Towards a Theory of Definable Sets

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The central goal of descriptive set theory is to develop the theory of definable sets.

- Traditionally, this is viewed as meaning definable sets of reals (or subsets of Polish spaces).
- Polish spaces (complete, separable, metric spaces) are the conventional spaces of analysis.

With the axiom of choice, AC, there are "pathological" sets of reals, e.g., sets which are not measurable, without the Baire property, etc.

Consider the following variation of the Vitali set.

Definition

For $x, y \in 2^\omega$,

$$xE_0\, y \Leftrightarrow \exists m\, \forall n \geq m\, (x(m) = y(m)).$$

Using AC, let $S \subseteq 2^\omega$ be a selector of E_0. Since E_0 has an invariant measure, S cannot be measurable. Also, S cannot have the Baire property.

It is reasonable to expect a structural theory to exists for sets which are reasonably definable.

Consider first sets of “reals” $A \subseteq X$, where X is Polish.

The most simply defined sets are the Borel sets, which are stratified in the ω_1-length Borel hierarchy.

$\boldsymbol{\Sigma}^0_1$ =open, $\boldsymbol{\Pi}^0_1$ =closed, and for $\alpha < \omega_1$:

$$\boldsymbol{\Sigma}^0_\alpha = \bigcup_\omega \boldsymbol{\Pi}^0_{<\alpha} \quad \boldsymbol{\Pi}^0_\alpha = \check{\boldsymbol{\Sigma}}^0_\alpha = \bigcap_\omega \boldsymbol{\Sigma}^0_{<\alpha} \quad \boldsymbol{\Delta}^0_\alpha = \boldsymbol{\Sigma}^0_\alpha \cap \boldsymbol{\Pi}^0_\alpha$$

$$\begin{array}{ccccccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & & & \Sigma^0_\alpha & & \\
 & \subseteq & & \supseteq & & \subseteq & & \supseteq & & & & \subseteq & & \supseteq & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & \cdots & \Delta^0_\alpha & & & & \Delta^0_{\alpha+1} \quad \cdots \quad \mathcal{B} = \bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha \\
 & \supseteq & & \subseteq & & \supseteq & & \subseteq & & & & \supseteq & & \subseteq & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & & & \Pi^0_\alpha & &
\end{array}$$

Figure: The Borel hierarchy.

The projective hierarchy is the natural extension of the Borel hierarchy.

They are defined by:

$$
\begin{aligned}
\boldsymbol{\Sigma}^1_1 \restriction X &= \exists^{\omega^\omega} \boldsymbol{\Pi}^0_1 \restriction (X \times \omega^\omega) \\
\boldsymbol{\Pi}^1_1 \restriction X &= \check{\boldsymbol{\Sigma}}^1_1 \\
\boldsymbol{\Sigma}^1_{n+1} \restriction X &= \exists^{\omega^\omega} \boldsymbol{\Pi}^1_n \restriction (X \times \omega^\omega)
\end{aligned}
$$

$$
\begin{tikzcd}[column sep=small]
& \Sigma^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^0_2 \arrow[dr, phantom, sloped, "\subseteq"] & & & & \Sigma^0_\alpha \arrow[dr, phantom, sloped, "\subseteq"] & & & & \Sigma^1_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^1_2 \arrow[dr, phantom, sloped, "\subseteq"] & & \\
\Delta^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^0_3 & \cdots & \Delta^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^0_{\alpha+1} & \cdots & \Delta^1_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_3 & \cdots \\
& \Pi^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^0_2 \arrow[ur, phantom, sloped, "\subseteq"] & & & & \Pi^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] & & & & \Pi^1_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_2 \arrow[ur, phantom, sloped, "\subseteq"] & &
\end{tikzcd}
$$

Figure: The Borel and projective hierarchies.

Beginning with the work of Cantor through the work of the classical descrptive set theorists up through the 40's, a theory of the Borel, analytic ($\boldsymbol{\Sigma}^1_1$), and coanalytic ($\boldsymbol{\Pi}^1_1$) sets was developed.

- Every $\boldsymbol{\Sigma}^1_1$ set in a Polish space is either countable or contains a perfect set.
- Every $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$ set (or provably $\boldsymbol{\Delta}^1_2$ set, which includes the C sets) is universally measurable and has the Baire property.
- Every $\boldsymbol{\Sigma}^1_1$ set $A \subseteq \omega^\omega$ is completely Ramsey.
- Every $\boldsymbol{\Sigma}^1_1$ or $\boldsymbol{\Pi}^1_1$ set is an ω_1 union of Borel sets, every $\boldsymbol{\Sigma}^1_2$ is an ω_1-union of Borel sets.
- Every $\boldsymbol{\Pi}^1_1$ set $A \subseteq X \times Y$ has a $\boldsymbol{\Pi}^1_1$-uniformization.

