Resonances in hyperbolic dynamics

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Wave scattering and decay

Waves scattered by obstacles $\emptyset \subset \mathbb{R}^d$:

$$(\partial_t^2 - \Delta_\Omega) u = 0, \quad u(0) = u_0, \quad \partial_t(0) = u_1,$$

$\Delta_\Omega$ Dirichlet Laplacian on $\Omega = \mathbb{R}^d \setminus \emptyset$ (ass. $\partial\Omega$ smooth, $\Omega$ connected).

Focus on a transient regime: $u_0, \ u_1 \in C_\infty^\infty(\Omega)$. Both:

- the local energy $\mathcal{E}_R(u(t)) = \frac{1}{2} \int_{B(0,R)} (|\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2) \, dx$

- the “correlation” $\langle f, u(t) \rangle_{\mathcal{D}, \mathcal{D}'}$, for a given $f \in C_\infty^\infty(\Omega)$,

will decay to zero as $t \to \infty$. 
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How fast is the decay? How does it depend on the obstacles?
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*How fast is the decay? How does it depend on the obstacles?*

- Large time asymptotics $\sim$ spectral problem.
To analyze the spectral properties of $\Delta_{\Omega}$, a central object is the \textit{resolvent operator} $R_{\Omega}(\lambda) = (-\Delta_{\Omega} - \lambda^2)^{-1}$.

- $-\Delta_{\Omega}$ selfadjoint on $L^2(\Omega)$, so $R_{\Omega}(\lambda)$ can be defined in $\{\text{Im} \lambda > 0\}$.
- $-\Delta_{\Omega}$ has continuous spectrum on $\mathbb{R}_+$, so $R_{\Omega}(\lambda)$ explodes as $\text{Im} \lambda \downarrow 0$.

Yet, $\forall \chi \in C^\infty_c(\mathbb{R}^d)$, the truncated resolvent $\chi R_{\Omega}(\lambda) \chi : L^2 \to L^2$ can be \textit{meromorphically continued} into $\{\text{Im} \lambda < 0\}$.

$\sim$ Resonances $\{\lambda_j\} = \text{discrete poles of finite multiplicities}$.
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\sim Resonances $\{\lambda_j\} = \text{discrete poles of finite multiplicities}.$

- If $\{0 \geq \text{Im} \lambda > -A\}$ contains finitely many resonances (resonance gap), one hopes to expand the correlation (with $u(0) = 0$, $\partial_t u(0) = u_1$) as:

$$
\langle f, u(t) \rangle = \sum_{\text{Im} \lambda_j > -A} e^{-it\lambda_j} \langle f, v_j \rangle \langle v_j, u_1 \rangle + O(e^{-tA}), \quad t \to \infty.
$$

$v_j \in C^\infty(\Omega)$ the \textit{resonant state} associated with $\lambda_j$, of \textit{lifetime} $|\text{Im} \lambda_j|^{-1}$. 
Resolvent and Resonances (odd dimension)

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*How does the shape of $\Omega$ influence the distribution of resonances?*
High frequency / semiclassical regime

- Half-wave equation $i\partial_t u = \sqrt{-\Delta_\Omega} u \equiv \text{semiclassical Schrödinger equation}$
  $i\hbar \partial_t u = P_h u$, with quantum Hamiltonian $P_h = \hbar \sqrt{-\Delta_\Omega} (= |\hbar D_x|)$.
  High-\(\lambda\) regime $\equiv$ semiclassical regime $\hbar = \lambda^{-1} \ll 1$, with $P_h \sim 1$.

$\implies$ **Semiclassical analysis**: the quantum dynamics is guided by the classical dynamics on the phase space $T^*\Omega = \{\rho = (x, \xi)\}$ generated by the Hamiltonian $p(x, \xi) = |\xi|$: the geodesic flow $\Phi^t : T^*\Omega \to T^*\Omega$.

Ex: wavepacket $u_{\rho_0}(x) = a \left( \frac{x-x_0}{\sqrt{\hbar}} \right) e^{i \frac{\xi_0 \cdot x}{\hbar}}$ localized in the $\sqrt{\hbar}$-nbhd of $x_0$; its $\hbar$-Fourier transform is localized in the $\sqrt{\hbar}$-nbhd of $\xi_0$.

