Interactions of solitons from the PDE point of view

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Outline

1. Collision of solitons for the generalized KdV equation

2. Interaction of solitons for the 5D energy critical wave equation
We consider the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x (\partial_x^2 u + g(u)) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}$$

with a general nonlinearity $g$, typically $g(u) = u^p$, for $p = 2, 3, 4, \ldots$ or $g(u) = u^p \pm u^q$.

We call soliton a solution of the form $u(t, x) = Q_c(x - ct)$, where $c > 0$ and $Q_c$ solves

$$Q''_c + g(Q_c) = cQ_c, \quad x \in \mathbb{R}.$$

Under suitable assumptions on $g$, for some range of $c$, there exists a unique positive solution of this equation.

The question of the orbital stability of the solution $u(t, x) = Q_c(x - ct)$ with respect to perturbation of initial data is well-known.
Collision problem for the gKdV equation

Let $u(t)$ be a solution of gKdV such that

$$u(t, x) \sim Q_{c_1}(x - c_1 t) + Q_{c_2}(x - c_2 t) \quad \text{as } t \to -\infty,$$

where $Q_{c_1}(x - c_1 t)$ and $Q_{c_2}(x - c_2 t)$ are two solitons with $0 < c_2 < c_1$. The existence and uniqueness of such a solution is known [YM '04].

We ask general questions about the collision of two solitons:

- What is the behavior of $u$ during and after the collision?
- Do the solitons survive the interaction/collision at the principal order?
- If yes, are the speeds, sizes and trajectories of the solitons modified?
- Is the collision elastic or inelastic?
In the integrable cases for gKdV \((g(u) = u^2, u^3, u^2 + au^3)\), there exist explicit multi-soliton solutions describing the interaction of solitons for all time. The collision is elastic.

[Fermi-Pasta-Ulam '55], [Zabusky-Kruskal '65], [Lax '68],
[GGKM '67-'74], [Hirota '71], [Miura '76]

By numerical simulations and experiments (in water tanks), it is expected that for several related non-integrable models, the collision is almost elastic but not exactly elastic.

[Eilbeck-McGuire '75], [Kivshar-Malomed '89], [Bona et al. '80],
[Shih '80], [Craig et al. '06]
Now, we describe recent PDE works aiming at describing rigorously the collision of two solitons for the quartic non-integrable gKdV

\[ \partial_t u + \partial_x (\partial_x^2 u + u^4) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R} \]

in the following two asymptotic regimes:

- Very different sizes: \( 0 < c_2 \ll c_1 \)
- Almost equal sizes: \( c_1 \sim c_2 \).

The interest of the quartic gKdV comes from its simplicity, and the fact that it cannot be considered perturbative of any integrable equation.
Two solitons with very different sizes for quartic gKdV

**Theorem (YM-Merle ’08)**

Assume $0 < c \ll 1$. Let $u$ be a 2-soliton of quartic gKdV:

$$\lim_{t \to -\infty} \| u(t) - Q(\cdot - t) - Q_c(\cdot - ct) \|_{H^1} \to 0$$

Then there exist $c_1^+ \sim 1$, $c_2^+ \sim c$ such that for

$$u(t, x) = Q_{c_1^+}(x - y_1(t)) + Q_{c_2^+}(x - y_2(t)) + w(t, x)$$

- **Stability**

  $$\sup_{t \in \mathbb{R}} \| w(t) \|_{H^1} \lesssim c^{\frac{1}{3}}, \quad \lim_{t \to +\infty} \| w(t) \|_{H^1(x \geq \frac{1}{10} ct)} = 0$$

- **Inelasticity**

  $$c_1^+ > 1, \ c_2^+ < c \quad \text{and} \quad \lim_{t \to +\infty} \| w(t) \|_{H^1} \neq 0.$$
It is a perturbative result: $0 < c \ll 1$

The two solitons are preserved through the collision

$$\sup_t \|w(t)\|_{H^1} \lesssim c^{\frac{1}{3}} \quad \text{and} \quad \|Q_c\|_{H^1} \sim c^{\frac{1}{12}}$$

