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LESSON STUDY IN MATHEMATICS: CURRENT STATUS AND FURTHER DIRECTIONS

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1 Introduction

Lesson study[1] (LS), job-embedded, teacher-oriented, student-focused teacher professional development approach, has been in place in Japan and China for over a century Chen and F. Yang [2013] and Lewis and Tsuchida [1997]. The approach has played crucial roles in transforming mathematics teaching, promoting students' learning, developing teachers, and implementing mathematics curriculum reforms in both countries system wide L. Gu and Y. Yang [2003], Huang, Gong, and Han [2016], Lewis and Takahashi [2013], and Lewis [2016]. It was the seminal work on introducing Japanese lesson study to the mathematics education community in the West in the late 1990s e.g. Lewis and Tsuchida [1997] and Stigler and Hiebert [1999] that has attracted educators around the world to adapt Japanese LS in their own countries, and consequently LS has spread the globe Lewis and Lee [2017]. LS has been widely incorporated in both pre-service teacher preparation and in-service teacher professional development programs. Although many positive effects of LS on improving teaching, improving students' learning and implementing new curriculum have been documented around the world, the challenges and obstacles of adapting LS in other education systems were revealed Huang and Shimizu [2016] and H. Xu and Pedder [2015]. The goals of this chapter are to provide an overview of LS with examples in several selected aspects and discuss the directions of further development of LS. Thus, the chapter is organized into six sections. The first section is about overview and conceptualization of Japanese LS. The second section focuses on introduction into Chinese LS. The third section shows a case of adaptation of LS with pre-service teacher in Malawi. The fourth section presents a case of adaptation of LS with in-service teachers in Thailand. The fifth section discusses theoretical and methodological issues of studies on LS with an example in Switzerland. The chapter ends with discussions about further directions of LS.

2 Overview of Japanese LS

A Japanese LS typically includes four steps Lewis [2016] as follows: (1) Goal Setting: Consider students' current characteristics, their long-term goals and development.

MSC2010: primary 18D50; secondary 55P62, 57Q45, 81T18, 57R56.

[1] Lesson Study is the English Translation of the Japanese term, *Jyugyou Kenkyuu*

Identify gaps between these long-term goals and current reality. Formulate the research theme. (2) Planning: Based on studies of teaching materials (textbooks, curriculum and teacher guides, relevant research), collaboratively plan a "research lesson" to address the identified goals. (3) Research Lesson: One team member teaches the research lesson while other members of the team observe. The observers collect data of students' learning. (4) Reflection: Share and discuss the data collected from research lesson, draw implications for redesign, teaching and student learning broadly, and write a report. The revision and re-teaching of the research lesson is optional Lewis, Perry, and Hurd [2009] and Murata [2011]. Murata [2011] identified major features of LS: Centering on teachers' interest, focusing on students' learning surrounding a research lesson, and collaborative processes, and reflective processes. Furthermore, Lewis and Takahashi [2013] illustrated four types of LS in Japan: local schools, districts, university-attached laboratory schools, and professional associations. Although different types of LS focus on different aspects of teaching or implementation of curriculum, they all provide opportunities to observe teaching and learning, to analyze and discuss data collected during the research lesson, and to network with other educators and build professional learning communities. Thus, LS in Japan has resulted in "teaching for understanding in both mathematics and science, successfully spreading some major instructional innovations" Lewis [2015, p. 50] and is a type of improvement science Langley, Moen, K. M. Nolan, T. W. Nolan, Norman, and Provost [2009].

Since adaptation of Japanese LS globally, a great deal of studies have documented the major benefits of implementing LS including: teacher collaboration and development of a professional learning community, development of professional knowledge, practice and professionalism, more explicit focus on pupil learning, and improved quality of classroom teaching and learning H. Xu and Pedder [2015]. Regarding LS with in-service mathematics teachers, Huang and Shimizu [2016] highlight the following effects of LS: promoting teacher learning, improving teaching and student learning, implementing curriculum, sharing instructional products, and building connections between theory and practice.

However, researchers also identified obstacles and challenges when adapting LS in other countries Fujii [2014], Huang and Shimizu [2016], Larssen et al. [2018], and Ponte [2017]. With in-service teachers, Fujii [2014] revealed six misconceptions of LS such as regarding LS as a workshop. Huang and Shimizu [2016] classified factors influencing the success of LS into two broad categories. At macro-level, these factors include cultural value in general, teaching and teacher learning culture in particular, teacher professional development system, professional learning community, and leadership of district leaders and school leaders. At micro-level, these factors include appropriate content and pedagogical knowledge of teachers, developing inquiry stances such as critical lens as researcher, critical lens as curriculum developer, and critical lens as student; classroom observation with a focus on student learning, teachers' commitment and so on.

With regarding LS with pre-service teachers, Ponte [2017] identified the challenges such as defining the aims of LS, establishing the relationships among participants, scaling up of LS, and adapting or simplifying lesson studies for the particular purpose of educating future teachers. Larssen et al. [2018] put forward the challenges in adapting LS with initial teacher education (ITE) programs including how to prepare student-teachers to observe (professional noticing being a promising option), the wide variation in the focus of classroom observation in ITE lesson studies, and discussion of what is understood by learning needs to stand at the heart of preparation for lesson studies in ITE. To maximum the benefits of LS and to address challenges facing implementing LS, it is critical to conceptualizing LS by consideration of the variation and adaptation of LS Huang and Shimizu [2016].

While there are many projects related to LS outside of Japan, they have been met with varying degrees of success, often because they diverge from authentic Japanese LS. In order to maximize the impact of LS, Japanese mathematics education researchers and researchers of mathematics LS in the US worked collaboratively to examine the process of LS in Japanese schools e.g. Fujii [2016], Takahashi [2014b,a], and Watanabe [2014]. As a result of the collaboration Collaborative Lesson Research (CLR) was proposed as a program to adapt LS to schools outside of Japan, Takahashi and McDougal [2016]. The term is drawn from Catherine Lewis's original translation of *jyugyou kenkyuu* as "Lesson Research," which we reverse in order to emphasize the research aspect Lewis and Hurd [2011] and Takahashi, Watanabe, Yoshida, and Wang-Iverson [2005]. CLR is defined as having the following components:

1. A clear research purpose
2. *Kyouzai kenkyuu*, the "study of teaching materials"
3. A written research proposal
4. A live research lesson and discussion
5. Knowledgeable others
6. Sharing of results

CLR is an investigation undertaken by a group of educators, usually teachers, using live lessons to answer shared questions about teaching and learning. If any component is missing, then the activity cannot be called CLR. CLR has been piloting in several US urban school districts and schools in UK, Qatar, and other countries, Takahashi and McDougal [2018].