Moschovakis isolated the key notion of a scale $\{\varphi_n\}$ on a set $A \subseteq X$.

Definition

Let $\kappa \in \mathrm{On}$. $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that

$$\begin{aligned} A &= p[T] \\ &= \{x \colon \exists f \in \kappa^\omega\ \forall n\ (x \restriction, f \restriction n) \in T\} \end{aligned}$$

Definition

A semi-scale on A is a sequence or norms $\varphi_n \colon A \to \kappa$ such that if $\{x_m\} \subseteq A$, $x_m \to x$, and for each n, $\varphi_n(x_m)$ is eventually constant, say equal to λ_n, then $x \in A$.

$\{\varphi_n\}$ is a scale if in addition $\varphi_n(x) \leq \lambda_n$.

Fact

A is κ-Suslin iff A has a κ-semiscale iff A has a κ-scale.

We say $\{\varphi_n\}$ is a $\boldsymbol{\Gamma}$-scale if all of the φ_n are $\boldsymbol{\Gamma}$-norms, that is, the relations $<^*$, $\leq^*$ are in $\boldsymbol{\Gamma}$ where

$$x <^* y \Leftrightarrow (x \in A) \wedge (y \notin A \vee (y \in A \wedge \varphi(x) < \varphi(y)))$$

Theorem (ZF)

We have $Sca(\boldsymbol{\Pi}^1_1)$, $Sca(\boldsymbol{\Sigma}^1_2)$. Also, $S(\omega) = \boldsymbol{\Sigma}^1_1$, and $\boldsymbol{\Sigma}^1_2 \subseteq S(\omega_1)$.

Beginning with the work of Gödel it was realized that ZFC was not sufficient to extend the theory past the first levels of the projective hierarchy.

- It is consistent with ZFC that there is a $\mathbf{\Pi}^1_1$ set without the perfect set property.
- It is consistent with ZFC that there is a $\mathbf{\Delta}^1_2$ set which is not Lebesgue measurable and doesn't have the Baire property.

Two extensions of ZF are determinacy axioms and large cardinal axioms.

Results of Martin, Steel, and Woodin establish the exact conection between determinacy hyptheses and large cardinal axioms.

Theorem (Martin, Steel)

ZFC $+ \exists n$ *Woodin cardinals +measurable* $\Rightarrow \boldsymbol{\Sigma}^1_{n+1}$*-determinacy.*

Theorem (Woodin)

ZFC $+ \exists \omega$ *Woodin cardinals +measurable* $\Rightarrow L(\mathbb{R})$*-determinacy.*

Theorem (Woodin)

ZFC $+ \exists \lambda$ *limit of Woodins and cardinals strong to* $\lambda \Rightarrow Con(\mathrm{AD}_{\mathbb{R}})$.

If X is a set and $A \subseteq X^\omega$, we have the game $G(A)$ on the set X:

$$
G(A) \quad \begin{array}{c|ccccc}
\mathrm{I} & x(0) & & x(2) & & \\
 & & & & & \cdots \quad x \in \omega^\omega \\
\mathrm{II} & & x(1) & & x(3) &
\end{array}
$$

Figure: The game $G(A)$. I wins iff $x \in A$.

Martin (75) showed that in ZFC all Borel games $A \subseteq X^\omega$ (for any X, where X^ω has the product of the discrete topology on X) are determined.

AD is the axiom that all games on ω are determined.

- Was introduced by Mycielski and Steinhaus in the 60's.
- Contradicts AC, but reasonable to assume all "definable" games are determined.
- If all games in $L(\mathbb{R})$ are determined, then $L(\mathbb{R}) \models$ AD.

$\mathrm{AD}_{\mathbb{R}}$ is the axiom that all games on $\mathbb{R}$ are determined. This is strictly stronger than AD (Solovay).

Under AD, there sets of reals fall into a single hierarchy, the Wadge hierarchy, which refines and extends the Borel/projective hiearchies.

Martin and Moschovkis in late 60's established "periodicity theorems" which propagated the scale property up the projective hierarchy assuming PD.

Work of Kechris, Kunen, Martin, Moschovakis, Solovay, Steel, Woodin showed how the "scale theory" could be extended through the projective sets assuming PD, and through the Wadge hierarchy assuming AD.