$\equiv u_{\rho_0}$ is microlocalized in the $\sqrt{\hbar}$-nbhd of $\rho_0 = (x_0, \xi_0) \in T^*X$.

$\implies u(t) = e^{-itP_h/\hbar} u_{\rho_0}$ wavepacket microlocalized near $\rho_t = \Phi^t(\rho_0)$.

Its microscopic shape follows $d\Phi^t(\rho_0)$: dispersion.
Our analysis generalizes to potential scattering on $\mathbb{R}^d$: Schrödinger equation

$$i\hbar \partial_t u = P_h u, \quad P_h = -\hbar^2 \Delta + V(x),$$

with potential $V \in C^\infty_c(\mathbb{R}^d)$.

The center of the wavepacket $u(t) = e^{-itP_h/\hbar}u_{\rho_0}$ follows the Hamiltonian flow generated by $p(x, \xi) = |\xi|^2 + V(x)$ on $T^*\mathbb{R}^d$.

Fixing an energy $E > 0$, the resonances $\{z_j(h)\}$ (poles of $\chi(P_h - z)^{-1}\chi$) are studied in a neighbourhood $E + \mathcal{O}(\hbar)$, in the limit $\hbar \searrow 0$. 
Scattering and resonances

Semiclassical regime

Chaotic trapped set

Normally hyperb. trapped set

High frequency / semiclassical regime (2)

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Let us go back to the case of hard obstacles.
Distribution of resonances vs. classical trapped set

We need to understand the long time asymptotics of the (broken) geodesic flow $\Phi^t$ on $p^{-1}(1) = S^*\Omega$: a question of classical dynamics.

Focus on the interaction region $B(0, R)$:

- most trajectories spend a finite time in the interaction region before escaping to $|x| \to \infty$
- there may exist trapped trajectories: trapped set $K \overset{\text{def}}{=} \{ \rho \in S^*\Omega, \Phi^t(\rho) \not\to \infty, t \to \pm\infty \}$
- $K$ compact, flow-invariant subset of $S^*\Omega$.

Main idea: the distribution of the resonances near the real axis depends on the geometric and dynamical properties of $K$. 
Distribution of resonances vs. trapped set (2)

Case 0: $K$ empty (ex: convex obstacle).

**No** resonances in $\{| \text{Im } \lambda | \leq C \log \text{Re } \lambda \}$.

[Lax-Phillips, Vainberg, Morawetz, Melrose, Ralston, Strauss]

Case 1: $K$ contains an elliptic (=stable) periodic orbit.

No dispersion $\implies$ one can construct quasimodes

$\| (\Delta_\Omega + \lambda^2) v_\lambda \| = O(\lambda^{-\infty})$ microlocalized on $K$, and deduce the existence of nearby resonances with $| \text{Im } \lambda_j | = O(\lambda^{-\infty})$

[Ralston, Lazutkin, Popov, Vodev, Stefanov, Tang-Zworski]
Distribution of resonances for a single hyperbolic orbit

Case 2: 2 convex obstacles (on $\mathbb{R}^2$) [IKAWA, GÉRARD, GÉRARD-SJÖSTRAND].

$K =$ single hyperbolic periodic orbit $\gamma$, Lyapounov exponent $\nu_\gamma > 0$.

- Hyperbolic dispersion: $|\langle u_{\rho_\gamma}, e^{-itP_h/h} u_{\rho_\gamma} \rangle| \leq C e^{-t\nu_\gamma / 2}, \forall t > 0$

  $\sim$ Asymptotic resonance gap $-\nu_\gamma / 2$.

A more precise analysis (Quantum Birkhoff normal form) shows that resonances near $\lambda \gg 1$ make up a deformed half-lattice.
A first generalization of the 1-orbit case: we enter the realm of *quantum chaos*
Chaotic trapped set

**Case 3:** \( N \geq 3 \) convex obstacles on \( \mathbb{R}^d \) (far enough from each other) 
\( K \) a fractal hyperbolic repeller, \( \Phi^t \upharpoonright K \) Axiom A flow. (Plot © Leon Poon).