3 Introduction into Chinese LS

3.1 Introduction. Chinese students' strong performance in mathematics and science in international assessments Fan and Zhu [2004] and OECD [2014] have attracted scholars to explore mathematics education in China from various perspectives Leung [2005], Y. Li and Huang [2013, 2018], Ma [1999], and Stevenson and Stigler [1992]. In particular, Ma's seminal study Ma [1999] reveals a paradox of Chinese mathematics teachers: the elementary mathematics teachers possessed profound understanding of fundamental mathematics (PUFM) while they received very limited academic training. Researchers Fang and Paine [2008] and Ma [1999] speculated that Chinese mathematics teachers extensively studied curriculum and other instructional materials, collaboratively planned lessons, and frequently observed public lessons within school-based teaching research groups may have promoted their subject matter knowledge and pedagogical knowledge. Yet how and why they have developed that knowledge still remains a significant research problem to explore. Recently, researchers Chen [2017], Fang [2017], and Huang, Fang, and Chen [2017] have started to investigate the core component of teaching research activities, known as Chinese LS (CLS hereafter), from different lens. The major findings of relevant studies on CLS are reviewed and presented in three aspects: historical and cultural perspectives of CLS, underlying infrastructures of CLS, and studies on CLS.

3.2 Chinese LS from historical and cultural perspectives. Chinese Lesson Study (CLS), an approach known for teacher professional development in China, includes cycles of collaborative activities, such as lesson planning, delivering planned lessons along with team observation of classroom teaching, post-lesson debriefing (including talk lesson) and reflection, followed by continued revisions for improvement Huang and Bao [2006] and Y. Yang and Ricks [2013]. It can be generally seen as different from Japanese LS Lewis [2016] in four major aspects Huang and Han [2015] and Huang, Fang, and Chen [2017]. First, for a long time, CLS is content-focused, oriented to developing best teaching strategies for specific subject contents for student learning, while Japanese lesson study was regarded as focusing on general and long-term education goals. Second, there are various kinds of lesson studies for teachers at different stages of their professional development in China with a focus on demonstrating and/or developing exemplary lessons to demonstrate effective teaching. Yet, Japanese model focuses more on the process of teacher learning than the product of a “perfect lesson”. Third, rehearsal teaching is repeated multiple times until teachers involved feel satisfied with the goals they set out to achieve, while Japanese teachers might or might not repeat teaching the improved lesson. Fourth, knowledgeable others could be involved throughout the entire process of LS, while Japanese LS, knowledgeable others may also be involved, but not necessary, or not throughout the entire process.

From a historical perspective, Li (in press) illustrated how Chinese LS has evolved through 3 major stages and finally developed its own characteristics. At the first phase (1896–1949), the initial forms of LS activities were first introduced and practiced. During that period, the affiliated schools to normal university provided a venue for pre-service teachers' teaching practicum experience. The goals of student teachers' teaching practices were: (1) experienced teachers observe and provide feedback on lessons taught by student teachers; (2) experienced teachers demonstrate how to teach to student teachers. These two types and their variations have been and are currently practiced in schools in China. At the second stage (1949–1999), adapted from former Soviet Union, Chinese government officially established a nationwide teaching research group (TRG) system. Teaching Research Groups focused on three main themes: (1) in-depth analysis of textbooks, other instructional materials, and pedagogy; (2) collective lesson planning by teachers in the same group; and (3) observing an exemplary lesson taught by expert teachers or an experimental lesson involving new teaching strategies; teaching or observing a public lesson, followed by the audience commenting on the lesson; mutually observing lessons taught by other teachers in the same group, and providing feedback

S. Li [2014]. At the third stage (2000–), to address the challenges of implementing new curriculum, teacher educators have developed their own style of LS, by adopting ideas of community of practices Lave [1988] and Japanese LS. The most recognized work was done by the Action Research Project team at Qingpu County in Shanghai L. Gu and Y. Yang [2003]. Based on over 20 years of teaching experiment and empirical research in mathematics classrooms, the project team formulated an Action Education model for teacher professional development that incorporates LS as the main platform for teacher learning, planning, teaching, reflections, and new behaviors F. Gu and L. Gu [2016] and Huang and Bao [2006].

From a cultural perspective, Chen [2017] explained the major features of Chinese LS. Based on a careful analysis of classic texts on Chinese cultural thoughts and related literature, ground approaches to analyzing the data collected from Chinese teachers' LS activities and interviews with teachers, the characteristics of CLS could be interpreted from three aspects. First, in terms of their actions, the Chinese teachers enact their understanding of teaching in public lessons through unity of knowing and doing (知行合一) more than conceptual explication. Second, with regard to their thinking, the Chinese teachers use practical reasoning (实践推理) in deliberate practice of repeated teaching through group inquiry and reflection. Third, a tendency of emulating those better than oneself (见贤思齐) is evident in novice teachers' learning from "good" examples by expert teachers. These cultural interpretations can contribute to a deeper understanding about the persistence and importance of LS in the Chinese education history. These cultural roots help to explain the legitimacy of why the repeated teaching is stressed, why learning from experts or peers is valued.

3.3 Two infrastructures laying a foundation for conducting LS in China system wide. A teaching research system consisting of hierarchical Teaching Research Offices (TROs, hereafter) from national to district levels is one of the most important infrastructures for supporting teacher professional development and implementing curriculum reform in China Huang, Ye, and Prince [2016] and Wang [2013]. "Teaching research activities" refer to various types of professional development organized by the TROs. Mathematics *Teaching Research Specialists* (TRSs, hereafter) employed in TROs are officially responsible for organizing teaching research activities at different levels. Based on official documents and requirements of TRSs in China, Huang, S. Xu, and Su [2012] found that Chinese TRSs should have expertise in conducting effective teaching, doing teaching research, effectively organizing school-based teaching activities, and evaluating teachers' teaching and students' learning. Huang, Zhang, Mok, Zhou, Wu, and Zhao [2017] further identified that TRSs valued the following knowledge and skills including subject matter and interdisciplinary knowledge, student learning, mathematics instruction, evaluating student learning, evaluating teacher instruction and mentoring teaching research activities, and leadership in cultivating master teachers and being educational policy consultants.

Correspondingly, a professional promotion system for primary and secondary teachers has motivated teachers to participate in various teaching research activities Huang, Ye, and Prince [2016]. Similar to university promotion system, there are different professional ranks for secondary and primary teachers in China. The senior-rank level includes full-senior and senior teachers. The intermediate-rank level is called level 1 teacher, and the primary-rank level consists of level 2 and level 3 teachers. There are specific criteria for each level. For example, full-senior teachers should meet the following criteria: (1) have high, professional aspirations and firm professional beliefs; have experience working as a teacher for a long time and serve as a guide and steering role in prompting students' growth, and have been an excellent class supervisor and student counselor, and have made a great accomplishment in educating students; (2) have a profound understanding and mastery of curriculum standards and subject knowledge; achieved excellent performance in education and teaching, demonstrated an adept in teaching arts, and developed a unique teaching style; (3) have an ability to organize and guide education and teaching research; achieved creative results in educational ideas, curriculum reform, teaching methods, and applied them in teaching practices, and exerted a demonstration and steering role; (4) Make exceptional contributions to mentoring and cultivating teachers at level 1, 2 and 3; maintain a high reputation in subject teaching, and have been well-recognized as an education and teaching expert; and (5) normally hold a bachelor or above degree, and have served as an advanced teacher at least five years. For another example, a level 3 teacher (entry level) should meet the following criteria: (1) basic mastery of the principles and methods of educating students, and should be able to appropriately educate and guide students; (2) have educational, psychological and pedagogical knowledge, and basic mastery of subject matter knowledge and pedagogical knowledge in the subject being taught, and be able to teach a subject; and (3) holds an associate degree or above, and one year successful teaching probation. Thus, the system defines what professional knowledge and skills are required in order to get promotion to a level. Moreover, senior teachers are required to mentor and supervise novice teachers.

