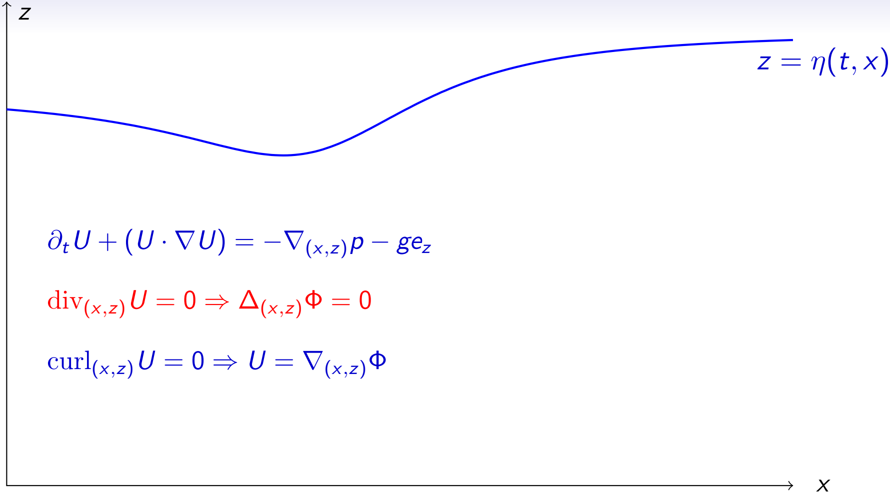
$$\mathrm{div}_{(x,z)} U = 0$$

$$\mathrm{curl}_{(x,z)} U = 0 \Rightarrow U = \nabla_{(x,z)} \Phi$$



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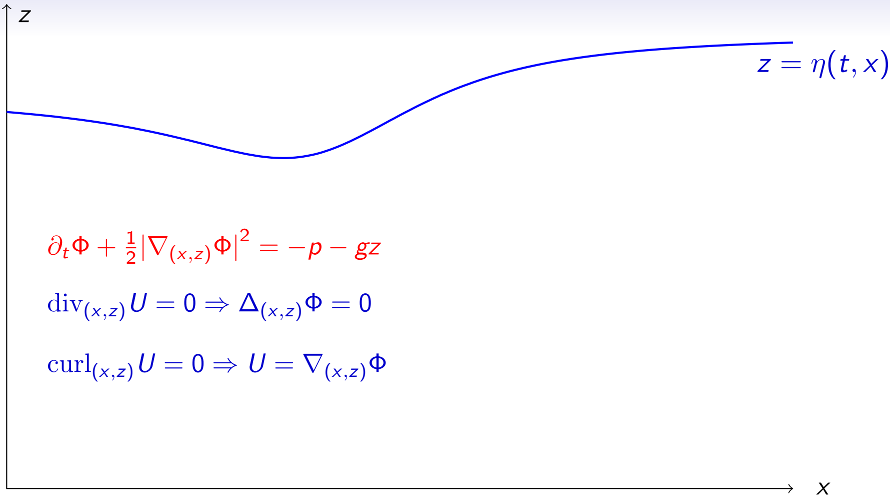

$$\partial_t \Phi + \frac{1}{2} |\nabla_{(x,z)} \Phi|^2 = -p - gz$$

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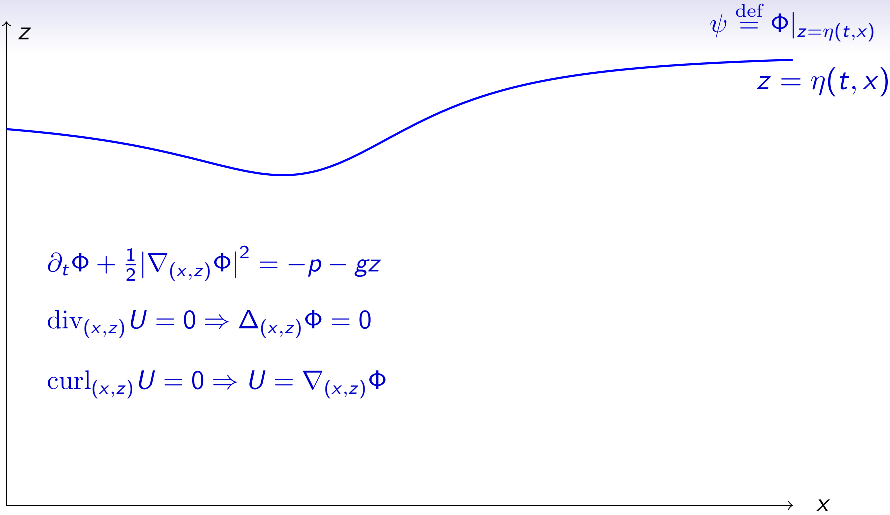

