Proof-theoretic Methods in Nonlinear Analysis

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Early history

Hilbert's Program:

Establish that uses of higher noneffective/transfinite **(,,ideal") principles** $\mathcal{I}$ in proofs of combinatorial/finitistic **(,,real") propositions** $\mathcal{P}$ can be **eliminated**, at least in principle.

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Though in this specific form, in general impossible (Gödel), the basic approach is largely correct for existing ordinary mathematics: **proof-theoretic tameness** of ordinary mathematics!

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'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true?' (G. Kreisel)

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- Bounds on the number of solutions in Roth's theorem (Kreisel, Luckhardt; **Herbrand-Analysis**, first **polynomial bound**).

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- Interesting proofs that use **WKL** but allow for a **WKL-elimination** due to their logical form: **uniqueness statements** ($\in \forall \to \forall$).
- Moduli and constants of **strong unicity in best Chebycheff approximation** (later with P. Oliva **L^1-approximation**).

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Under appropriate conditions $(\mathbf{x_n})$ converges to a **fixed point** of $\mathbf{T}$ (or zero of $\mathbf{F} : \mathbf{X} \to \mathbb{R}$ or minimizer of convex l.s.c. $\mathbf{f} : \mathbf{X} \to \overline{\mathbb{R}}$).

Why rewarding for proof mining?

- $\|x_n - Tx_n\| \to 0 \in \forall\exists$ if $(\|x_n - Tx_n\|)$ nonincreasing.

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- The **finitary proof-theoretic analysis** often **generalizes** to **geodesic settings** (Hadamard spaces, CAT(κ)-spaces).
- Extracted bounds are **highly uniform**.

Since 2004: rates of metastability

If one studies the convergence of (x_n) itself, **in general no computable rate of convergence possible**.

Let (x_n) be a Cauchy sequence in a metric space (X, d), i.e.

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A bound $\Phi(k, g)$ on '$\exists n$' in the latter formula is a **rate of metastability** (**Kreisel's no-counterexample interpr., 1951**).

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This asked for **general logical metatheorems** to account for this.

From case studies to logical metatheorems

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$\forall \underline{\mathbf{x}} \in \mathbf{P}, \mathbf{K}, \mathbf{X}, \mathbf{X}^{\mathbf{X}}, \mathbf{X}^{\mathbb{N}} \ldots \exists \mathbf{n} \in \mathbb{N}\, \mathbf{A}(\underline{\mathbf{x}}, \mathbf{n})$-theorems.

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- bounded subsets of **abstract** metric structures X.

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$\mathcal{A}^\omega[X, \|\cdot\| \ldots]$ results by adding constants with axioms expressing that $(X, \|\cdot\|, \ldots)$ is a normed, uniformly convex, Hilbert space.

Keeping track of uniform bounds: majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$:

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$$\mathbf{T}^* \gtrsim_{\mathbf{X}\to\mathbf{X}} \mathbf{T} \equiv \forall \mathbf{n} \in \mathbb{N}, \mathbf{x} \in \mathbf{X}\ (\mathbf{n} \geq \|\mathbf{x}\| \to \mathbf{T}^*(\mathbf{n}) \geq \|\mathbf{T}(\mathbf{x})\|) .$$

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Metric setting: reference point $\mathbf{a} \in \mathbf{X}$: $\mathbf{x} \geq \mathbf{d}(\mathbf{a}, \mathbf{y})$.

Theorem (K., Trans.AMS 2005, Gerhardy/K.,Trans.AMS 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{z} := z_1, \ldots z_k$ variables ranging over X, $\mathbb{N} \to X$ or $X \to X$.

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then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every normed space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\underline{z}^* \gtrsim_{\underline{\tau}} \underline{z}$:

$$\forall \mathbf{y} \in \mathbf{K} \exists \mathbf{v} \leq \mathbf{\Phi}(\mathbf{r_x}, \underline{\mathbf{z}}^*)\, \mathbf{A}_\exists(\mathbf{x}, \mathbf{y}, \underline{\mathbf{z}}, \mathbf{v}).$$

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then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every normed space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\underline{z}^* \gtrsim_{\underline{\tau}} \underline{z}$:

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If only WKL and Σ^0_1-IA used: $\mathbf{\Phi}$ is **primitive recursive**.

Theorem (K., Trans.AMS 2005, Gerhardy/K.,Trans.AMS 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{z} := z_1, \ldots z_k$ variables ranging over X, $\mathbb{N} \to X$ or $X \to X$.
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If only WKL and Σ^0_1-IA used: $\boldsymbol{\Phi}$ is **primitive recursive**.
Usually: Φ has very low complexity: **'proof-theoretic tameness'** of ordinary mathematics!