**Hyperbolicity:** (un)stable directions transversely to each trajectory:

at each point \( \rho \in K \), \( T_\rho S^* \Omega = E^u_\rho \oplus E^s_\rho \oplus V(\rho) \),
\[ \forall t > 0, \quad \| d\Phi^t \upharpoonright E^s_\rho \| \leq C e^{-\nu t}, \quad \| d\Phi^{-t} \upharpoonright E^u_\rho \| \leq C e^{-\nu t}. \]
Quantitatively: unstable Jacobian \( J^u_t(\rho) = | \det( d\Phi^t \upharpoonright E^u_\rho ) | > 1. \)
Second ingredient of chaos: complexity

**Complexity**: infinitely many trapped orbits (can be indexed by infinite nonrepeating sequences . . . 1212323123 . . .).

**Topological pressures** measure the interplay between **complexity** and **hyperbolicity**:

\[
P(s) \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \log \sum_{\gamma: T \leq T_\gamma \leq T+1} (J_\gamma^u)^{-s}
\]

We will be interested in the sign of \(P(s = 1/2)\).

- if \(P(1/2) < 0\), we’ll say that \(K\) is **thin**
  (in \(d = 2\), \(P(1/2) < 0 \iff \dim_H K < 1 + 2 \times 1/2\)).
Resonance gap for thin chaotic trapped sets

**Theorem ([IKAWA,GASPARD-RICE,BURQ,N-ZWORSKI])**

Assume the trapped set $K$ is a hyperbolic repeller with $\mathcal{P}(1/2) < 0$. Then, $\exists C_0, \forall \epsilon > 0$, the strip \( \{ \text{Re} \lambda \geq C_0, \ 0 \geq \text{Im} \lambda \geq \mathcal{P}(1/2) + \epsilon \} \) contains no resonances. In this strip \( \| \chi R_{\Omega}(\lambda) \chi \|_{L^2 \to L^2} \leq C \lambda^N \).
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"Proof": we want to show that any initial state $u_0 \in C^\infty_c(B(0, R))$ decays faster than $e^{t\mathcal{P}(1/2)}$. 
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1. **Hyperbolicity** $\implies$ wavepackets microlocalized on $K$ disperse exponentially fast, thus “leak out” of $B(0, R)$ after a time $C \log \lambda$.
2. **Interferences** between wavepackets on nearby trajectories may reduce the global leakage from $K$.
3. **Sum** the contributions of many trajectories. If $\mathcal{P}(1/2) < 0$, dispersion beats interferences $\Rightarrow$ any state near frequency $\lambda$ will leak out at rate $\|u(t)\|_{L^2(B(0,R))} \leq C e^{(t-C \log \lambda)\mathcal{P}(1/2)}$. 
Dynamical consequences of the resonance gap

- \((d \geq 3\) odd): Exponential decay of the local energy
  \[ \forall s > 0, \exists \alpha_s, C > 0, \text{ for any } u_0 \in H^s(\Omega) \text{ supported in } B(0, R), \]
  \[ \mathcal{E}_R(u(t)) \leq Ce^{-\alpha_s t} \| u_1 \|_{H^s}^2, \quad \forall t > 0. \]
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- Local smoothing for the (nonsemiclassical) Schrödinger equation
  
  \[ \forall T > 0, \forall \epsilon > 0, \exists C > 0, \forall u_0 \in L^2, \]
  
  \[ \int_0^T \| \chi e^{it\Delta} u_0 \|_{H^{1/2-\epsilon}} \leq C \| u_0 \|_{L^2} \]
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Strichartz estimates with no loss [BURQ-GUILLARMOU-HASSELL]
Other models of chaotic scattering

- obstacles \( \sim \) "bumpy potential", \( P_h = -h^2 \Delta + V(x) \).
- \( K_E \) depends on the energy \( E > 0 \).
Other models of chaotic scattering

- obstacles \( \leadsto \) "bumpy potential", \( P_h = -h^2 \Delta + V(x) \).
  \( K_E \) depends on the energy \( E > 0 \).
  \implies \text{Resonance gap if } \mathcal{P}(1/2, K_E) < 0 \text{ [N-ZWORSKI]}.

- \( X = \Gamma \backslash \mathbb{H}^2 \) hyperbolic surface of infinite area, \( \Delta_X \) Laplace-Beltrami op.

The resonances \( \lambda_j^2 = s_j(1 - s_j) \), with \( \{s_j\} \) the zeros of

\[
Z_{\text{Selberg}}(s) = \prod_{\gamma} \prod_{k \geq 0} \left( 1 - e^{-(s+k)T_{\gamma}} \right), \quad (\gamma = \text{primitive closed geodesics}).
\]

[PIAFFTERSON, SUILLIVAN]: \( \text{Re } s_j - 1/2 \leq \min(0, \delta_{\Gamma} - 1/2) \), with \( \delta_{\Gamma} \in (0, 1) \)
the dimension of the limit set \( \Lambda_{\Gamma} \). This is exactly our pressure bound.
How sharp is the pressure bound?

