Analytic study of hyperbolic flows and applications

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August 2, 2018
Introduction

Applications of analytic methods for hyperbolic flows to study:

1) **Inverse problems**: lens rigidity problem
2) **Riemannian geometry**: marked length spectrum rigidity
3) **Dynamical systems**: meromorphic extension of Ruelle zeta function
4) **Topological invariants in dynamical systems**: Reidemeister torsion and Ruelle function
Boundary rigidity problem (Michel conjecture)

- \((M, g)\): smooth compact manifold with \(\partial M\) strictly convex
- \(d_g : M \times M \rightarrow \mathbb{R}^+\) the Riemannian distance
- \(\beta_g := d_g|_{\partial M \times \partial M}\) the restriction to \(\partial M\)

Boundary rigidity pb: does \(\beta_g\) determine \(g\) up to isometries fixing \(\partial M\)?
Lens rigidity problem

- \((M, g)\): smooth compact manifold with \(\partial M\) strictly convex
- \(SM := \{(x, v) \in TM; g_x(v, v) = 1\}\)
- \(\varphi_t : SM \rightarrow SM\) geodesic flow
- for \((x, v) \in \partial SM\), let \(\ell_g(x, v) := \) length of geodesic \(\gamma_{(x, v)}\)
- for \((x, v) \in \partial SM\), let \(S_g(x, v) := \varphi_{\ell_g(x,v)}(x, v)\) scattering map

Lens rigidity prb: does \((\ell_g, S_g)\) determine \(g\) up to isometries fixing \(\partial M\)?
The linearised operator - X ray transform

We linearise the non-linear map $g \mapsto \beta_g$:

- $\mathcal{G}$ = set of geodesics $\gamma$ with endpoints on $\partial M$
- Linearised operator: called X-ray transform on 2-tensors:

$$I_2 : C^0(M; S^2 T^* M) \to L^\infty_{\text{loc}}(\mathcal{G}),$$

$$I_2 f(\gamma) := \int_{\gamma} f = \int_0^{\ell_g(\gamma)} f_{\gamma}(t) (\dot{\gamma}(t), \dot{\gamma}(t)) dt$$

Linearised pb:

- s-injectivity: $\ker I_2 = \{ \mathcal{L}_V g; V \in C^1(M; TM), V|_{\partial M} = 0 \}$?
- Stability estimates: $\|I_2 f\| \geq C\|f\|$ for $f \perp \ker I_2$?
Positive results

- \textbf{dim 2:} Otal ('90), Croke ('90) if $K_g \leq 0$ & simply connected.
  Pestov-Uhlmann ('03) if no conjugate points & simply connected (\textit{simple metrics}).

- \textbf{dim }n > 2: Stefanov-Uhlmann-Vasy ('17) if strictly convex foliation. Satisfied if $(M, g)=$ topological ball and $K_g \leq 0$

Our contribution

**Theorem (G ’17)**

*On negatively curved surfaces with \( \partial M \) convex, \( S_g \) determines \((M, g)\) up to conformal diffeomorphisms fixing \( \partial M \). Moreover \( I_2 \) is \( s \)-injective with stability estimates.*

**Remark:** First general result for non simply connected manifolds.

**Main difficulty:** some geodesics are trapped.
Marked length spectrum rigidity

- \((M, g)\) closed Riemannian manifold with \(K_g < 0\)
- \(C := \text{set of free homotopy classes on } M\)
- each \(c \in C\) contains a unique closed geodesic \(\gamma_c\)
- marked length spectrum:

\[
L_g : C \to \mathbb{R}^+, \quad L_g(c) := \ell_g(\gamma_c)
\]

Conjecture (Burns-Katok ’85): \(L_g = L_g'\) implies \(g\) isometric to \(g'\).
The linearised operator - X ray transform

- $\mathcal{G} =$ set of closed geodesics $\gamma$ on $M \simeq \mathbb{C}$
- Linearisation at $g$: the X-ray transform on 2-tensors

$$I_2 : C^0(M; S^2 T^* M) \to L^\infty(\mathcal{G}),$$

$$I_2 f(\gamma) := \frac{1}{\ell_g(\gamma)} \int_0^{\ell_g(\gamma)} f_\gamma(t)(\dot{\gamma}(t), \dot{\gamma}(t))dt$$

Linearised prb:

- s-injectivity ker $I_2 = \{\mathcal{L}_V g; V \in C^1(M; TM)\}$ ?
- Stability estimates: $\|I_2 f\| \geq C\|f\|$ for $f \perp \text{ker} I_2$?
Positive results