The collision is almost elastic since $\|w(t)\|_{H^1} \ll \|Q_c\|_{H^1}$

...but not elastic since

$$\|w(t)\|_{H^1} \not\to 0 \text{ as } t \to +\infty \text{ and } c_1 > 1 \text{ and } c < c$$

This proves the nonexistence of a pure 2-soliton solution in this regime

More generally, for a large class of nonlinearities $g$, collisions for gKdV are elastic if and only if the equation is integrable [Muñoz ’10]
Collision of two solitons with different sizes

The first part of Theorem 10 has been extended to (1.3) with general nonlinearity \( f \) for which solitons are stable in Weinstein's sense. The second part, i.e. the inelastic nature of the collision, has been proved for general non integrable nonlinearities for small solitons by Muñoz [71] (in that work, both solitons are small, and one is much smaller than the other one).

The proof of Theorem 10 in [59] is long and involved and beyond the scope of this course. We just sketch the main steps of the proofs in [59], and refer to the review paper [57] and to the original paper for details.

1. First, we construct an approximate solution to the problem in the collision region, i.e. in the time interval \( -c_1 + (1/2), +c_1 - (1/2) \). The approximate solution has the form of a series in terms of \( c = c_2/c_1 \) and involves a delicate algebra.

2. Second, using asymptotic arguments similar to the ones presented in the previous sections, we justify that the solution \( U(t) \) is close to the approximate solution (so that the description of the collision given by the approximate solution is relevant) and we control the solution in large time, i.e. for \( |t| > c_1 - (1/2) \).

3. Finally, we prove the inelastic character of the collision by a further analysis of the approximate solution. The defect is due to a nonzero extra term in the approximate solution after recomposition of the series. Thus, the defect is a direct consequence of the algebra underlying the construction of the approximate solution.
Similar question for 1D NLS equation

**Theorem (Perelman '11)**

Let $\nu > 0$, $0 < c \ll 1$. There exists a solution $u$ of

$$i\partial_t u + \Delta u + |u|^2 u + f(|u|^2)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R},$$

where $|f(u)| \lesssim_0 u^2$, $f(u) \lesssim_\infty |u|^q$ ($q < 2$), satisfying

- **Two-soliton at $-\infty$**

$$\lim_{t \to -\infty} \|u(t) - e^{i\nu t} Q - e^{i\Gamma_{\nu}(t, \cdot)} Q_c(\cdot - \nu t)\|_{H^1} \to 0$$

- **Splitting of the small soliton after the collision**

$$u(t, x) \sim e^{i\Gamma(t, x)} Q(x - \sigma(t)) + \psi^+(t, x) + \psi^-(t, x)$$

where $\psi^\pm$ are solutions of the cubic NLS corresponding to the transmitted part and the reflected part of the small soliton.

See [Holmer-Marzuola-Zworski '07] for splitting by impurity.
Two solitons with almost equal sizes for gKdV

In the integrable case, using explicit formulas, it is known that two-solitons with almost equal speeds do not cross and remain at a large distance for all time [LeVeque '87].

For the quartic gKdV, a similar intuition allows to use perturbation theory and show repulsive interactions of two solitons.

Theorem (Mizumachi ’03)

Let

\[ u_0 \sim Q(x) + Q(x + Y_0) \quad \text{with } Y_0 \gg 1. \]

Then, the corresponding solution of quartic gKdV satisfies, for some \( c_1 > c_2 \) close to 1, for large time,

\[ u(t, x) = Q_{c_1}(x - c_1 t - y_1) + Q_{c_2}(x - c_2 t - y_2) + w(t, x), \]

where \( w \) is uniformly small.
Theorem (YM-Merle ’11)

Let \(0 < \mu \ll 1\). Let \(u\) be a two-soliton of the quartic gKdV

\[
\lim_{t \to -\infty} \| u(t) - Q_{1-\mu}(\cdot - (1-\mu)t) - Q_{1+\mu}(\cdot - (1+\mu)t) \|_{H^1} = 0
\]