The theory established a connection between pointclasses $\boldsymbol{\Gamma}$ and certain ordinals $\delta(\boldsymbol{\Gamma})$ associated to them.

Definition

$$\delta(\boldsymbol{\Gamma}) = \sup\{|\preceq| : \preceq \text{ is a } \boldsymbol{\Delta} \text{ prewellordering of } \omega^\omega\}$$

$$
\begin{tikzcd}
 & \Sigma^1_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^1_2 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^1_3 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^1_4 \arrow[dr, phantom, sloped, "\subseteq"] & & \Sigma^1_5 \\
\Delta^1_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_3 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_4 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \Delta^1_5 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & \\
 & \Pi^1_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_2 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_3 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_4 \arrow[ur, phantom, sloped, "\subseteq"] & & \Pi^1_5
\end{tikzcd}
$$

Figure: Scale pattern under PD.

- Sca($\boldsymbol{\Pi}^1_{2n+1}$), Sca($\boldsymbol{\Sigma}^1_{2n+2}$), with scales into $\boldsymbol{\delta}^1_{2n+1} = \delta(\boldsymbol{\Pi}^1_{2n+1})$.
- $\boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$, where $\mathrm{cof}(\lambda_{2n+1}) = \omega$.
- $S(\lambda_{2n+1}) = \boldsymbol{\Sigma}^1_{2n+1}$, $S(\boldsymbol{\delta}^1_{2n+1}) = \boldsymbol{\Sigma}^1_{2n+2}$.

This gives that every $\mathbf{\Sigma}^1_{2n+2}$ set is a δ^1_{2n+1} union of $\mathcal{B}_{\delta^1_{2n+1}}$ sets.

- $\mathcal{B}_\kappa$ is smallest collection closed under $< \kappa$ unions and intersections).
- In fact, the "codes" are effective.

Key Point: Assuming AD, an intricate connection emerges between the theory of the projective sets, the projective ordinals, and the cardinal structure below the projective ordinals.

- Work of Kechris, Kunen, Martin computed the δ^1_n for $n \leq 4$ and established their properties.
- J developed a theory of descriptions which computed all the δ^1_n, and determined the cardinal structure through their supremum.

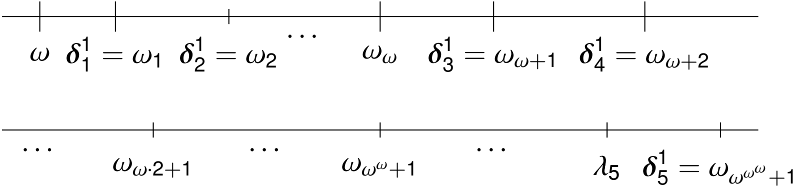


Figure: The cardinal structure below the projective ordinals.

Much of the combinatorics is described in terms of partition relations. A key fact is that the δ^1_{2n+1} have the strong partition property:

$$\delta^1_{2n+1} \to (\delta^1_{2n+1})^{\delta^1_{2n+1}}$$

Some facts about the cardinal structure:

- There are $2^{n+1} - 1$ many regular cardinals strictly between δ^1_{2n+1} and δ^1_{2n+3}.
- These regular cardinals are computed explicitly and they satisfy $\kappa \to (\kappa)^\delta$ where δ is the largest odd projective ordinal below κ.
- The cofinalities of the cardinals between δ^1_{2n+1} and δ^1_{2n+3} can be computed explicitly (the cofinalities are all above δ^1_{2n+1}).

Corollary

Assume ZFC $+$ AD$^{L(\mathbb{R})}$. Then every projective set is $\aleph_\omega$-Borel.

The inductive analysis extends a ways past the projective hierarchy, but not high into the Wadge hierarchy.

Some test questions:

Conjecture

Assuming AD, there does not exists a cardinal $\kappa < \Theta$ such that κ, κ^+, κ^{++}, and κ^{+++} are all regular.

Conjecture

Assuming AD, every regular Suslin cardinal has the strong partition property.

Recent advances in inner-model theory have made new connections with the theory of determinacy models.

Theorem (Steel)

Assume $\mathrm{AD} + V = L(\mathbb{R})$. *Then every regular* $\kappa < \Theta$ *is measurable.*

Theorem (J, Ketchersid, Schlutzenberg, Woodin)

Assume $\mathrm{AD} + V = L(\mathbb{R})$. *Then every* $\kappa < \Theta$ *is Jonssón.*

Question

Can we use these methods to extend the inductive $L(\mathbb{R})$ analysis?

