Small scales and singularity formation in fluid mechanics

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The background

Euler equation 1755. Set on domain $D \subset \mathbb{R}^n$, $n = 2, 3$.

$$
\partial_t u + (u \cdot \nabla) u = \nabla p, \quad \nabla \cdot u = 0, \quad u \cdot n|_{\partial D} = 0, \quad u(x, 0) = u_0(x)
$$

Navier-Stokes equation 1845.

$$
\partial_t u + (u \cdot \nabla) u - \Delta u = \nabla p
$$

Global regularity vs finite time blow up? The story is very different in dimensions two and three. Key quantity: vorticity $\omega = \text{curl} u$.

Euler equation in vorticity form:

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2D Euler equation: history

The 2D Euler equation:

$$\partial_t \omega + (u \cdot \nabla) \omega = 0, \quad u = \nabla^\perp (-\Delta_D)^{-1} \omega.$$ 

Here $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$, $\Delta_D$ is the Dirichlet Laplacian. Trajectories

$$\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x), t), \quad \Phi_0(x) = x.$$ 

Then $\omega(\Phi_t(x), t) = \omega_0(x)$, $\omega(x, t) = \omega_0(\Phi_t^{-1}(x))$. $L^\infty$ norm is conserved!

Theorem (Wolibner; Hölder 1933)

Let $D$ be smooth, compact, $\omega_0 \in C^1(D)$. Then there exists unique solution of the 2D Euler equation $\omega(x, t) \in C^1$. Moreover,

$$\|\nabla \omega(\cdot, t)\|_{L^\infty} \leq (1 + \|\nabla \omega_0\|_{L^\infty}) \exp C\|\omega_0\|_{L^\infty} t.$$
2D Euler equation: is double exp real?

Key ingredient in global regularity proof: **Kato inequality**.
Recall $u = \nabla^\perp (-\Delta_D)^{-1} \omega$, so $\partial_j u_i$ are Riesz transforms of $\omega$. Then

$$
\|\nabla u\|_{L^\infty} \leq C \|\omega\|_{L^\infty} \left( 1 + \log \left( 1 + \frac{\|\nabla \omega\|_{L^\infty}}{\|\omega\|_{L^\infty}} \right) \right)
$$

It is exactly the log term that leads to double exponential upper bound.

But can such fast growth happen?

Yudovich '74: some infinite growth of $\nabla \omega$.

Nadirashvili '91: linear growth of $\|\nabla \omega\|_{L^\infty}$.

Bahouri-Chemin '94. “Singular cross” flow. $\omega$ odd in $x_1$, $x_2$, $\equiv 1$ on $(0, \pi)^2$, periodic.

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u_1(x_1, 0) = cx_1 \log x_1 + O(x_1)!
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2D Euler growth examples

Denisov 2010s: example set on $\mathbb{T}^2$ where $\| \nabla \omega \|_{L^\infty}$ grows superlinearly in time. Also, given $T > 0$, can find initial data leading to double exponential burst of growth over $[0, T]$.

**Theorem (K-Sverak '14)**

Let $D$ be unit disk. There exist $\omega_0 \in C^\infty(D)$ with $\| \nabla \omega_0 \|_{L^\infty} > 1$ such that

$$\| \nabla(\cdot, t) \|_{L^\infty} \geq \| \nabla \omega_0 \|_{L^\infty} \exp \left( c \| \omega_0 \|_{L^\infty} t \right) \text{ for all } t \geq 0.$$ 

Inspired by Luo-Hou numerical experiments for 3D Euler (more later). Growth happens at the boundary!

Xu '16: extension to any regular domain with symmetry axis.

Zlatos '15: exponential growth examples for $\| \nabla^2 \omega \|_{L^\infty}$ on $\mathbb{T}^2$.  

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Main Lemma: Biot-Savart

Assume $\omega_0$ is odd with respect to $x_1$. Analyze
$u = \nabla \perp (-\Delta_D)^{-1} \omega$.