There exist \(\mu_1(t), \mu_2(t), y_1(t), y_2(t)\) such that

\[
u(t, x) = Q_{1+\mu_1(t)}(x - y_1(t)) + Q_{1+\mu_2(t)}(x - y_2(t)) + w(t, x)
\]

- **Stability**

\[
\lim_{t \to +\infty} \| w(t) \|_{H^1(x > \frac{1}{10} t)} = 0, \quad \sup_{t \in \mathbb{R}} \| w(t) \|_{H^1} \leq \mu^2
\]

\[
\min_t (y_1(t) - y_2(t)) \sim 2|\ln \mu|
\]

\[
\lim_{t \to +\infty} \mu_1(t) = \mu_1^+ \sim \mu, \quad \lim_{t \to +\infty} \mu_2(t) = \mu_2^+ \sim -\mu
\]

- **Inelasticity**

\[
\mu_1^+ > \mu, \quad \mu_2^+ < -\mu, \quad \lim_{t \to +\infty} \inf \| w(t) \|_{H^1} > 0
\]
- It is a perturbative result: $0 < \mu \ll 1$
- The two solitons are preserved through the collision
- The collision is almost elastic $\|w(t)\|_{H^1} \ll 1$
- ...but not elastic since

$$\|w(t)\|_{H^1} \not\to 0 \text{ as } t \to +\infty \text{ and } \mu_1^+ > \mu \text{ and } \mu_2^+ < -\mu$$

- We again deduce the nonexistence of a pure 2-soliton in this regime.
Collision of two solitons with almost equal sizes

Inelasticity of the interaction.

\[ \lim \inf_{t \to +\infty} \|w(t)\|_{H^1} \geq c \mu_0^3, \]  

\[ \mu_1 + 1 \geq \mu_0 + c \mu_0^5, \]  

\[ \mu_2 \leq -\mu_0 - c \mu_0^5. \]  

It follows immediately from the lower bound (10.14) that no pure 2-soliton exists, which was a new result in this regime.

Comments on the results:

1. For the specific solution \( U(t) \) considered in Theorem 11, the dynamics of the parameters \( \mu_j(t), y_j(t) \) are closely related to the function \( Y(t) = Y_0 + 2 \ln(\cosh(\mu_0 t)) \) which solves

\[ \ddot{Y} = 2 \alpha e^{-Y}, \]  

\[ \lim_{\pm \infty} \dot{Y} = \pm 2 \mu_0, \]  

\[ \dot{Y}(0) = 0. \]  

More information on the behavior of \( U(t) \) and the parameters \( \mu_j(t), y_j(t) \) in terms of \( Y(t) \) is available in [60].

2. Theorem 11 answers the two questions raised before concerning the interaction of two solitons of almost equal speeds. The lower bounds in estimates (10.14) and (10.15) measure the defect of \( U(t) \) at +\( \infty \); in other words, they quantify in the energy space \( H^1 \) the inelastic character of the collision of 2 solitons of (10.8) in the regime where \( \mu_0 \) is small.

The behavior of the solution \( U(t) \) considered in Theorem 11 is represented schematically by the following picture:

- Non-zero residual
- \( \mu_0 > 1 + \mu_0 < 1 - \mu_0 < 1 - \mu_0 \)
- \( t \to -\infty \)
- \( t \sim 0 \)
- \( t \to +\infty \)
A non-perturbative inelasticity result for quartic gKdV

**Theorem (YM-Merle ’15)**

Assume that

$$\frac{3}{4} < c < 1.$$  

Let $u$ be a two-soliton of the quartic gKdV

$$\lim_{t \to -\infty} \| u(t) - Q(\cdot - t) - Q_c(\cdot - ct) \|_{H^1} = 0.$$  

Then $u$ is not a multi-soliton as $t \to +\infty$.