Let X be a Polish space and E a definable equivalence relation on X. The quotient space X/E is in general not a Polish space nor is wellorderable.

- "Borel" is often taken as a stand-in for "definable."

If G acts on X, $G \curvearrowright X$, then we have a corresponding orbit equivalence relation

$$xE_G y \leftrightarrow \exists g\, (g \cdot x = y).$$

We say $(X, E) \leq (Y, F)$, E is (Borel) reducible to F, if $\exists f \colon X \to Y$ with

$$x_1 \, E \, x_2 \text{ iff } f(x_1) F f(x_2).$$

Note if we can surject $\mathbb{R}$ onto a set A then A is such a quotient space, so any set in $L_\Theta(\mathbb{R})$ is of this form.

We say (X, E) is smooth if (X, E) is reducible to equality on a Polish space. That is, X/E is isomorphic to a subset of a Polish space.

The Harrington-Kechris-Louveau dichotomy says that if the borel equivalence relation (X, E) is not smooth, then E_0 embeds into E.

An important subclass of equivalence relations are the countable ones: (X, E) is countable if every E class $[x]_E$ is countable.

By Feldman-Moore every countable Borel equivalence relation is induced by the Borel action of a countable group G.

- Thus we study countable Borel equivalence relations by the algebraic complexity of the group G.

Definition

A countable Borel equivalence relation is hyperfinite if it an increasing union $E = \bigcup_n E_n$ of finite equivalence relations E_n (all classes finite).

Question

For which countable groups G are all the Borel actions of G hyperfinite?

Conjecture (Kechris, Weiss)

The Borel actions of an amenable group G are hyperfinite.

There has been a progression of results on this problem.

- Slaman, Steel Every Borel action of $\mathbb{Z}$ is hyperfinite (and conversely).
- Weiss Every Borel action of $\mathbb{Z}^n$ is hyperfinite.
- J, Kechris, Louveau Every Borel action by a finitely generated group of polynomial growth (almost nilpotent) is hyperfinite.
- J, Gao Every Borel action by an abelian group is hyperfinite.

- Scott, Schneider Every action by a countable locally nilpotent group is hyperfinite.
- Conley, J, Marks, Seward, Tucker-Drob There is a finitely generated solvable group of exponential growth all of whose Borel actions are hyperfinite.

The proofs revolve around understanding the types of marker structures that can be put on the equivalence classes.

Also, there are many interesting combinatorial questions about the quotient spaces for even $\mathbb{Z}^n$ actions. They are related to marker structure questions.

In the last few years some new methods such as orthogonal markers/decompositions and hyperaperiodic points/2-colorings have been developed.

Using these methods we can compute the continuous and Borel chromatic numbers for $F(2^{\mathbb{Z}^n})$.

Theorem (Gao, J, Krohne, Seward)

For $n \geq 2$ we have:

$$3 = \chi_b(F(2^{\mathbb{Z}^n})) < \chi_c(F(2^{\mathbb{Z}^n})) = 4.$$

The continuous results depend on the notion of hyperaperiodic point. [notion due to Gao-J-Seward and independently to Glasner and Uspenski].

Definition

$x \in 2^G$ is a hyperaperiodic point if $\overline{[x]} \subseteq F(2^G)$.

Thus, $\overline{[x]} \subseteq F(2^G)$ is a compact subflow in $F(2^G)$.

Definition

$x \in 2^G$ is a 2-coloring iff for all $s \neq e_G$ there is a finite $T \subseteq G$ such that

$$\forall g \in G \, \exists t \in T \, (x(gt) \neq x(gst))$$

Fact

x is a hyperaperiodic point iff x is a 2-coloring.

Theorem (Gao-J-Seward)

For every countable group G there is a 2-coloring $x \in 2^G$.

- For simple groups such as $\mathbb{Z}^n$ it is easy to construct 2-colorings directly.
- 2-colorings for groups such as $\mathbb{Z}^n$, $n \geq 2$, may be constructed having additional properties such as periodicity in one direction for any section.

By constructing specialized 2-colorings we can prove a general result analyzing continuous equivariant maps from $F(\mathbb{Z}^n)$ into subshifts $S \subseteq b^{\mathbb{Z}^n}$ of finite type. $S = S(b; p_1, \ldots, p_k)$.

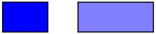
The result is stated in terms of certain finite graphs $\Gamma_{n,p,q}$ built from rectangular tiles. For $n = 2$ there are 12 tiles forming the $\Gamma_{n,p,q}$.