**Lemma (Main Lemma)**

Fix small $\gamma > 0$. For $x \in D_1^{\gamma}$, $|x| \leq \delta$, we have

$$u_1(x) = -\frac{4}{\pi} x_1 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4} \omega(y, t) \, dy + B_1 x_1.$$  

Moreover, if $x \in D_2^{\gamma}$, $|x| \leq \delta$, we have

$$u_2(x) = \frac{4}{\pi} x_2 \int_{Q(x_1,x_2)} \frac{y_1 y_2}{|y|^4} \omega(y, t) \, dy + B_2 x_2.$$  

Here $\|B_{1,2}\|_{L^\infty} \leq C(\gamma) \|\omega_0\|_{L^\infty}$.

$D_+ = \{(x_1, x_2) \in D \mid x_1 \geq 0\}$. Set the origin at the bottom of the disk!
The 2D Euler example

Denote

$$\Omega(x, t) = \frac{4}{\pi} \int_{Q(x_1, x_2)} \frac{y_1 y_2}{|y|^4} \omega(y, t) \, dy.$$ 

Corollary (of Main Lemma)

Exponential growth is easy!

Take $0 \leq \omega_0 \leq 1$ in $D_+$, and $\omega_0(x) = 1$ if $x_1 \geq \delta$.

Then by incompressibility $\Omega(x, t) \geq c \log \delta^{-1}$ if $|x| \leq \delta$, for all times.

If $\delta$ is chosen small enough, Main Lemma gives $u_1(x) \leq -Cx_1$ for all times, all $|x| \leq \delta$. In particular the characteristic along the boundary converges to the origin at an exponential rate.

For double exponential growth, have to control a region where $\omega(x, t) = 1$ approaching the origin. “Comparison principle” helps!
The 3D Euler equation: history

Now consider 3D Euler equation:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \quad u = K_D \ast \omega, \quad \omega(x, 0) = \omega_0.$$ 

Known: local well-posedness in appropriate spaces.

Beale-Kato-Majda '84: at blow up time $T$, must have

$$\int_0^T \|\omega(\cdot, s)\|_{L^\infty} \, ds = \infty.$$ 

Constantin-Fefferman-Majda '96 (later refinements by Hou and collaborators): conditions on vorticity direction sufficient for regularity.

A selection of numerical simulations looking for singular scenarios:

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A selection of numerical simulations looking for singular scenarios:

- Brachet-Meiron-Orszag-Nickel-Morf-Frisch '83
- Grauer-Sideris '91
- Pumir-Siggia '92
- Caflisch '93
- Kerr '93
- E-Shu '94
- Boratav-Pelz '94
- Pelz-Gulak '97
- Ohkitani-Gibbon '00
- Hou-Li '06
- Larios-Petersen-Titi-Wingate '15.

Singularity formation
The 3D Euler equation: the Hou-Luo scenario

Luo-Hou '14: axi-symmetric flow in a cylinder. Denote

\[ D_t = \partial_t + u^r \partial_r + u^z \partial_z. \]

3D Euler equation in cylindrical coordinates:

\[
D_t \left( \frac{\omega^\phi}{r} \right) = \frac{\partial_z (ru^\phi)^2}{r^4}; \quad D_t (ru^\phi) = 0
\]

\[
(u^r, u^z) = (-r^{-1} \partial_z \psi, r^{-1} \partial_r \psi),
\]

\[ L \psi = \frac{\omega^\phi}{r}, \quad L \psi = -\frac{1}{r} \partial_r \left( \frac{1}{r} \partial_r \psi \right) - \frac{1}{r^2} \partial_z^2. \]

Fast growth of \( \omega^\phi \) is observed near a ring of boundary hyperbolic points of the flow.