3.4 Studies on the effect of Chinese lesson study (CLS) in mathematics education. Studies have shown the effects of CLS on many educational areas such as curriculum reform Chen and F. Yang [2013], L. Gu and Y. Yang [2003], Fang [2017], and Cravens and Wang [2017], improving teaching Huang and Y. Li [2009] and Y. Yang [2009], promoting teacher learning Huang, Su, and S. Xu [2014], Huang and Y. Li [2009], and Huang and Han [2015], and student learning Huang, Gong, and Han [2016] and L. Y. Gu and Wang [2003]. In Huang and Han [2015]'s study, the authors evidenced how teachers developed their instructional expertise through cross-district LS. Huang, Gong, and Han [2016] reported how teachers improved their instruction that promoted student learning through CLS. Fang [2017] illustrated that some school-based teaching research projects and activities have translated the official curriculum into classroom instruction based on CLS. Moreover, Cravens and Wang [2017] further revealed that how expert teachers played crucial roles in making transaction of curriculum from intended to enacted, and promoting junior/novice teachers' growth district-wide while conducting teaching research activities. Moreover, Huang et al. (in press) demonstrated the mechanism of transfer of a curriculum idea into classroom practice through CLS. Combined, Huang, Fang, and Chen [2017] argued that CLS is a deliberate practice from a deliberate practice for developing instructional expertise; a research methodology for linking research and practice, and an improvement science for instruction and school improvement system wide.

3.5 Discussion and conclusion. Researchers have interpreted the major features of CLS from historical and cultural perspectives. The persistence and predominance of repeated teaching, learning from knowledgeable others and peers, perfecting research lessons through deliberate practice are legitimate in CLS due to a cultural value of teachers' learning. Meanwhile, research has documented the effect of CLS in various aspects. However, there are several issues that need to be addressed in order to further develop and maximize the effects of LS in China. First, due to the emphasis of perfecting research lesson and learning from others, Chinese LS often focuses on improving teacher's instructional techniques, with less attention to student learning. It is still a challenge how to elicit and use student thinking in larger classrooms in China. Second, due to the popularity of using online systems, it is quite common to watch online videos developed by expert teachers in China. However, it is largely unknown how the nature of teacher's learning is impacted through online systems or how to use online systems for enhancing LS. Finally, although a few studies explored the adaptation of CLS in Italy and the US Bartolini Bussi, Bertolini, Ramploud, and Sun [2017] and Huang, Haupt, and Barlow [2017], it is a new area to explore the value and possibility of adaptation of CLS in other countries.

4 Adaptation of LS with pre-service teachers in Malawi

4.1 Background. LS has been introduced in many contexts outside of Japan with the aim of improving teaching e.g. Dudley [2014] and Lewis and Hurd [2011]. However, research has shown that there are some challenges and misconceptions in adapting and implementing LS outside of Japan Fujii [2014]. Furthermore, it has been observed that some fundamental aspects of LS are not easily adapted into new contexts e.g. Cheung and Wong [2014], Ponte [2017], and H. Xu and Pedder [2015].

In more recent decades, LS has also been introduced and researched in teacher education. For example in Norway, a study on LS in initial teacher education found that in planning their research lessons, student teachers did not always start with a research question. In addition, student teachers' mathematics research lessons were not structured in a way that would make students' learning visible. In many cases students were working on tasks individually. Consequently, there was lack of observation of students learning Bjuland and Mosvold [2015].

In Canada, Chassels and Melville [2009] researched LS to teacher education focusing on affordances and challenges. They found that LS raised student teachers' consciousness about the needs of students, the importance of teaching strategies that address students' needs, and value of working collaboratively to improve teaching. The challenges that were observed were related to constraints in time and administrative structures to support the LS. These challenges seem to be common to new contexts as they were also observed in the Norwegian context Bjuland and Mosvold [2015].

In their review of literature of LS in teacher education, Larssen et al. [2018] show that there are differences in adaptations of LS in different contexts in terms of how the LS cycle is conducted and the discussion among researchers. For instance, they found that there was no common understanding of the process of observation. Ponte [2017] reviewed research on LS in secondary teacher education and observed that adaptation of LS in new contexts is not yet well understood, and thus called for further research with a critical view in exploring affordances and challenges in LS.

LS has been adapted in some schools in a few African countries such as Malawi, Uganda and Zambia e.g. Fujii [2014, 2016] and Ono and Ferreira [2009], However, H. Xu and Pedder [2015] observed that between 2001 and 2013 there were only two studies published from the African context. Furthermore, Larssen et al. [2018] observed that there were no studies from African context of LS in initial teacher education. Therefore, researching LS in teacher education in African contexts is important to contribute to worldwide understanding of adaptation of LS in different contexts.

4.2 The study. In the remainder of this section, we discuss findings of a study that explored how mathematics teacher educators in Malawi understand LS and how they intended to implement the LS in their teaching. The study is part of a wider project whose aim is to improve quality of mathematics teacher education in Malawi. One objective is to enhance capacity of mathematics teacher educators through professional development. The professional development is in the form of LS where mathematics teacher educators at each teacher college work together to plan and implement one LS cycle. The entire process takes 7 months; it starts with a three-day workshop in May where teacher educators are introduced to LS and ends with another three-day workshop in November where they report about their LS. After the first workshop, mathematics teacher educators from each college work together to draft their research lesson plan and send it to the authors who act as knowledgeable others Takahashi [2014b], and comment on the lesson plans. The teacher educators then revise the lesson plans following the comments. The process of drafting and refining the lesson plan takes at least 10 weeks. The research lessons are then taught and video recorded and post lesson discussions are also videotaped. At the second workshop, teacher educators report on their LS cycle and describe what they learned from their research lesson.

For the purpose of this paper, data was collected from five teacher education colleges and the unit of analysis is the teacher educators' written lesson plans; the draft lesson plan and the revised lesson plan. We analyzed the lesson plans using qualitative content analysis both deductively and inductively, and we focused on three aspects, namely, research question, prediction and observation.