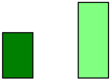
Theorem

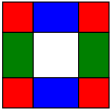
Let S be a subshift of finite type. TFAE

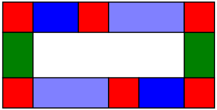
1. *There is a continuous equivariant map $f\colon F(2^{\mathbb{Z}^n}) \to S$.*
2. *There are $n, p, q \in \mathbb{Z}^+$ with $n < p, q$, $(p, q) = 1$, $n \geq \max |p_i| - 1$, and a $g\colon \Gamma_{n,p,q} \to b$ which respects S.*

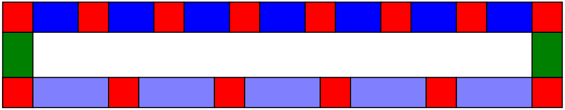
They are built from a rectangular tiles of size $n \times n$: 

Tiles of sizes $(p - n) \times n$ and $(q - n) \times n$: 

Tiles of sizes $n \times (p - n)$ and $n \times (q - n)$: 

Four Torus tiles: 

Four Commutativity tiles: 

Four long tiles: 

Example

There is no continuous tiling of $F(2^{\mathbb{Z}^2})$ by tiles of sizes $d \times d$, $d \times (d+1)$, and $(d+1) \times d$.

Proof.

Consider the $p \times p$ torus tile for $p > d$ a prime. This cannot be tiled by these tiles as then $d | p^2$, a contradiction.

$\square$

Remark

The proof that $\chi_c(F(2^{\mathbb{Z}^2})) > 3$ uses the long tiles. One gets a contradiction by consider the “slope” of the cocycle corresponding to a chromatic 3-coloring along the edges of this tile.

Connection between logic and dynamics

Using the tile analysis we can show the following.

Theorem

For $n \geq 2$ the subshift problem for $F(2^{Z^n})$ is undecidable.

Theorem

For $n \geq 2$ the graph homomorphism problem for $F(2^{\mathbb{Z}^n})$ is undecidable.

The graph homomorphism result uses the following results which give positive and negative conditions for the existence/non-existence of a continuous graph homomorphism.

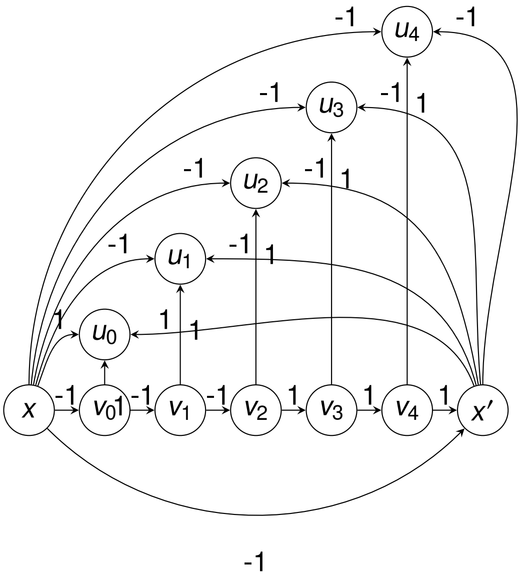
For Γ a graph let $\pi_1(\Gamma)$ denote the homotopoy group of Γ, and let $\pi_1^*(\Gamma) = \pi_1(\Gamma)/N$, where N is the normal subgroup generated by the 4-cycles in Γ.

Theorem (positive condition)

If there is an odd cycle γ in Γ of finite order in $\pi_1^*(\Gamma)$, then there is a continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to Γ.

Theorem (negative condition)

Suppose that for every $N > 0$ there are relatively prime $p, q > N$ such that for every p cycle γ in Γ, γ^q is not a pth power in $\pi_1^*(\Gamma)$. Then there is no continuous graph homomorphism from $F(2^{\mathbb{Z}^2})$ to Γ.


-1
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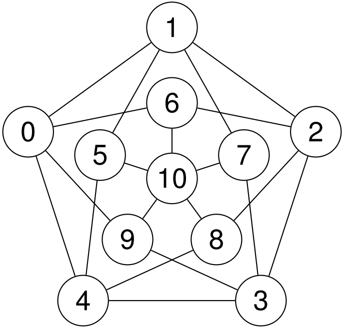

Figure: The Grötzsch Graph. The odd cycle $\gamma = (0, 1, 2, 3, 9, 0)$ has order 2.