E. Saw et al '16: extreme dissipation regions in turbulence often feature hyperbolic points.
A good proxy for 3D axi-symmetric Euler equation away from the rotation axis is the 2D inviscid Boussinesq system. 

\[
\begin{align*}
\partial_t \omega + (u \cdot \nabla) \omega &= \partial_{x_1} \theta \\
\partial_t \theta + (u \cdot \nabla) \theta &= 0 \\
u &= \nabla \perp (-\Delta)^{-1} \omega.
\end{align*}
\]

Global regularity for this system is on the list of “Eleven great problems of mathematical hydrodynamics” by Yudovich. 

When \( \theta \) is constant, get the 2D Euler equation. The 2D Euler double exponential growth example happens in similar geometry!
1D models of the Hou-Luo scenario

Main difficulties in analysis of the Boussinesq system (relative to Euler):

- Vorticity may grow, affecting the “error” terms estimates.
- Vorticity is no longer sign definite.

The 1D models of the Hou-Luo scenario.

\[
\begin{align*}
\partial_t \omega + u \partial_x \omega &= \partial_x \theta; \\
\partial_t \theta + u \partial_x \theta &= 0.
\end{align*}
\]

Hou – Luo model: \( u_x = H \omega \); CKY model: \( u(x) = -x \int_x^1 \frac{\omega(y)}{y} \, dy \).

Derivation of the Hou-Luo model is based on boundary layer assumption on structure of vorticity. It is used to close the Biot-Savart law: \( \omega(x_1, x_2, t) = \omega(x_1, t) \chi_{[0,a]}(x_2) \).

The CKY model is “almost local” and its Biot-Savart law is inspired by 2D Euler Main Lemma.
1D models of the Hou-Luo scenario

**Theorem (Choi-K-Yao ’15; Choi-Hou-K-Luo-Sverak-Yao ’16)**

Both models are locally well-posed, but there exist initial data leading to finite time blow up. For the CKY model, \( \int_0^T \| \omega(\cdot,t) \|_{L^\infty} \, dt = \infty \) at blow up time; for Hou-Luo model, \( \int_0^T \| u_x(\cdot,t) \|_{L^\infty} \, dt = \infty \).

Recall trajectories

\[
\frac{d\Phi_t(x)}{dt} = u(\Phi_t(x), t), \quad \Phi_0(x) = x.
\]

If \( \psi(x,t) = -\log \Phi_t(x) \), then \( \partial_t^2 \psi \gtrsim e^{C\psi}! \)

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If \( \psi(x, t) = -\log \Phi_t(x) \), then \( \partial_t^2 \psi \geq e^{c\psi} \).

Other work

**Tao ’16** - Fourier side-inspired models. Inspiration going back to shell models, in particular Katz-Pavlovic ’05. Blow up in the bulk, for 3D “averaged” Navier-Stokes and Euler equations (still conserving energy). Philosophy of self-similar cascade of energy to higher modes, “perfect fluid Turing machine” replicating itself on a sequence of converging times in smaller size and faster velocity.

**Brenner, Hormoz, Pumir ’16** - suggestion of a particular implementation for the real equations: vortex filaments $\mapsto$ vortex sheets $\mapsto$ smaller scale vortex filaments.

**Elgindi, Jeong ’17** - blow up for singular solutions in a sector (less than $\pi$). Vorticity is degree zero homogeneous in radial variable (and so only bounded, not continuous at the origin), odd (and so the flow is hyperbolic near the origin). Finite time blow up in a sense that gradients of the velocity and density tend to infinity. Elgindi-Jeong are able to cut off solutions at infinity to make energy finite.
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The modified SQG equation: history

One of the key difficulties in the Hou-Luo scenario for the actual 2D Boussinesq system: growth of $\omega$, which makes error terms in the Main Lemma uncontrollable. Heuristic estimates using numerical data suggest that “helpful” for blow up and “opposing” terms are of the same order.

A similar story in a different setting: the modified SQG patches.

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \quad u = \nabla^\perp(-\Delta)^{-1+\alpha}\omega, \quad 0 \leq \alpha \leq 1/2.$$  

The value $\alpha = 0$ - 2D Euler, $\alpha = 1/2$ - SQG.