4.3 Findings.

	Draft lesson plan	**Revised lesson plan**
Research question	2 had research question though not explicit. 2 had lesson titles which implied question but not clearly 1 had no research question	2 more explicit research question 2 still lesson titles but more explicit 1 still no research question
Prediction	1 had prediction, 2 had assumptions of students' knowledge, 2 had no prediction nor assumptions	1 had more specific prediction, 2 had same assumptions of students' knowledge 2 still had neither predictions or assumptions
Observation	4 had few points of observation (average 3) 1 had no explicit observation	3 of the 4 had more and well-focused points of observations (average 8) 1 not revised 1 still no explicit observation

4.3.1 Research Question, Prediction and Observations in Teacher Educators' lesson plans. Across the research lesson plans from the five teacher colleges, we observe that it is difficult for the teacher educators who are newcomers to LS to understand the importance of a research question as a starting point for planning their research lessons. This supports findings in Norway where Bjuland and Mosvold [2015] found that student teacher did not always see a research question as an important starting point for their research lesson. For prediction, we observe that it was not easy for the teacher educators to predict how students would respond to their mathematical tasks. It appears that prediction is difficult for teacher educators just as it is difficult for student teachers as observed by Munthe, Bjuland, and Helgevold [2016]. For observation, we observe that this was better and more commonly understood by the teacher educators than research question and prediction. As we can see from the table all draft lesson plans except one had some points of observations and the revised plans improved in both the number of observation points and the focus. However, two of the five teacher colleges still had challenges in developing explicit points of observations. This seems to be related to the lack of a clear research question which is important in focusing research lesson on students learning and consequently what and how to observe the students learning.

4.4 Conclusion. Mathematics teacher educators understanding of LS varied across the five teacher colleges, especially in terms of research question and prediction. The variation highlights the complexity of LS in new contexts and by inexperienced participants. Nevertheless, it is important to note that in general the teacher educators' lesson plans improved from the draft to the revised version in all three aspects. The improvement varied across the colleges and in some cases was very minimal. The little and varied improvement further highlights the complexity of LS in new contexts. We view the improvement as emphasizing the importance of knowledgeable others in LS especially in new contexts and by inexperienced participants. The lack of much improvement challenges us of the crucial role of knowledgeable others that would make a difference to the quality of LS in new contexts.

5 Adaptation of LS with in-service teachers in Thailand

5.1 Introduction. After the release of *"The Teaching Gap"* by Stigler and Hiebert [1999], Japanese LS has become well-known around the world, Inprasitha [2015]. Many countries have been trying to adapt this approach. Thailand has also been adapting this Japanese LS since 2002, Inprasitha [2003]. At the early state, the research project had focused on introducing "new school mathematics" to teaching mathematics. In 2002, a LS team of student teachers pioneered in using "open-ended problems" as new mathematical contents for developing rich mathematical activity to challenge their students. Later during 2003–2005, a LS team of in-service teachers at a number of schools had been challenged to use 5 open-ended problems in their traditional classrooms, see Inprasitha, Loipha, and Silanoi [2006]. Since 2006, Japanese mathematics textbooks have been used in the first two LS project schools. At present, Thai version of Japanese mathematics textbooks are being used in 120 LS project schools across the country. Through an analysis of the project mathematics textbook, it concludes that focusing on students' ideas can provide a new approach for teachers to teach new school mathematics, rather than just follow mathematical topics as they appear in the traditional textbooks. In this way, it is found that it is useful in helping to bridging the gap, as Klein (1902–1908); 2004 cited in Biehler and Peter-Koop [2008] put it more than one hundred years ago: *"there is a gap between university mathematics and school mathematics"*.

For years, school mathematics has not had its own unique identity Kilpatrick [1992] and Sierpinska [1994]. It has been taught on the demand of university mathematics. For example, in order to prepare school students to learn calculus at the university level, they have to learn pre-calculus at the school level. In other words, school mathematics is what teachers simplify university mathematics to teach at the school level. Pòlya [1945] mentioned that the opportunity is lost if the student regards mathematics as a subject in which he has to earn so-and-so much credit and which he should forget after the final examination as quickly as possible. Furthermore, mathematics has been viewed as a static arena Dossey [1992] and Ernest [1988] and school mathematics as content is viewed separately from teaching mathematics. Teachers just transmit the content to the students. Treating school mathematics and pedagogy this way hinders the development of both school mathematics and pedagogy to teach mathematics for many decades in many countries. This is especially true in the case of Thailand and some developing economies, or even in the United States Stigler and Hiebert [1999]. Thus, this paper provides new ideas on how we treat school mathematics, how it relates to teaching mathematics, and how we can support teachers with teaching new school mathematics through an analysis of the project mathematics textbook.

5.2 A new idea for school mathematics. Many mathematicians, philosophers, and mathematics educators have long been proposing that we could have an alternative approach of school mathematics. Pioneered with Felix Klein (1902–1908); 2004 cited in Biehler and Peter-Koop [2008], as a mathematician marks the history of mathematics education by highlighting the gap between university and school mathematics. He proposed analytical geometry as school mathematics, which is more accessible to school students, rather than traditional calculus. Followed by another great mathematician, George Pòlya [1945], who proposed a new aspect of mathematics that is school mathematics as doing problem solving. This idea later became an agenda for action of school mathematics highlighted by NCTM (1980) as *"the central of school mathematics is mathematical problem solving."* To name just a few, ideas of school mathematics as "a quasi-mathematics, experimental mathematics etc., Lakatos [1976] are in concerted with those of Klein, Pólya and Freudenthal.

A mathematics educator and philosopher like Paul Ernest Ernest [1989, 1991, 1992] proposes a philosophical view on school mathematics seen by different groups of teachers as follows: 1) mathematics as an instrumentalist, 2) mathematics as Platonist, and 3) mathematics as problem solving. The third view is a new view for school mathematics emerged from students as his/her own experiences in solving mathematics by themselves. In the late the 20th century, it became clear that *"problem solving approach is the central issue for school mathematics around the world."*

5.3 Mathematical problem solving as school mathematics. Traditionally, when we think about mathematical problem solving, mathematics as a content is on one side and solving that mathematical problem is on the other side. Goldin and McClintock [1979] and Lester [1994] put it this way: there was a general agreement that problem difficulty is not so much a function of various task variables such as content and context variables, structure variables, syntax variables and heuristic behavior variables as it is of characteristics of the problem solver. Similarly, Nohda [1982, 1991, 2000] put it, "Opening up the hearts of students toward mathematics. In the open-approach method, it is intended to provide students with rich situations by using open problems that have possibility to serve for individual differences among students both in their abilities and interests and in the development of mathematical ways of thinking, and to support the investigative process of solving and generating problems". And the most well-known citations are from those ideas of *"metacognition as a driving forces of mathematical problem solving*" Lester [1985], articulated by Schoenfeld and Herrmann [1982] and Silver [1985]. This quote informs us that an idea of mathematics, especially school mathematics is not a separate entity from students' problem solving. Steffe [2002] has confirmed that there is such a thing as 'students' mathematics, or mathematics of students'. From these points of views, school mathematics is intertwined between mathematical contents and students' problem solving and if we should put these ideas in school contexts, especially, contexts for teaching and learning mathematics. School mathematics could be