- The strategy is not perturbative and the condition $\frac{3}{4} < c < 1$ seems to be technical. However, considering all $0 < c < 1$ is challenging.
- The proof (by contradiction) does not give any information on the collision process but only says that the collision is inelastic.
- The main idea is to study precisely the (exponential like) asymptotic behavior of $u(t, x)$ for any $t$ but only for large $x$. 
Find a situation where this non-perturbative approach would allow to treat all two-soliton collisions.

Since the restriction comes from weak exponential tails for the quartic gKdV it is natural to address a problem where solitons have algebraic decay.
1. Collision of solitons for the generalized KdV equation

2. Interaction of solitons for the 5D energy critical wave equation
The 5D energy critical nonlinear wave equation

We consider the 5D NLW model

\[ \partial_t^2 u - \Delta u - |u|^4 \frac{4}{3} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^5. \]

Let

\[ W(x) = \left( 1 + \frac{|x|^2}{15} \right)^{-\frac{3}{2}}, \quad \Delta W + W^{\frac{7}{3}} = 0, \quad x \in \mathbb{R}^5. \]

For \( \ell \in \mathbb{R}^5, |\ell| < 1 \), let

\[ W_\ell(x) = W \left( \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell(\ell \cdot x)}{|\ell|^2} + x \right). \]

Then \( u(t, x) = \pm W_\ell(x - \ell t) \) is a solution of 5D NLW.
Inelasticity of two-soliton collisions for 5D NLW

Theorem (YM-Merle ’17)

For $k = 1, 2$, let $\lambda_k > 0$, $y_k \in \mathbb{R}^5$, $\epsilon_k = \pm 1$, $|\ell_k| < 1$, $\ell_1 \neq \ell_2$, and

$$W_k(t, x) = \frac{\epsilon_k}{\lambda_k^2} \mathcal{W}_{\ell_k} \left( \frac{x - \ell_k t - y_k}{\lambda_k} \right).$$

Then there exists a solution $u$ of 5D NLW such that

- **Two-soliton as $t \to -\infty$**

$$\lim_{t \to -\infty} \| \nabla_{t, x} u(t) - \nabla_{t, x} (W_1(t) + W_2(t)) \|_{L^2} = 0$$

- **Dispersion as $t \to +\infty$. Assume that $\epsilon_1 = \epsilon_2$ or $\lambda_1 \neq \lambda_2$. For $A \gg 1$,**

$$\liminf_{t \to +\infty} \| \nabla_x u(t) \|_{L^2(|x| > |t| + A)} \gtrsim A^{-\frac{5}{2}}$$
The result holds for any two-soliton configuration, except the special case $\epsilon_1 \lambda_1 + \epsilon_2 \lambda_2 = 0$, where some cancellation takes place. We also expect the result to be true in this case, but at the cost of more advanced computations.

The exact global behavior of the solution as $t \to +\infty$ is not known: blow-up, dispersion, soliton decomposition, etc.. However, the dispersive behavior is universal. In particular, the solution is not a pure two-soliton.

For 3D and 4D, the existence of such two-solitons is not clear due to strong interactions.

This inelasticity result is related to the soliton resolution conjecture proved in the radial case for 3D NLW [Duyckaerts-Kenig-Merle ’13] and in the general case in 3, 4 and 5D for a subsequence of time by [DKM-Jia ’17].
Strategy of the proof

The overall strategy of the proof follows the one for quartic gKdV, but here it can be carried out for any choice of parameters.

First, we construct a refined approximate solution of the two-soliton problem as $t \sim -\infty$ with an explicit tail for the radial component of the solution at the leading order.