As special case of **general logical metatheorems** one has:

Corollary (Gerhardy/K., TAMS 2008)

If $\mathcal{A}^\omega[X, \|\cdot\|]$ proves

$$\forall \mathbf{x} \in \mathbf{P}\, \forall \mathbf{y} \in \mathbf{K}\, \forall \mathbf{z} \in \mathbf{X}\, \forall \mathbf{T} : \mathbf{X} \to \mathbf{X}\, \big(\mathbf{T} \text{ n.e.} \to \exists \mathbf{v} \in \mathbb{N}\, \mathbf{A}_\exists\big),$$

then one can extract a **computable functional** $\Phi : \mathbb{N}^\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t. for all $\mathbf{x} \in \mathbf{P}, \mathbf{b} \in \mathbb{N}$ in all normed spaces $\mathbf{X}$

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Similar for Hilbert spaces and uniformly convex spaces (then bound depends on modulus of convexity).

Abstract metric and normed structures

Abstract metric and normed structures

- **Examples of admissible classes of spaces X :** metric, hyperbolic, CAT(0), CAT($\kappa > 0$), normed, their completions, Hilbert, uniformly convex, uniformly smooth, abstract $C(K), L^P$ spaces. Also **several** spaces $X_1, \ldots, X_n$ (Günzel/K.)

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- **Uniform continuity** of the constants **not necessary**, but convenient as it implies **full extensionality** (if not: weak extensionality rule!).

Relations to positive bounded logic

Theorem (Günzel/K., Adv. Math. 2016)

All structures X treated in positive bounded logic have a logical metatheorem of the type above.

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- $\exists$-UB$^{\mathbf{X}}$ proves the **equivalence between positive bounded formulas and their approximations** (Günzel/K.).
- Proof-theoretic metatheorems are **effective quantitative** replacements of **noneffective uses of ultraproducts**.

A polynomial rate of asymptotic regularity in Bauschke's solution of the 'zero displacement conjecture'

Consider a Hilbert space $\mathbf{H}$ and nonempty closed and convex subsets $\mathbf{C_1, \ldots, C_N \subseteq H}$ with metric projections $\mathbf{P_{C_i}}$, define $\mathbf{T := P_{C_N} \circ \ldots \circ P_{C_1}}$. In 2003 Bauschke proved the 'zero displacement conjecture':

$$\mathbf{\|T^{n+1}x - T^n x\| \to 0 \quad (x \in H).}$$

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Proof uses abstract theory of maximal monotone operators: Minty's theorem, Brézis-Haraux theorem, Rockafellar's maximal monotonicity and sum theorems, Bruck-Reich theory of strongly nonexpansive mappings, conjugate functions, normal cone ...).

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Logical metatheorems, therefore, guarantee the extractability a uniform rate of asymptotic regularity which only depends on the error $\varepsilon > 0$, $N \in \mathbb{N}$ and **majorants** for $x \in H$ and $P_{C_1}, \ldots, P_{C_N}$: $b \geq \|x\|$ and $K \geq \|c_1\|, \ldots, \|c_N\|$ for some **arbitrary** points $c_1 \in C_1, \ldots, c_N \in C_N$:

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Since the mappings P_{C_i} are nonexpansive, the corollary guarantees a computable $\Phi(\varepsilon, N, b, K)$ s.t. for $b \geq \|x\|$

$$\forall \varepsilon > 0\, \forall n \geq \Phi(\varepsilon, N, b, K)\ (\|T^{n+1}x - T^n x\| < \varepsilon).$$

Theorem (K. FoCM 2018)

$$\Phi(\varepsilon, N, b, K) := \left\lceil \frac{18b + 12\alpha(\varepsilon/6))}{\varepsilon} - 1 \right\rceil \left\lceil \left(\frac{D}{\omega(D, \tilde{\varepsilon})} \right) \right\rceil$$

is a **rate of asymptotic regularity** in Bauschke's result, where

$$\tilde{\varepsilon} := \frac{\varepsilon^2}{27b + 18\alpha(\varepsilon/6)}, \; D := 2b + NK, \; \omega(D, \tilde{\varepsilon}) := \frac{1}{16D}(\tilde{\varepsilon}/N)^2.$$

$$\alpha(\varepsilon) := \frac{(K^2 + N^3(N-1)^2 K^2) N^2}{\varepsilon}.$$

Here $b \geq \|x\|$ and $K \geq \left(\sum_{i=1}^{N} \|c_i\|^2 \right)^{\frac{1}{2}}$ for some $(c_1, \ldots, c_N) \in C_1 \times \ldots \times C_N$.

Proximal Point Algorithm (PPA)

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K.,Leuştean, Nicolae, Comm.Contem.Math 2018: explicit rates of asymptotic regularity and metastability.

The uniform case

$\mathbf{A} : \mathbf{H} \to 2^{\mathbf{H}}$ is **uniformly monotone** on $\mathbf{C} \subseteq \mathbf{H}$ with modulus ρ if

$$\forall x, y \in C\, \forall u \in A(x), v \in A(y) \Big(\|x - y\| \geq 2^{-k} \to \langle x - y, u - v \rangle \geq 2^{-\rho(k)} \Big).$$

Then $\mathbf{A}$ has a unique zero in $\mathbf{C}$ and ρ gives rise to a **modulus of uniqueness** (see K./Koutsoukou-Argyraki JMAA 2015 even for accretive operators in general Banach spaces).