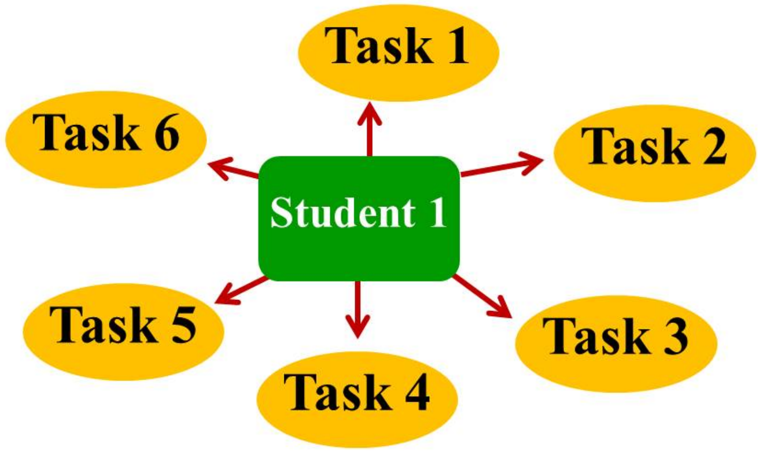

Figure 1: One student engaging in multiple tasks

seen as components of these following: 1) students' ideas emerge while they engage in mathematical tasks or problem situations, 2) students' individual differences become rich resources of mathematical aspects, 3) students' cognitive development become constraints of school mathematics at each grade level. These ideas contrast to the way typical mathematics textbooks provide school mathematics according to mathematical topics from the authors or the teachers' points of views, which is a rather definitional approach.

5.4 Distinguishing between mathematical tasks and students' authentic problem or real problem. Usually, mathematics teachers take for granted the distinction between students' authentic or real mathematical problems and others' sources of mathematical problems such as teachers' or mathematics textbooks. A typical approach to dealing with students' individual differences is for teachers to provide each student with "a number of mathematical problems" expecting that each student should have the opportunity to solve those problems according to their capability (See Fig. 1). However, students are busy with solving those problems superficially and teachers do not have much time to observe students' solving processes. Accordingly, students do not engage deeply with the problems while the teachers do not have time to learn from students' ideas. Thus, students' individual differences become a hindrance for teaching and learning mathematics. On the contrary, more than 100 years ago Japanese teachers developed innovations for dealing with this issue (See Fig. 2).

Figure 2 shows an innovation for engaging a number of students in a class with a particular task or problem. In this situation, the teachers have much more time in the

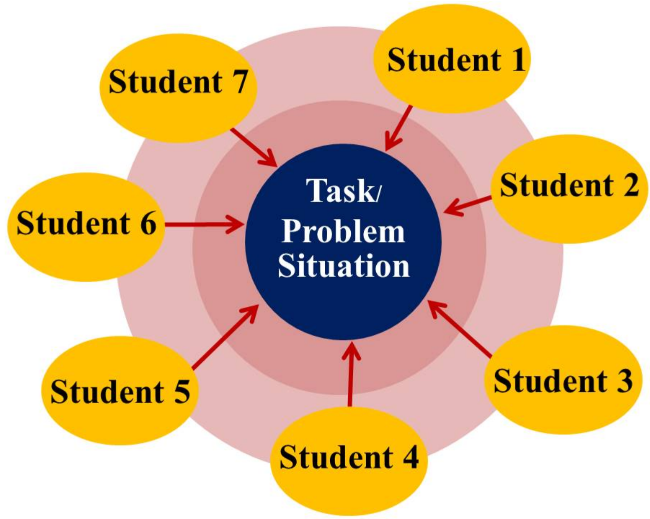

Figure 2: One problem engaged by multiple students

class to engage students to collaboratively solve a particular problem and provide time for teachers to invest in students' individual differences.

5.5 Students' ideas as an origin of mathematical problem solving. Typically, mathematical problem solving focuses more on the problem-solving phase but ignores the problem posing phase Silver [1985]. Worse than that, in the problem-solving phase, teachers emphasize on getting right or wrong answers instead of the process of students' problem-solving. To complement these two phases, Brown and Walter [1990] highlight the importance of problem posing phase. They put it this way: there are two phase of problem posing: problem accepting and problem challenging. To engage students in mathematical tasks or problem situations in order that they come to accept the problem or they have their "own problem" is very crucial for future problem-solving phase. According to figure 2, if the teachers carefully 'designed' mathematical task or problem situations based on an anticipation of students' ideas, it will guarantee that students have been provided multiple chances to develop their own problems. Mathematical tasks or problem situations which are embedded in the students' real world will be easily accessible for most of the students in these classes. Each student's ideas emerge in collaborative problem solving and become rich resources for productive discussions in various aspects of a particular task or problem situation. Step 3 and 4 of innovation for teaching mathematics like Open Approach Inprasitha [2011, 2017] provide chances for the students in the class to compare and discuss among many ideas, later each student becomes 'aware of' each idea as a 'how to' for future solving an 'unknown' task or problem situation. Thus, students' ideas are an origin of mathematical problem solving.

5.6 Open approach LS: Focusing on Japanese mathematical textbooks. The unique adaptation of Japanese LS in Thailand since 2002 is the focus of introducing new school mathematics as an innovation for teaching mathematics through problem-solving. Between 2002–2005, the selected open-ended mathematical problems had been used in a number of schools in the pioneered phase. Since 2006, 4 steps of Open Approach as an innovation for teaching mathematics have been incorporated into 3 steps of LS Inprasitha [2011]. Reading mathematics textbooks Plianram and Inprasitha [2012] is the main activity of LS team in the LS process implemented in a number of the project schools. For the first 6 years (2006–2010), most in-service teachers found it very difficult to read mathematics textbooks (translation version) due to many constraints such as: 1) the rigidity of indicators of curriculum standards, and 2) the teachers' familiarity with reading content through topics. However, Japanese mathematics textbooks are structured based on students' ideas beginning with students' real-world problems and progressing towards typical mathematics textbook problems. If the teachers cannot anticipate the students' ideas when they engage in the given problem task or situation, it is very difficult for the teachers to follow students' problem-solving process in the classroom. However, through working in LS team and teaching through 4 steps of Open Approach, the teachers have improved their deep pedagogical content knowledge on teaching new school mathematics. The weekly cycle of Open Approach LS Inprasitha [2017] is a promising innovation for teaching and learning mathematics in Thailand.