Second, we observe that this tail has a dispersive nature which sends for any positive time some energy at the exterior of large cones. For this, we use the finite speed of propagation and the method of channels of energy of [DKM ’14].
Channels of energy [DKM ’14], [Kenig-Lawrie-Schlag ’15]

Any radial energy solution $U_L$ of the 5D linear wave equation

\[
\begin{cases}
\partial_t^2 U_L - \Delta U_L = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^5 \\
U_L|_{t=0} = U_0 \in \dot{H}^1, \quad \partial_t U_L|_{t=0} = U_1 \in L^2
\end{cases}
\]

satisfies, for some $C > 0$, for any $R > 0$, either

\[
\liminf_{t \to -\infty} \int_{|x| > |t| + R} |\partial_t U_L(t)|^2 + |\nabla U_L(t)|^2 \geq C \| \pi_R \perp (U_0, U_1) \|^2_{(\dot{H}^1 \times L^2)(|x| > R)}
\]

or

\[
\liminf_{t \to +\infty} \int_{|x| > |t| + R} |\partial_t U_L(t)|^2 + |\nabla U_L(t)|^2 \geq C \| \pi_R \perp (U_0, U_1) \|^2_{(\dot{H}^1 \times L^2)(|x| > R)}
\]

where $\pi_R \perp (U_0, U_1)$ denotes the orthogonal projection of $(U_0, U_1)^T$ onto the complement of the plane

\[
\text{span} \left\{ (|x|^{-3}, 0)^T, (0, |x|^{-3})^T \right\} \quad \text{in} \quad (\dot{H}^1 \times L^2)(|x| > R).
\]
Approximate solution with correction terms

Let

\[ W_k = \frac{\epsilon_k}{\lambda_k^{3/2}(t)} W \left( \frac{x - \ell_k t - y_k(t)}{\lambda_k(t)} \right), \quad \vec{W}_k = \left( \begin{array}{c} W_k \\ -\ell_k \cdot \nabla W_k \end{array} \right) \]

We construct \( \vec{W} = \left( \begin{array}{c} W \\ X \end{array} \right) = \sum_{k=1,2} \left( \vec{W}_k + c_k \vec{v}_k \right) \) such that

\[
\begin{align*}
\partial_t W &= X - \sum M_k + O(t^{-4}) \\
\partial_t X &= \Delta W + |W|^4 W + \sum (\ell_k \cdot \nabla) M_k + O(t^{-4})
\end{align*}
\]

where the modulation terms are

\[
M_k = \left( \frac{\dot{\lambda}_k}{\lambda_k} - \frac{a_k}{\lambda_k^{5/2} t^2} \right) \left( \frac{3}{2} W_k + (x - \ell_k t - y_k) \cdot \nabla W_k \right) + (y_k \cdot \nabla) W_k,
\]

Observe that \( (\vec{v}_k)_{k=1,2} \) remove all first order interaction terms of size \( t^{-3} \).
Let $A \gg 1$. At the time $T = -A^{11/12}$, the asymptotic behavior of the approximate solution writes, for $|x| > A$,

$$W(T) = W_1(T) + W_2(T) + \frac{\alpha}{|T||x|^3} + \frac{\beta}{|x|^4} + \text{non-radial + lower order}$$

$$c_1 v_1 + c_2 v_2$$

The two terms $\frac{\alpha}{|T||x|^3}$ and $\frac{\beta}{|x|^4}$ are due to the nonlinear interactions of the two-solitons and they are computed by the Duhamel formula.

Note that the term $\frac{\alpha}{|T||x|^3}$ is not dispersive.

We check that $\beta \neq 0$ if and only if $\epsilon_1 \lambda_1 + \epsilon_2 \lambda_2 \neq 0$.

Under this condition, the term $\frac{\beta}{|x|^4}$ is dispersive.
To compute the tail, a typical model question is to determine the asymptotics of the main orders of the radial part of $v_\ell$, solution of a linear wave equation with source:

$$v_\ell(t) = \int_0^{+\infty} \frac{\sin(s\sqrt{-\Delta})}{\sqrt{-\Delta}} f_\ell(t + s)ds$$

where

$$f_\ell(t, x) = t^{-3}\langle x_\ell \rangle^{-3} + \ell t^{-2} \partial_{x_1} \langle x_\ell \rangle^{-3}$$

and

$$x_\ell = \left( \frac{1}{\sqrt{1 - |\ell|^2}} - 1 \right) \frac{\ell(\ell \cdot (x - \ell t))}{|\ell|^2} + x - \ell t.$$