Non-linear problem:
- dim 2: Otal ’90, Croke ’90
- dim $n > 2$ and $g$ is conformal to $g'$: Katok ’88
- dim $n > 2$ when $g$ is locally symmetric and $K_g < 0$, Besson-Courtois-Gallot ’95, Hamenstädt ’99

Linearised problem:
- s-injectivity of $l_2$ when $K_g < 0$: Guillemin-Kazhdan ’80, Croke-Sharafutdinov ’98
- dim 2: s-injectivity of $l_2$ when $g$ has Anosov geodesic flow: Paternain-Salo-Uhlmann ’14
Our contributions

Theorem (G-Lefeuvre ’18)

Let \((M, g)\) be either

- a closed surface with Anosov geodesic flow, or
- a closed manifold of \(\text{dim } n > 2\) with \(K_g \leq 0\) and Anosov geodesic flow.

There is a \(C^k\) neighborhood \(U\) of \(g\) such that if \(g' \in U\) and \(L_g = L_{g'}\), then \(g\) is isometric to \(g'\).

Remark: the proof gives

- stability estimates quantifying how close are isometry classes of \(g, g'\) in terms of \(L_g/L_{g'}\)
- Hölder stability estimates for \(I_2\): \(\exists C > 0, \forall f \perp \ker I_2, \exists C \beta \geq C\|f\|_{H^{-1-\varepsilon}}, \alpha \sim 1/2\)
Using compactness results of Hamilton, we get the finiteness statement

**Corollary (G-Lefeuvre ’18)**

\[ \forall B \subset C^k \text{ bounded, } \exists \text{ at most finitely many isometry classes of negatively curved manifolds with same marked length spectrum and curvature in } B \text{ if } k \text{ is large enough.} \]

**Remark:** first general results in dim \( n > 2 \).

**Main difficulty:** pass from linear to non-linear problem
Axiom A flows

- $\mathcal{M}$ a smooth compact manifold with or without boundary
- $X$ a smooth non-vanishing vector field on $\mathcal{M}$, with flow $\varphi_t$, such that $\partial \mathcal{M}$ is strictly convex for the flow lines of $X$ (or $\partial \mathcal{M} = \emptyset$).
- $K := \bigcap_{t \in \mathbb{R}} \varphi_t(\mathcal{M}^\circ)$ the trapped set, is closed flow-invariant, contains the closed orbits ($K = \mathcal{M}$ if $\partial \mathcal{M} = \emptyset$)
- Assume $K$ is hyperbolic for $\varphi_t$: i.e. flow-invariant splitting
  \[ \exists \nu > 0, \quad T_K \mathcal{M} = \mathbb{R}X \oplus E_s \oplus E_u \]
  \[ \| d\varphi_t \|_{E_s} \leq Ce^{-\nu t}, \quad \forall t \gg 1, \quad \| d\varphi_t \|_{E_u} \leq Ce^{-\nu |t|}, \quad \forall t \ll -1 \]
- if $\partial \mathcal{M} = \emptyset$, the flow is said Anosov
**Examples:** geodesic flow on $\mathcal{M} = SM$ if $(M, g)$ has negative curvature and either $\partial M$ strictly convex or empty.
Ruelle zeta function: for $\Re(s) \gg h_{\text{top}}$

$$\zeta(s) := \prod_{\gamma \in P} \left(1 - e^{-s\ell(\gamma)}\right)$$

with $P$ the set of prime closed orbits, $\ell(\gamma) =$ period of $\gamma$.

**Question (Smale '67):** does $\zeta(s)$ extend meromorphically to $s \in \mathbb{C}$?
Results

- Ruelle ('76), Rugh ('95), Fried ('96): $\zeta(s)$ extends meromorphically to $s \in \mathbb{C}$ if the flow is real analytic.
- Giuletti-Liverani-Pollicott ('13), Dyatlov-Zworski ('16): $\zeta(s)$ extends meromorphically to $\mathbb{C}$ in the smooth Anosov case.
- Faure-Tsujii ('15): related result for semiclassical zeta function.