6 Methodological and theoretical issues of studying LS

6.1 Introduction. With the increased popularity of LS around the world there have been calls to deepen our understandings of how it contributes to teacher learning Lewis, Perry, and Hurd [2009] and Widjaja, Vale, Groves, and Doig [2017]. While studies have demonstrated the potential of LS to develop teacher community and enhance teacher knowledge e.g. Lewis and Perry [2017], Ni Shuilleabhain [2016], and Warwick, Vrikki, Vermunt, Mercer, and van Halem [2016], research has also highlighted the need to explore the theoretical underpinnings of mathematics teacher learning in LS Clivaz [2015, 2018], Miyakawa and Winsløw [2009], and H. Xu and Pedder [2015]. Many discussions about theoretical and methodological issues can be found in two recent books: *Mathematics lesson study around the world: Theoretical and methodological issues* Quaresma, Winsløw, Clivaz, Ponte, Ni Shuilleabhain, and Takahashi [2018] and *Theory and practices of lesson study in mathematics: An international perspective* Huang, Takahashi, and da Ponte [2019]. The latter comprises the main content of discussion of this panel. In this book, several theoretical frameworks have been called to analyze in detail what is happening in LS in terms of teacher learning: knowledge integration environment, self-determination theory, self-efficacy theory and pedagogies of practice Lewis, Friedkin, Emerson, Henn, and L. Goldsmith [2019], cultural-historical activity theory Wei [2019], deliberate practice Han and Huang [2019], interconnected model of professional growth Widjaja, Vale, Groves, and Doig [2019], Teaching for Robust Understanding Schoenfeld, Dosalmas, Fink, Sayavedra, Tran, Weltman, and Zuniga-Ruiz [2019] or, Theory of Didactical Situation Bahn and Winsløw [2019] and Clivaz and Ni Shuilleabhain [2019].

As an illustration of "finding an appropriate theoretical perspective to approach LS and glean its advantages Wei [2019]" this panel contribution highlights the types of knowledge employed by teachers in LS (for the full paper, please see Clivaz and Ni Shuilleabhain [2019]. Based on a case study conducted with eight primary school teachers in Switzerland Ni Shuilleabhain and Clivaz [2017], we detail their participation across one cycle of LS utilizing a combination of the theoretical frameworks of and Ball, Thames, and Phelps [2008] and Margolinas, Coulange, and Bessot [2005]. This fine-grained analysis demonstrates the constituents of both subject and pedagogical content knowledge employed by teachers, at varying levels of pedagogical activity, for each phase of the LS cycle. In this case study teachers' pedagogical content knowledge, particularly related to their consideration of content, was the most utilized form of knowledge incorporated in their LS work. Teachers' values and considerations about teaching and learning was also apparent throughout their planning and reflection of the research lesson. These findings provide a detailed representation of the types of knowledge employed by teachers across the phases of LS and contribute to our understanding of how and what teachers may learn in their participation in LS. In addition, our analysis demonstrates that each phase of LS need not necessarily take place in succession, but rather occur in a confluence of teachers' conversations over one full cycle.

6.2 Theoretical framework. Ball, Thames, and Phelps [2008] developed a practice based theory of the knowledge required "to carry out the work of teaching mathematics", presented as a framework of Mathematical Knowledge for Teaching (MKT) (p. 395). This model built on Shulman's theoretical suggestion of PCK as a specific type of knowledge unique to teachers and distinguished it from subject matter or content knowledge (1986, 1987). In their model, Ball and her colleagues highlighted particular categories of knowledge within the subject matter delineations and PCK delineations:

Subject Matter Knowledge

- Common Content Knowledge (CCK)
- Horizon Content Knowledge (HCK)
- Specialized Content Knowledge (SCK)

Pedagogical Content Knowledge

- Knowledge of Content and Teaching (KCT)
- Knowledge of Content and Students (KCS)
- Knowledge of Content and Curriculum (KCC).

In their review of the conceptualization and evidence of PCK in mathematics education research, Depaepe, Verschaffel, and Kelchtermans [2013] noted the MKT model as "probably the most influential re-conceptualizations of teacher PCK within mathematics education" (p. 13). However, Steinbring [1998] and Margolinas [2004] suggest that in Shulman's proposed framework of teacher knowledge Shulman [1986], on which the MKT framework is modelled, fixed categories of teacher knowledge are "not a good model for teacher's activity, which is more complicated" Margolinas, Coulange, and Bessot [2005, p. 207].

In the 1970s, Brousseau's theory of didactical situations Brousseau [1997] first modelled a learning situation by focusing on student learning without explicitly incorporating the role of the teacher Bloch [2005]. However, in analyzing student learning from the 1990s, the importance of the teacher's role became increasingly evident in the theorization of classroom situations Bloch [1999], Dorier [2012], and Roditi [2011]. This new lens provided opportunity to introduce a situated theory embedded in the context of the classroom, through which the various levels of practices, skills and knowledge required of mathematics teachers could be analyzed. Based on this theory of didactical situations, Margolinas [2002] developed a model of a mathematics teacher's milieu to describe the teacher's activities, both in and outside of the classroom. This model was designed to acknowledge the complexity of teachers' actions and while also capturing the broad range of activities contained in teaching and learning Margolinas, Coulange, and Bessot [2005]. Centering on the action of the classroom, the model depicts the various levels at which a teacher must situate themselves within their pedagogical practices. In this model (see Fig. 3), level +3 refers to teachers' values and conceptions about learning and teaching, level +2 concerns teachers' actions and discourses about the global didactic project, level +1 pertains to the local didactic project, level 0 is the didactic action, and level –1 deals with the observation of pupils' activity. The teacher's point of view can be related to his or her considerations and reflections at different levels of generality. Observing students' work (including noticing student talk) relates to a more deliberate focus of the teacher on individual students and, hence, relates to level -1. Planning the local didactic project (about the lesson) refers to the preparation and sequencing of content within the lesson and, hence, is placed at level +1. At level +2, the teacher considers the didactic project in a more holistic or global sense (e.g. teaching a particular element of a topic as one lesson in a series of lessons or throughout a term). While at level +3, the teacher draws upon their beliefs about the teaching and learning of mathematics, which can be related to how the didactic projects may be constructed and to how the teacher will engage with individual students. The model is not intended as a linear interpretation of teachers' work, but rather identifies the multidimensional tensions involved in teaching Margolinas, Coulange, and Bessot [ibid.]. At every level the teacher not only has to deal with the current, most prescient, level of activity, but also with the levels directly before and after and, in some instances, with levels extending beyond.