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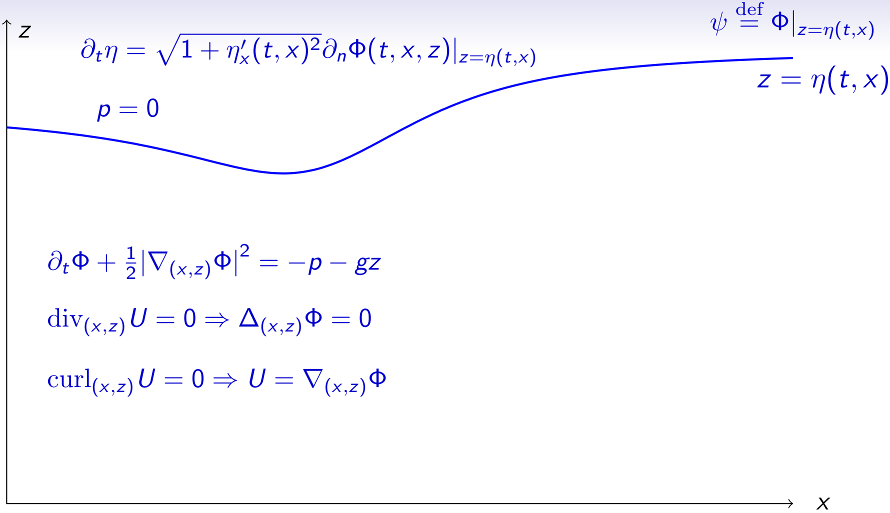

Define the Dirichlet-Neuman operator $\psi \to G(\eta)\psi$ as follows:
Given ψ, let Φ be the solution of $\Delta\Phi = 0$, with
$\Phi|_{z=\eta(t,x)} = \psi(t,x)$, $\nabla\Phi(x,z) \to 0$ if $z \to -\infty$. Set then
$G(\eta)\psi = \sqrt{1+\eta_x'(t,x)^2}\,\partial_n \Phi(t,x,z)|_{z=\eta(t,x)}$.

The Craig-Sulem-Zakharov formulation of the water waves equation

The above equations may be reformulated as a system on (η, ψ). One gets:

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\text{(WW)} \qquad
\begin{aligned}
\partial_t \eta &= G(\eta)\psi \\
\partial_t \psi &= -g\eta - \frac{1}{2}|\nabla_x \psi|^2 + \frac{\big(G(\eta)\psi + \nabla_x \eta \cdot \nabla_x \psi\big)^2}{2(1 + |\nabla_x \eta|^2)}.
\end{aligned}
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Remark: One may also consider a more general equation, taking into account surface tension at the interface between the two fluids. One gets then a variant, the gravity capillary water waves equation above, where $\kappa > 0$.

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$$(\text{GCWW}) \qquad \begin{aligned} \partial_t \eta &= G(\eta)\psi \\ \partial_t \psi &= -g\eta - \frac{1}{2}|\nabla_x \psi|^2 + \frac{\big(G(\eta)\psi + \nabla_x \eta \cdot \nabla_x \psi\big)^2}{2(1+|\nabla_x \eta|^2)} \\ &\qquad + \kappa \,\mathrm{div}\left(\frac{\nabla_x \eta}{\sqrt{1+|\nabla_x \eta|^2}}\right). \end{aligned}$$

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2. Local existence theory

Local existence theory is highly non trivial. It has been solved in full generality by Sijue Wu, after preliminary works of V. Nalimov and H. Yosihara.

Key difficulty: The system is *not* an hyperbolic quasi-linear system. It may be written formally as

$$\partial_t \begin{bmatrix} \eta \\ \psi \end{bmatrix} = A(\eta, \psi, D_x) \begin{bmatrix} \eta \\ \psi \end{bmatrix} \quad (D_x = \frac{1}{i}\frac{\partial}{\partial x})$$

where $A(\eta, \psi, D_x)$ is a 2×2 matrix of pseudo-differential operators of order 1 i.e. for V in $\mathcal{S}$

$$A(\eta, \psi, D_x)V = \frac{1}{2\pi} \int e^{ix\xi} A(\eta, \psi, \xi) \hat{V}(\xi)\, d\xi.$$

Problem: The symbol $A(\eta, \psi, \xi)$ is a two by two matrix whose eigenvalues have real parts going to infinity when $|\xi|$ goes to infinity.