**Theorem (Dyatlov-G ’16)**

For smooth Axiom A flows, $\zeta(s)$ extends meromorphically to $\mathbb{C}$. 
Fried conjecture

- $M$ closed and $\varphi_t$ Anosov flow
- $\rho : \pi_1(M) \to U(N)$ a unitary representation
- twisted Ruelle zeta function: for $\text{Re}(s) \gg 1$

$$\zeta_{\rho}(s) := \prod_{\gamma \in \mathcal{P}} \det(1 - \rho(\gamma)e^{-s\ell(\gamma)})$$

Conjecture (Fried '84): the order of $\zeta_{\rho}$ at $s = 0$ is topological. If $\rho$ is acyclic, then $|\zeta_{\rho}(0)|$ (or its inverse) is the Reidemeister torsion.
Positive results

- Fried ('86), Moscovici-Stanton ('91)/Shen ('16): true for geodesic flow of hyperbolic manifolds and locally symmetric spaces.
- Sanchez-Morgado ('96): if \( \dim(M) = 3 \), true for volume preserving analytic Anosov flow when \( \rho \) acyclic and if \( \exists \gamma \in \mathcal{P}, \ker(\rho(\gamma) - \text{Id}) = 0. \)
- Dyatlov-Zworski ('17): if \( \dim(M) = 3 \), \( \rho = \text{Id} \) and contact Anosov flow,
  \[
  \zeta_\rho(s) = s^{b_1(M) - 2} A(s), \quad A(0) \neq 0.
  \]
Our contribution

Theorem (Dang-G-Rivière-Shen ’18)

For $\rho$ acyclic, Fried conjecture holds true in the following cases:

1) $\dim \mathcal{M} = 3$, smooth volume preserving Anosov flow s.t. $\exists \gamma \in \mathcal{P}$, $\ker(\rho(\gamma) - \Id) = 0$.

2) $\dim \mathcal{M} = 5$, for smooth flows close to geodesic flows of 3-dim hyperbolic manifolds.

Remark: first positive results of Fried conjecture for non-analytic Anosov flows, and first results for non-locally symmetric cases in $\dim \mathcal{M} > 3$
Analytic methods for the previous problems

- $\mathcal{M}$: smooth compact manifold with boundary
- $X$: smooth vector field on $\mathcal{M}$ with flow $\varphi_t$
- $\partial \mathcal{M}$ strictly convex for the flow lines of $X$ (or $\partial \mathcal{M} = \emptyset$)

\[ \partial_0 \mathcal{M} = \{ y \in \partial \mathcal{M}; X(y) \text{ tangent to } \partial \mathcal{M} \} \]
\[ \partial_- \mathcal{M} = \{ y \in \partial \mathcal{M}; X(y) \text{ pointing inside } \mathcal{M} \} \]
\[ \partial_+ \mathcal{M} = \{ y \in \partial \mathcal{M}; X(y) \text{ pointing outside } \mathcal{M} \} \]
Boundary value problems, well-posedness

Let $\mu$ be smooth measure invariant by $\varphi_t$, $V \in C^\infty(\mathcal{M})$ a potential.

**Question:** for $f \in C^\infty(\mathcal{M})$ (or $L^p(\mathcal{M}), H^s(\mathcal{M}),...$), can we solve the linear PDE (transport equation)

$$(X + V)u = f, \quad u|_{\partial - \mathcal{M}} = 0$$

in a given functional space? Is the solution unique? singularities of $u$?

**Remark:**
- In $\mathcal{D}'(\mathcal{M})$, no uniqueness: if $V = 0$ and $X$ has a periodic orbit $\gamma$ not intersecting $\partial \mathcal{M}$, then $X \delta_\gamma = 0$.
- if $\partial \mathcal{M} = \emptyset$ and $\varphi_t$ is ergodic, $\ker X \cap L^p(\mathcal{M}) = \mathbb{R}$
Trapped sets

Define the exit times from $\mathcal{M}$

$$\ell_+ : \mathcal{M} \to [0, \infty], \quad \ell_+(y) = \sup\{t \geq 0; \varphi_t(y) \in \mathcal{M}^\circ\} \cup \{0\}.$$  

$$\ell_- : \mathcal{M} \to (-\infty, 0], \quad \ell_-(y) = \inf\{t \leq 0; \varphi_t(y) \in \mathcal{M}^\circ\} \cup \{0\}.$$  

Introduce the

- forward/backward trapped set $\Gamma \pm := \{y \in \mathcal{M}; \ell_\pm = \pm \infty\}$,
- trapped set $K := \Gamma_- \cap \Gamma_+$
Resolvents (case $V = 0$)

Add a damping: let $\lambda \in \mathbb{C}$, $\text{Re}(\lambda) > 0$ and define the operators

$$R_+(\lambda)f(y) = \int_0^{\ell_+(y)} e^{-\lambda t} f(\varphi_t(y)) \, dt,$$

$$R_-(\lambda)f(y) = -\int_0^{\ell_-(y)} e^{\lambda t} f(\varphi_t(y)) \, dt.$$  

bounded on $L^2(M, \mu)$. They solve the boundary value pb

$$\begin{cases}
(-X - \lambda) R_-(\lambda)f = f \\
(R_-(\lambda)f)|_{\partial_-M} = 0
\end{cases}, \quad \begin{cases}
(-X + \lambda) R_+(\lambda)f = f \\
(R_+(\lambda)f)|_{\partial_+M} = 0
\end{cases}$$
Extension to the complex plane