In our research Ni Shuilleabhain and Clivaz [2017], we proposed a combination Prediger, Bikner-Ahsbahs, and Arzarello [2008] of these two existing frameworks of Mathematical Knowledge for Teaching Ball, Thames, and Phelps [2008] and Levels of Teacher Activity Margolinas, Coulange, and Bessot [2005] to analyze the knowledge incorporated by teachers in two case studies. The graphical representation of this framework (Fig. 3) shows that the categorization of knowledge lies in one plane (the *egg*), while the levels of activity are characterized in a contrasting cross-sectional plane (the *cake*). In this chapter, we develop this work and employ the framework as a tool to

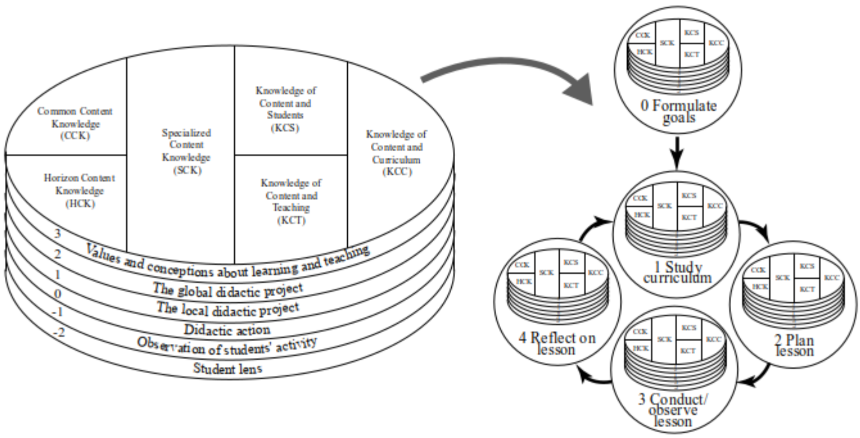

Figure 3: MKT and levels of teacher activity in a cycle of LS.

further detail and analyze mathematics teachers' knowledge in various phases of planning, conducting, observing, and reflecting on teaching in one case-study cycle of LS (see Fig. 3).

6.3 Context and methodology. Eight generalist grade 3–4 primary school teachers from the Lausanne region, French-speaking part of Switzerland, were introduced to LS and conducted four LS cycles over two school years. The group was facilitated by two university teacher educators, one specialist in teaching and learning and one specialist in mathematics didactics. All meetings (37 of an average of 90 minutes duration) and research lessons (8) were transcribed and coded in a qualitative analysis software (NVivo). Student work, teachers observations during lessons, and lesson plans were also recorded and coded Clivaz [2016]. In the comprehensive analysis, see Clivaz and Ni Shuilleabhain [2019] we detail the different forms of Mathematical Knowledge for Teaching Ball, Thames, and Phelps [2008] and Levels of Teacher Activity Margolinas, Coulange, and Bessot [2005] that occur over each phase of the LS cycle. Through the use of quotes and graphical data we explicate teachers' knowledge recorded in their participation in LS.

6.4 Some results. Several graphical representations in this chapter demonstrate the repartition of each of the components of MKT and Levels of Teacher Activity across a cycle of LS. Our research demonstrates that all levels of teacher activity, from the values and conceptions about learning and teaching to seeing mathematics through the eyes of the student, are afforded opportunities of articulation in teachers' participation in the collaborative work of LS, making implicit teacher knowledge explicit Fujii [2018] and Warwick, Vrikki, Vermunt, Mercer, and van Halem [2016]. Analysis of the data also evidences the presence of all categories of MKT across the phases of the cycle,

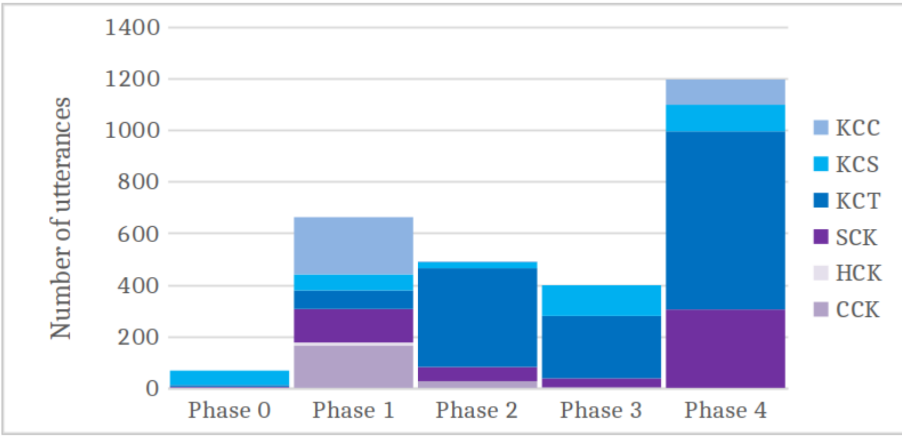

Figure 4: MKT expressed in one cycle of LS.

particularly those of KCS and KCT (types of pedagogical content knowledge) and SCK (a form of subject matter knowledge) (see Fig. 4).

Combining both frameworks, our data shows a prevalence of KCT within the LS cycle, which may support other research findings which have demonstrated changes to teachers' classroom practices as a result of their participation in LS e.g. Batteau [2017], L. T. Goldsmith, Doerr, and Lewis [2014], Shuilleabhain and Seery [2018], and Takahashi and McDougal [2018]. Furthermore, our graphical representations (for example, Fig. 4), depict the benefit of participation in LS, where teachers have opportunity to utilize almost all elements of their MKT across each phase of the cycle and across all levels of teacher activity. An advantage of participating in LS may be the fact that it encourages teachers to incorporate, draw on, and potentially develop their knowledge at various levels, by openly articulating their knowledge through active participation across each of the phases.

In analyzing the types of knowledge occurring across each of the LS meetings, our research demonstrates that the phases of LS do not necessarily occur in a strict chronological or sequential order, but rather take place at varied points throughout the cycle. This may be an important finding in facilitating and analyzing LS, where teachers can articulate goals, student learning, or subject topics at all point throughout their LS conversations.

This articulation of teacher knowledge in LS, and the way that teachers are propelled and compelled teachers to express this professional knowledge, is a key element of this model of professional development. Further investigation and theorization of teacher knowledge and teacher learning is required in order to sustain LS as a professional situation where teachers can develop their knowledge in and for teaching.

7 Further directions of LS

Based on literature reviews, the selected case studies, and the panel debates during the conference, the discussions about the future directions of LS are organized into three aspects: (1) conceptualization of LS; (2) research on LS; and (3) adaptation of LS.

7.1 Conceptualization of LS. Based on Japanese LS and its adaptation globally, Takahashi and McDougal [2018] proposed a collaborative lesson research (CLR) model (see section 2). The model includes key elements of LS: (1) A clear research purpose, (2)*Kyouzai kenkyuu*, the "study of teaching materials", (3) A written research proposal; (4) A live research lesson and discussion; (5) Knowledgeable others; (6) Sharing of results. Along these key components, the panel discussed the following questions: (1) Does Lesson studies need specific instructional guiding theories? (2) Is repeated teaching of the same lesson optional or necessary? (3) Is perfecting a research lesson not one of the goals of LS? (4) Does knowledgeable other need intervene the process of LS? (5) What is the specificity of LS in terms of teacher learning? Debates surrounding these questions could advance the conceptualization of LS.