(Example: $\partial_t u = a(D_x)u$ or $\partial_t \hat{u}(t, \xi) = a(\xi)\hat{u}(t, \xi)$ so that $\hat{u}(t, \xi) = e^{ta(\xi)}\hat{u}(0, \xi) \notin \mathcal{S}'$ for $t \neq 0$ if $\mathrm{Re}\, a(\xi) \to \infty$ when $|\xi| \to +\infty$).

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Solution: Define from (η, ψ) a "good unknown" from which the problem will be hyperbolic.

• Nalimov, Yosihara, Wu: Use of the Lagrangian formulation of the problem.

• Lannes (Alinhac) defined $(\eta, \psi) \to \big(\eta, \omega = \psi - (B(\eta)\psi)\eta\big)$, where B is defined in the following way:

Analytic expression:

$$B(\eta)\psi = \frac{G(\eta)\psi + \partial_x\eta\partial_x\psi}{1 + (\partial_x\eta)^2}.$$

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Then $\partial_t \begin{bmatrix}\eta\\ \omega\end{bmatrix} = \tilde{A}(\eta, \omega, D_x)\begin{bmatrix}\eta\\ \omega\end{bmatrix}$ where $\tilde{A}(\eta, \omega, \xi)$ has eigenvalues with *bounded real part*.

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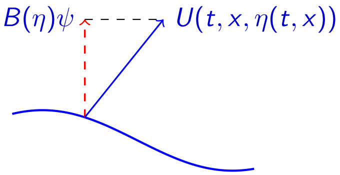

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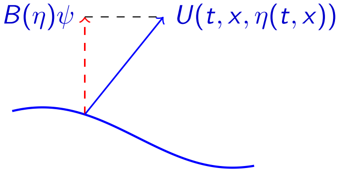

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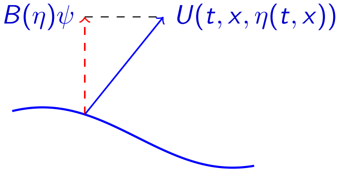

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3. Global solutions for small smooth decaying data

As (WW) is a dispersive equation, solutions with small smooth decaying initial data are $O(t^{-\frac{d}{2}})$ in L^∞ as time goes to infinity (where d is the dimension). One may thus expect that the nonlinearity is a small perturbation, negligible enough to allow for global existence of solutions. One expects the problem to be easier in higher space dimension.

• Case $d = 2$: S. Wu and Germain-Masmoudi-Shatah proved independently global existence for small, smooth, decaying initial data.

• Case $d = 1$: S. Wu showed that solutions corresponding to smooth, decaying data of size ϵ exist over a time interval of length at least $e^{c/\epsilon}$.

Later on, global existence has been obtained by several authors: Ionescu-Pusateri, Alazard-D., Ifrim-Tataru.

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Theorem (Global existence for gravity WW)

There are indices of regularity and decay $s \gg s_1 \gg 1$ *and* $0 < \epsilon_0 \ll 1$ *such that for any* (η_0, ψ_0) *satisfying*

$$(x\partial_x)^p \eta_0 \in H^{s-p},\ |D_x|^{\frac{1}{2}}(x\partial_x)^p \psi_0 \in H^{s-p-\frac{1}{2}},\ 0 \le p \le s_1$$
$$|D_x|^{\frac{1}{2}}(x\partial_x)^p \omega_0 \in H^{s-p}(\mathbb{R}),$$

of norm $O(\epsilon)$ *in those spaces (with* $\epsilon < \epsilon_0$*), the* (WW) *equation with data* (η_0, ψ_0) *has a unique global solution. Moreover, if* $u(t,x) = |D_x|^{\frac{1}{2}}\psi + i\eta$, *then*

$$u(t,x) = \frac{\epsilon}{\sqrt{t}} \alpha_\epsilon\left(\frac{x}{t}\right) \exp\left[i\frac{t}{4|x/t|} + i\frac{\epsilon^2}{64}\frac{|\alpha_\epsilon(x/t)|^2}{|x/t|^5}\log t\right] + O_{L^\infty}\left(\epsilon t^{-\frac{1}{2}-\theta}\right)$$

for some $\theta > 0$, *where* $(\alpha_\epsilon)_\epsilon$ *is bounded in* $C^0 \cap L^\infty$.