**Theorem (Dyatlov-G ’16)**

If the trapped set $K$ is hyperbolic, there exists $c > 0$ so that for each $s > 0$ the operator $R_{\pm}(\lambda)$ extends to $\text{Re}(\lambda) > -cs$ meromorphically in $\lambda$ with finite rank poles and maps

$$H_0^s(\mathcal{M}) \to H^{-s}(\mathcal{M}).$$

We obtain a description of wave-front set of the Schwartz kernel $R_{\pm}(\lambda; x, x')$ in terms of stable/unstable bundles and $\Gamma_{\pm}$.

In the Anosov case:
* meromorphic extension by Butterley-Liverani ’07, Faure-Sjöstrand ’11
* the wave-front set analysis done by Dyatlov-Zworski ’16.
(cf. Nonnemacher’s talk on resonances)
Tools used for this result

- Microlocal calculus - analysis in phase space
- Escape functions (cf. Faure-Sjöstrand)
- Use of anisotropic Sobolev spaces (cf. Kitaev, Blank, Keller, Liverani, Gouëzel, Baladi, Tsujii, Faure, Roy, Sjöstrand, etc)
- Set up of a Fredholm theory for the operator $-\mathbf{X} + \lambda$
- Propagation estimates: Hörmander propagation + propagation at radial sets (cf. Melrose, Vasy, Dyatlov-Zworski)
Applications to previous theorems

Lens rigidity problem:

1) let $\mathcal{M} = SM$, $\lambda = 0$: we deduce (microlocal) regularity of solutions of $Xu = f$ with $f \in \mathcal{C}^\infty(\mathcal{M})$ satisfying $u|_{\partial_{\pm}\mathcal{M}} = 0$. This is the key pb for description of ker $I_2$ in trapped case.

2) Deduce that the operator $I_2^* I_2 = \pi^* (R_+ (0) - R_- (0)) \pi^*$ is an elliptic pseudo-differential operator of order $-1$ on $(\ker I_2)^\perp \implies$ stability estimates for $I_2$.

($\pi^* : \mathcal{C}^\infty (M; S^2 T^* M) \to \mathcal{C}^\infty (SM)$ natural operator, $\pi^*$ its adjoint)
Marked length spectrum rigidity:

Use the operator $\Pi := R_+(0) - R_-(0)$ in Anosov case.
It is related to $l_2$ through a (positive) Livsic theorem (Lopes-Thieullen ’03)
It satisfies stability estimates: for $s > 0$, $f \in H^{-s-1}(M; S^2 T^* M)$ with $f \perp \ker l_2$

$$C \| l_2 f \|_{\ell_\infty}^{1-\nu(s)/2} \| f \|_{C^\alpha}^{(1+\nu(s))/2} \geq \| \Pi \pi^* f \|_{H^{-s}(SM)} \geq \frac{1}{C} \| f \|_{H^{-s-1}(M)}.$$ 

Here $\alpha > 0$, $\nu(s) = \mathcal{O}(s)$ as $s \to 0$. 
Ruelle zeta function - question of Smale:

1) Consider resolvent of $L_X$ acting on bundle of $k$-forms $w$ with $i_X w = 0$,

2) Define $Z_k(\lambda)$ by

$$\frac{Z'_k(\lambda)}{Z_k(\lambda)} := \int_M \text{Tr}(R_{-}^{(k)}(\lambda; x, x)) d\mu(x)$$

and show this is meromorphic with simple poles and integer residues $\Rightarrow Z_k(\lambda)$ is meromorphic.

3) Using Guillemin trace formula, write Ruelle function as

$$\frac{\zeta'(\lambda)}{\zeta(\lambda)} = \sum_{k=0}^{n-1} (-1)^k \frac{Z'_k(\lambda)}{Z_k(\lambda)}.$$
Fried conjecture

1) Same as above (with $\rho$-twisted Lie derivative)

2) Show that $\zeta_{\rho}(0)$ is continuous with respect to $X$, approximate with analytic flow and use Sanchez-Morgado result.

3) Show that $\zeta_{\rho}(0)$ is locally constant with respect to $X$ if no pole at $\lambda = 0$ for resolvents $R^{(k)}(\lambda)$ (variation formula), and vary near hyperbolic metrics.
OBRIGADO !