Constantin-Majda-Tabak '94; Cordoba '98, Cordoba-Fefferman '02.

A special class of initial data: patches, $\omega_0 = \sum_{j=1}^{N} \theta_j \chi_{\Omega_j}(x)$.

An SQG patch in nature
The regularity question in patch context refers to regularity of the domain boundaries $\partial \Omega_j(t)$ and lack of touching by different patches.

2D Euler: existence and uniqueness by Yudovich '63. Global well-posedness in $\mathbb{R}^2$ or $\mathbb{T}^2$: Chemin '93, Bertozzi-Constantin '93, Serfati '94.

With boundary - limited results, Morgulis '97, Depauw '99, Dutrifoy '03. Related results: Danchin '00.

Patches for modified SQG: local regularity Rodrigo '05, Gancedo '08. Cordoba, Fontelos, Mancho, Rodrigo '05: numerical evidence of corner formation and patch touching.

Let us consider patches in half-plane $D$, with $u \cdot n|_{\partial D} = 0$.

**Theorem (K-Ryzhik-Yao-Zlatos '16)**

Let $\alpha = 0$ (2D Euler), half-plane setting. If $\omega_0(x)$ is $C^{1,\gamma}$ patch for some $\gamma > 0$, then there exists a unique global $C^{1,\gamma}$ patch solution $\omega(x,t)$. 

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The Euler and SQG patches - history

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Alexander Kiselev (Duke University) Singularity formation
Theorem (K-Yao-Zlatos ’16)

If $0 < \alpha < 1/24$ the following holds. If $\omega_0$ is an $H^3$ patch, then there exists a unique local $H^3$ patch solution.

Why local well-posedness for patches with boundary is hard?
If $\alpha > 0$, $u \in C^{1-2\alpha}$ only, not Lipschitz.

However: can show normal to patch component of $u$ has better regularity.

$$u_2(x_1, x_2) = \int_{\Omega_j} \frac{x_1 - y_1}{|x - y|^{2+2\alpha}} \omega(y) \, dy$$

Due to the no-penetration boundary condition, a patch touching the boundary is equivalent to reflected patch touching the original. Estimates near the point of touching are hard!
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The modified SQG patches: finite time blow up

Theorem (K-Ryzhik-Yao-Zlatos ’16)

If \( 0 < \alpha < 1/24 \), there exist initial data \( \omega_0 \in H^3 \) such that the corresponding patch solution blows up in finite time.

The initial data: odd, two patches.

Patch structure:

On the right, \( K(0) \) is the initial barrier and

\[
K(t) = \{ x : x_2 \leq x_1, X(t) \leq x_1 \leq 1 \}, \quad X' = -\frac{1}{50} X^{1-2\alpha}, \quad X(0) = 3\epsilon.
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The modified SQG patches: finite time blow up

Note that $K(t)$ arrives at $x_1 = 0$ in time $\tau \sim \epsilon^{2\alpha}$.

Plan: show that $K(t) \subset \Omega(t)$ while the patch $\Omega(t)$ remains regular.

To the contrary, suppose barrier and patch touch. In this case, one can identify regions in the plane which contribute to “good” (for blow up) and “bad” parts to $u$ at touch point.

After cancellations, $u_1^{\text{good}}$ and $u_1^{\text{bad}}$ can be bounded by integrals of $\frac{x_2 - y_2}{|x - y|^{2+2\alpha}}$ over $G$ and $B$ respectively.

For small $\alpha$, kernel is long range and $G$ wins. For $\alpha$ closer to $1/2$, $B$ wins!

\[
\begin{align*}
  u_1^{\text{bad}}(x) &\leq \frac{1}{\alpha} \left( \frac{1}{1 - 2\alpha} - 2^{-\alpha} \right) x_1^{1-2\alpha} \\
  u_1^{\text{good}}(x) &\leq -\frac{1}{6 \cdot 20^\alpha} x_1^{1-2\alpha}.
\end{align*}
\]