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Leuştean/Sipoş (JNVA 2018): quantitative analysis of PPA in **CAT(0)-spaces**.

Further applications in convex optimization

- Quantitative convex feasibility results in Hilbert (M.A.A. Kkan/K. NA 2014), CAT(κ)-spaces (K. Israel J. Math. 2016).
- Quantitative analysis of Yamada's hybrid steepest descent algorithm (Körnlein, PhD Thesis 2016).
- Rates for the composition of two mappings in CAT(0)-spaces (K., Lopéz-Acedo, Nicolae, Optimization 2017).
- Quantitative properties of proximal maps in uniformly convex spaces (Bačák/K., J. Convex Anal. 2018).
- Rate of convergence using moduli of regularity and Fejér monotonicity (K., Lopéz-Acedo, Nicolae, arXiv:1711.02130).
- Quantitative analysis of Lion-Man games in geodesic spaces (K., Lopéz-Acedo, Nicolae, arXiv:1806.04496).

The Mean Ergodic Theorem

X **Hilbert space**, $f : X \to X$ **linear** and $\|\mathbf{f(z)}\| \leq \|\mathbf{z}\|, \ \forall \mathbf{z}.$

$$\mathbf{A_n(z)} := \frac{\mathbf{1}}{\mathbf{n+1}}\mathbf{S_n(z)}, \text{ where } \mathbf{S_n(z)} := \sum_{\mathbf{i=0}}^{\mathbf{n}} \mathbf{f^{(i)}(z)} \quad (\mathbf{n} \geq \mathbf{0})$$

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Since the proof formalizes in $\mathcal{A}^\omega[\mathbf{X}, \|\cdot\|, \eta]$ guaranteed:

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$\mathbf{f}^* := \mathbf{id}$ majorizes $\mathbf{f}$: $\mathbf{f(0) = 0}$.

Theorem (K./Leuştean, Ergodic Theor. Dynam. Syst. 2009)

X uniformly convex Banach space, η a modulus of uniform convexity and $f : X \to X$ as above, $b > 0$.

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where

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$$\mathbf{M} := \left\lceil \frac{\mathbf{16b}}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{\mathbf{16}} \eta \left(\frac{\varepsilon}{\mathbf{8b}} \right), \quad \mathbf{K} := \left\lceil \frac{\mathbf{b}}{\gamma} \right\rceil,$$
$$\mathbf{h}, \tilde{\mathbf{h}} : \mathbb{N} \to \mathbb{N}, \; \mathbf{h}(\mathbf{n}) := \mathbf{2}(\mathbf{Mn} + \mathbf{g}(\mathbf{Mn})), \; \tilde{\mathbf{h}}(\mathbf{n}) := \max_{\mathbf{i} \le \mathbf{n}} \mathbf{h}(\mathbf{i}).$$

Bounding the number of fluctuations

We say that (x_n) admits k ε-fluctuations if there are $i_1 \leq j_1 \leq \ldots i_k \leq j_k$ s.t. $\|x_{j_n} - x_{i_n}\| \geq \varepsilon$ for $n = 1, \ldots, k$.

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$(A_n(z))$ admits at most

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many fluctuations.

For the Hilbert space case: first fluctuation bounds by Jones, Ostrovskii, Rosenblatt 1996.

Extensions to Nonlinear Ergodic Theorems

- **Metastability for Baillon's nonlinear ergodic theorem** (**K., Comm.Contemp.Math. 2012**).
- **Metastability for Wittmann's strong nonlinear ergodic theorem on Halpern iterations** (**K., Adv.Math. 2011**).
- **Generalization of the above to CAT(0)-spaces** (Saejung 2010) (**K., Leuștean: Adv.Math. 2012**).
- **Generalization to CAT(κ) spaces** $\kappa > 0$ (**Leuștean, Nicolae Ergodic Theor. Dynam. Syst. 2016**).
- **Metastable version of strong nonlinear ergodic theorem** for maps $\forall \mathbf{u}, \mathbf{v} \in \mathbf{C}\ (\|\mathbf{f}(\mathbf{u}) + \mathbf{f}(\mathbf{v})\| \leq \|\mathbf{u} + \mathbf{v}\|)$ (covers Baillon's result for n.e. odd operators): **Safarik, J.Math.Anal.Appl. 2012**.

Further applications in nonlinear analysis

- Explicit moduli and rates for **nonlinear semigroups** (K.,Koutsoukou-Argyraki, JMAA 2016).
- Rates of convergence for the asymptotics of **abstract Cauchy problems** given by **accretive** operators in Banach spaces (K.,Koutsoukou-Argyraki, JMAA 2015).
- Rates of metastability of the path (z_t) of **resolvents of nonexpansive and pseudocontractive operators** T in Hilbert and CAT(κ)-spaces (K. Adv.Math. 2010, Leuștean,Nicolae ETDN 2016).

  Recent breakthrough (K.,Sipoș 2018): rates for $\lim\limits_{t\to 1^-} z_t$ in uniformly smooth and convex Banach spaces.