Regarding the question if guiding theories for conducting LS are needed or not, there are different interpretations. Typically, in Japanese LS, there is not a general learning theory as guiding principle. However, in learning study, the theory of variation is explicitly claimed as the only guiding instructional principle Marton and Pang [2006]. In Malawi, multiple theories could be used to guide LS. For example, during lesson plan stage, MKT Ball, Thames, and Phelps [2008] could be helpful to identify what kind of knowledge teachers need to have in order to develop a lesson plan effectively. While, evaluating research lesson, the theories about discourse analysis Adler and Ronda [2015] could be very helpful to analyze the nature of interaction between teacher and students. In Thailand, the community of practice Wenger [1998], metacognitive theory, and open approach of teaching Becker and Shimada [1997] were used to guide LS. In China, both learning trajectory Simon [1995] and teaching through variation are used to guide LS as well Huang, Gong, and Han [2016]. However, if we classify theories into local theories (how teachers teach a lesson based on their local tradition and cultural value, such as Japanese structured problem solving approach Stigler and Hiebert [1999], and Chinese teaching through variation F. Gu, Huang, and L. Gu [2017] and global theories (general theories about teaching and learning such as situated learning theory Lave [1988] and learning trajectory Simon [1995]), then all LS are guided by certain theories. If knowledgeable others who are mathematics specialists focus on improving teaching practice only, then, local theories are often used implicitly; if knowledgeable others who are mathematics educators from university, focus on both improving teaching practice and developing some theoretical knowledge, then, some grand theories may be used explicitly. So, whether theories are used explicitly, it depends on the "research purpose".

With regard to if repeated teaching is necessary, there are different opinions. Japanese LS does not require repeated teaching of the same lesson. However, LS in China, UK, and Sweden prefer repeated teaching. In China and Sweden, repeated teaching of the same content to different groups of students is required. Chen [2017] explained this specification from a cultural perspective and argued that it is related to the Chinese philosophy of unity of knowing and doing, and learning through deliberate practice. Practically, teachers want to see if suggested changes work in their classes. In Sweden, learning study is a combination of Japanese LS and design-research Cobb, Jackson, and Dunlap [2016], so the orientation of design research requires repeated teaching to testify any changes over the process. In Switzerland, repeated teaching is option, but it is realized that repeated teaching with careful redesign could provide valuable learning opportunities for participating teachers. Thus, it is clear that if specific guiding theories are used and repeated teaching is required, then, LS could be conducted as a design research with the purpose of developing the connection between theory and practice. But, if LS is only focused on improving teaching practice its self, it could be seen as an action research Elliott [2012] with a group of teachers. So, a LS could be located between action research (without any explicit guiding theory and repeated teaching) and design research (both guiding theories and repeated teaching are required).

Perfecting research lesson as a goal during a LS is debatable. In Japanese LS, perfecting research lesson is not the goal of LS. Yet, in China, developing exemplary lessons (product) is the major goal of LS. In Malawi and Thailand, good lesson plans and/or videotaped lessons developed through LS often are for pre-service teachers preparation program. In Sweden, learning study emphasizes developing shareable instructional products (annotated lesson plan, students work, lesson videotaped lesson and theoretical analysis documents). In Switzerland, reaching a perfect lesson is clearly not the goal. Nevertheless designing a lesson that helps the most students to learn is the apparent goal whereas the real goal is to acquire professional knowledge. The Chinese practice of emphasizing development of an exemplary lesson is closely related to teachers' learning from "examples" X. Li [2019]. The positive effect of emphasizing developing good lessons is the possibility of developing sharable instructional products Runesson, Lövström, and Hellquist [2018] and then further improving teacher learning on larger scale. However, it may mislead teachers that there are perfect lessons, or that there is a best way to teach a certain topic. Thus, the LS will be misled to focus on teaching performance rather than student thinking and learning. Ideally, balancing the goals of perfecting a lesson (product) and maximizing teachers' learning opportunity (process) will be important.

Regarding the roles of knowledgeable others during LS, it is agreed that having knowledgeable others' involvement during LS is important. In China, the promoting system ensures there are master teachers in each school and that there are teaching research officers who are responsible for teaching research activity in each district. So, there are quite a number of knowledgeable others. However, in many countries, LS is new and there are not so many knowledgeable others available. For example, in Thailand, knowledgeable others are normally university professors. They not only take responsibility for administrating LS (aliasing with school principals, forming LS groups, and scheduling meetings) but also facilitating the LS process (content and pedagogy aspects). Thus, the role of the knowledgeable other is important in Thailand. Similarly, in Malawi, LS is new. Teachers need to learn what LS is, and learn how to conduct LS from knowledgeable others. In all situations, how to facilitate post-lesson debriefs is crucial. Often, teachers focus on superficial aspects with praise for teachers, rather than focus on critical aspects of student learning. Training on facilitating skills is needed. Thus, it is a challenging task to train qualified knowledgeable others for facilitating LS.

7.2 Research on LS. With regard to research on LS, both theoretical perspectives and research methodologies are critical. As section 6 illustrated, many theories such as knowledge integration environment, self-determination theory, self-efficacy theory, cultural-historical activity theory deliberate practice, interconnected model of professional growth, and Theory of Didactical Situation have been explored to examine LS as a system or a construction of individual components. In addition to case study, both qualitative and quantitative, and mixed methods are used to study LS. For those theoretical perspectives, more empirical studies are needed to examine the strengths and constraints of each theory when used to examine LS.

In return, LS provides an extraordinary field of research for qualitative studies about teachers in mathematics education. The observation of LS group provides insight on how teachers think about mathematics, on how they prepare lessons, on how they observe students, on how they reflect about a lesson, on how they use past experiences or on how they incorporate external (theoretical or practical) resources; all these observations being in a quasi-natural environment. Moreover, these topics are interconnected in teachers' talk and these connections can be studied in LS, presumably better than in other research design Clivaz [2016] and Stigler and Hiebert [2016]. Last but not least, research on LS is part of LS as a research. The two types of research are interconnected and they influence each other. There is a risk that these two types of research would intermingle, and this has to be avoided. If such a confusion is avoided, the connection of academic research on LS and teacher research is a promising way of bridging the research gap between academics and teachers.

7.3 Strategies for adaptation of LS. For the country where LS is new, it is critical for knowledgeable others to talk to different departments, and convince administrators that LS is a powerful and rewarding professional development approach to invest. The support from education administration is decisive for the success of LS. It is also useful to have a long-term plan and strategies to scale up LS. For example, several universities could work together to train Ph.D. Students to serve as knowledgeable others. It is important to get government funding. In addition, to have demo lessons given by master teachers, and invite potential participating teachers to watch demo lessons and participate in post-lesson debriefs facilitated by knowledgeable others to help teachers understand the LS process and obtain a preliminary experience in LS.

Overall, it is hard to identify research question for teachers. So, specific attention needs to be paid to developing teachers' awareness of and ability of identifying research questions. Moreover, challenging issues include how to use LS with poverty students and how to address equity when conducting LS. Much more research in this area is needed.

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