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of norm $O(\epsilon)$ *in those spaces (with* $\epsilon < \epsilon_0$*), the* (WW) *equation with data* (η_0, ψ_0) *has a unique global solution. Moreover, if* $u(t,x) = |D_x|^{\frac{1}{2}}\psi + i\eta$, *then*

$$u(t,x) = \frac{\epsilon}{\sqrt{t}}\alpha_\epsilon\Big(\frac{x}{t}\Big)\exp\Big[i\frac{t}{4|x/t|} + i\frac{\epsilon^2}{64}\frac{|\alpha_\epsilon(x/t)|^2}{|x/t|^5}\log t\Big] + O_{L^\infty}\big(\epsilon t^{-\frac{1}{2}-\theta}\big)$$

for some $\theta > 0$, *where* $(\alpha_\epsilon)_\epsilon$ *is bounded in* $C^0 \cap L^\infty$.

References: Ionescu-Pusateri: $s_1 = 1, s \gg 1$.
Alazard-D.: $1 \ll s_1 \ll s$.
Ifrim-Tataru: $s_1 = 1, s \sim 6$.

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Consider $p(\xi) = |\xi|^{\frac{1}{2}}$, set $D = \frac{1}{i}\partial$ and for $n \in \mathbb{N}, n \geq 2$ let u solve

$$\big(D_t - p(D_x)\big)u = N(u) \sim u^n.$$

Write the Sobolev energy inequality (with $s \gg 1$)

$$(*) \quad \|u(t,\cdot)\|_{H^s} \leq \|u(0,\cdot)\|_{H^s} + C\int_0^t \|u(\tau,\cdot)\|_{L^\infty}^{n-1}\|u(\tau,\cdot)\|_{H^s}\,d\tau.$$

if we assume $\|u(\tau,\cdot)\|_{L^\infty} = O(\epsilon\tau^{-\frac{1}{2}})$. If we had $n \geq 3$, then Gronwall inequality would imply

$$\|u(t,\cdot)\|_{H^s} \leq Ct^{C\epsilon^2}.$$

Problem: For the (WW) system, $n = 2$!
Solution: Use a Shatah-Simon normal form method i.e. replace u by $\Phi = u + E(u,u)$ in order that Φ satisfies $(*)$ with $n = 3$.
Difficulty: Has to be adapted to quasi-linear equations.

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Second step of the proof: Show that the solution does satisfy the dispersive estimate $\|u(t,\cdot)\|_{L^\infty} = O\big(\epsilon t^{-\frac{1}{2}}\big)$.

Idea: Obtain energy estimates for the action of the Klainerman vector field $Z = tD_t + 2xD_x$ on u:

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A Klainerman-Sobolev inequality allows one to deduce $\|u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}+C\epsilon^2})$, which is bad.

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Solution: Consider the PDE as a quantum problem, and deduce from it the corresponding classical problem, i.e. an ODE

$$\big(D_t - \omega(x)\big)v = \frac{1}{t}N(v, \bar{v}) + O(\epsilon t^{-1-\theta})$$

where N is a polynomial, ω a real valued function, $\theta > 0$ and $u(t,x) = \frac{1}{\sqrt{t}}v\Big(t, \frac{x}{t}\Big)$. Analysing the ODE, one shows that it may be reduced to

$$\partial_t f = i\omega(x)f + \frac{i}{t}\Psi(x)|f|^2 f + \text{integrable remainder}$$

for some real valued function Ψ. One then deduces that f and then v is bounded.

References to further work

Gravity water waves in infinite depth and space dimension $d = 1$ (two dimensional fluid): Xuecheng Wang proved global existence with infinite energy small decaying data.

Capillary water waves in infinite depth and space dimension $d = 1$: Ifrim-Tataru and Ionescu-Pusateri obtained global existence for small, smooth, decaying data.

Capillary-gravity water waves infinite depth in dimension $d = 2$ (three dimensional fluids): Deng-Ionescu-Pausader-Pusateri proved global existence for small, smooth, decaying data.

Gravity water waves in *finite* depth and space dimension $d = 2$ (three dimensional fluid): Xuecheng Wang proved global existence for small, smooth, decaying data.