[Bowen-Series]: $Z_{Selberg}(s) = \det(1 - \mathcal{L}_s)$, with $(\mathcal{L}_s)$ family of transfer operators associated with an expanding map on $\partial \mathbb{H}^2$.

- [Naud] (spectral anal. of $\mathcal{L}_s$, adapting [Dolgopyat]) proves partial cancellations when summing over trajectories $\Longrightarrow \text{Re } s_j \leq \delta - \epsilon_1$.

Conjecture [Jakobson-Naud] $\text{Re } s_j \leq \delta/2 + o(1)$

$\iff \text{Im } \lambda_j \leq -\gamma_{cl}/2 + o(1)$.

[dyatlov-zahl, Bourgain-Dyatlov, Jin-Zhang] Resonance gap for any value of $\delta \in (0,1)$: $\exists \epsilon(\delta) > 0$, $\text{Re } s_j \leq \frac{1}{2} - \epsilon(\delta)$ for $\text{Im } s_j \geq C$.

Use semiclassical methods combined with a Fractal Uncertainty Principle.
Let us finally consider another generalization of the 1-periodic orbit case.
Normally hyperbolic trapped set

\[ P_h = p(x, hD_x) \text{ on } \mathbb{R}^d, \text{ resonances } z_j(h) = E + \mathcal{O}(h) \text{ for some fixed } E \in \mathbb{R}. \]

**Case 4:** \( K_E \) **normally hyperbolic.**

\[ K^\delta_E = \bigcup_{|E' - E| \leq \delta} K_{E'} \] compact, \( 2d || \)-dimensional smooth symplectic submanifold of \( T^*\mathbb{R}^d \), with hyperbolic transverse dynamics:

\[ \forall \rho \in K_E^\delta, \quad T_\rho (T^*\mathbb{R}^d) = T_\rho K_E^\delta \oplus (T_\rho K_E^\delta)^\perp, \quad (T_\rho K_E^\delta)^\perp = E^u_\rho \oplus E^s_\rho \]

\( E^u_\rho / E^s_\rho \) transverse (un)stable subspaces of dimension \( d - d || \).

**Examples:** quantum dynamics of chemical reactions [GOUSSEV-et-al].
Wave propagation on Kerr black holes [WUNSCH-ZWORSKI,DYATLOV].
Normally hyperbolic trapped set: resonance gap

Quantitatively: *transverse* unstable Jacobian
\[
J_{t}^{u,\perp}(\rho) \overset{\text{def}}{=} |\det d\Phi_{t}|_{E^{u}_{\rho}} \geq C^{-1}e^{t\Lambda_{\perp}} \quad \text{for all } \rho \in K_{\delta}, \quad t \geq 0.
\]

**Theorem ([GERARD-SJÖSTRAND, WUNSCH-ZWORSKI, N-ZWORSKI, DYATLOV])**

*For \( h > 0 \) small enough, all resonances of \( P_{h} \) such that \( |\Re z_{j}(h) - E| \leq \delta \) satisfy \( \Im z_{j}(h) \leq -h\Lambda_{\perp}/2 + o(h) \).*

*Application*: Decay of (quasi-)linear waves on Kerr-de Sitter black holes \( \rightsquigarrow \) stability of Kerr de Sitter BH [DYATLOV, HINTZ-VASY].

"Proof": hyperbolic dispersion of wavepackets *transversely to \( K \).*

For any \( \rho, \rho_{0} \in K \),
\[
|\langle u_{\rho}, e^{-itP_{h}/h}u_{\rho_{0}} \rangle| \leq C e^{-t\Lambda_{\perp}/2}
\]
One surprising application: mixing of contact Anosov flows

$X$ compact, contact manifold, s.t. the Reeb flow $\varphi^t$ is uniformly hyperbolic (Anosov).

Ex: $Y$ compact Riemannian, negative curvature, $X = S^*Y$, $\varphi^t =$ geodesic flow.

$\implies \varphi^t$ ergodic and mixing (decay of correlations):

$$\forall f, u \in C^\infty(X), \quad C_{fu}(t) \overset{\text{def}}{=} \int f \ u \circ \varphi^t - \int f \int u \xrightarrow{t \to \infty} 0.$$  

Decay governed by Ruelle-Pollicott resonances $\lambda_j \in \{\text{Im} \lambda < 0\}$.

[DOGOPYAT,LIVERANI]: $\exists$ resonance gap $\implies$ exponential mixing.

(study the transfer operator $\mathcal{L}^t u = u \circ \varphi^t$ on anisotropic Banach/Sobolev spaces)

[TSUJII]: if $\tilde{J}_t^u(\rho) \geq C^{-1} e^{\tilde{\Lambda}}$, the asymptotic resonance gap $\geq \tilde{\Lambda}/2$.

$\implies \forall f, u \in C^\infty(X), \quad C_{fu}(t) = \sum_{\text{Im} \lambda_j \geq -\tilde{\Lambda}/2} e^{-i\lambda_j t} c_j(f,u) + O_{f,u}(e^{-\frac{\tilde{\Lambda}}{2} t}), \quad t \to \infty.$
Contact Anosov flow $\equiv$ semiclassical scattering

[FAURE-SJÖSTRAND, DYATLOV-ZWORSKI, FAURE-TSUJII, N-ZWORSKI]: this Anosov flow is actually similar to a quantum scattering problem.

- lift $\varphi^t : X \to X$ to the Hamiltonian flow $\Phi^t : T^* X \to T^* X$ generated by $p(x, \xi) = \langle \xi, v(x) \rangle$.

- the transfer operator $\mathcal{L}^t$ is equal to the quantum propagator $e^{-itP_h/h}$ for $P_h = \langle hD_x, v(x) \rangle \implies C_{fu}(t) \equiv$ quantum correlation.
  $\implies$ resonances of $P_h \equiv$ Ruelle-Pollicott resonances: $z_j(h) = h\lambda_j$.

- $\Phi^t \simeq$ scattering flow ($p^{-1}(E)$ infinite in the momentum direction).
  $K_E$ is normally hyperbolic ($E^u/s = \text{the lifts of } \tilde{E}^u/s$ on $(T_p K_{E})_\perp$), $\tilde{\Lambda} = \Lambda_\perp$.
  $\leadsto$ our resonance gap $\text{Im } z_j \leq -h\Lambda_\perp/2$ recovers Tsujii's one.
A few other developments

- Fractal hyperbolic trapped set:
  - *Count* resonances in strips beyond the gap: *Fractal Weyl upper bounds*. Lower bounds are difficult to obtain. [SJÖSTRAND, GUILLOPÉ-ZWORSKI, SJÖSTRAND-ZWORSKI, JAKOBSON-NAUD, ...]
  - Structure of the resonant modes $v_j(x)$? Microlocalized along the unstable manifold of $K$ [BONY-MICHEL, KEATING et al., N-ZWORSKI]

- Normally Hyperbolic trapped set:
  - Assuming *pinching* conditions for the transverse hyperbolicity, resonances live in distinct strips. [FAURE-TSUJI, DYATLOV]

\[
\begin{align*}
0 & \quad E-\delta & \quad E & \quad E+\delta \\
\frac{h\Lambda}{\lambda} & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]

- *Noncontact* Anosov flow: the trapped set becomes a *rough* manifold [FAURE-TSUJI]. Resonance gap?
A few other developments

- Fractal hyperbolic trapped set:
  - *Count* resonances in strips beyond the gap: *Fractal Weyl upper bounds*. Lower bounds are difficult to obtain. 
    [SJÖSTRAND, GUILLOPÉ-ZWORSKI, SJÖSTRAND-ZWORSKI, JAKOBSON-NAUD, ...]
  - Structure of the resonant modes $v_j(x)$? Microlocalized along the unstable manifold of $K$ [BONY-MICHEL, KEATING et al., N-ZWORSKI]

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\[
\begin{align*}
0 & \quad E-\delta & \quad E & \quad E+\delta \\
\frac{h\Lambda^2}{\sqrt{\epsilon}} & \quad \cdots & \quad \cdots & \quad \cdots
\end{align*}
\]

- Noncontact Anosov flow: the trapped set becomes a *rough* manifold [FAURE-TSUJII]. Resonance gap?

Thank you for your attention