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# SHTUKAS FOR REDUCTIVE GROUPS AND LANGLANDS CORRESPONDENCE FOR FUNCTION FIELDS

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**Abstract**

This text gives an introduction to the Langlands correspondence for function fields and in particular to some recent works in this subject.

We begin with a short historical account (all notions used below are recalled in the text).

The Langlands correspondence Langlands [1970] is a conjecture of utmost importance, concerning global fields, i.e. number fields and function fields. Many excellent surveys are available, for example Gelbart [1984], Bump [1997], Bump, Cogdell, de Shalit, Gaitsgory, Kowalski, and Kudla [2003], Taylor [2004], Frenkel [2007], and Arthur [2014]. The Langlands correspondence belongs to a huge system of conjectures (Langlands functoriality, Grothendieck's vision of motives, special values of L-functions, Ramanujan–Petersson conjecture, generalized Riemann hypothesis). This system has a remarkable deepness and logical coherence and many cases of these conjectures have already been established. Moreover the Langlands correspondence over function fields admits a geometrization, the "geometric Langlands program", which is related to conformal field theory in Theoretical Physics.

Let $G$ be a connected reductive group over a global field $F$. For the sake of simplicity we assume $G$ is split.

The Langlands correspondence relates two fundamental objects, of very different nature, whose definition will be recalled later,

- the automorphic forms for $G$,
- the global Langlands parameters , i.e. the conjugacy classes of morphisms from the Galois group $\mathrm{Gal}(\overline{F}/F)$ to the Langlands dual group $\widehat{G}(\overline{\mathbb{Q}_\ell})$.

For $G = GL_1$ we have $\widehat{G} = GL_1$ and this is class field theory, which describes the abelianization of $\mathrm{Gal}(\overline{F}/F)$ (one particular case of it for $\mathbb{Q}$ is the law of quadratic reciprocity, which dates back to Euler, Legendre and Gauss).

Now we restrict ourselves to the case of function fields.

In the case where $G = GL_r$ (with $r \geq 2$) the Langlands correspondence (in both directions) was proven by Drinfeld [1980, 1987b, 1988, 1987a] for $r = 2$ and by L. Lafforgue [2002] for arbitrary $r$. In fact they show the "automorphic to Galois" direction

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by using the cohomology of stacks of shtukas and the Arthur–Selberg trace formula, and deduce from it the reverse direction by using the inverse theorems of Weil, Piatetski–Shapiro and Cogdell (see Cogdell and Piatetski-Shapiro [1994] which are specific to the case of $GL_r$) as well as Grothendieck's functional equation and Laumon's product formula Laumon [1987b] (which are specific to the case of function fields). Other works using the Arthur–Selberg trace formula for stacks of shtukas are, in chronological order, Laumon, Rapoport, and Stuhler [1993] and Laumon [1997], Châu [1999, 2006] and Châu and Tuấn [2008], Tuấn [2007, 2009], Lau [2004, 2007], Kazhdan, Varshavsky [2004b], Badulescu and Roche [2017].

In V. Lafforgue [2012] we show the "automorphic to Galois" direction of the Langlands correspondence for all reductive groups over function fields. More precisely we construct a *canonical* decomposition of the vector space of cuspidal automorphic forms, indexed by global Langlands parameters. This decomposition is obtained by the spectral decomposition associated to the action on this vector space of a commutative algebra $\mathcal{B}$ of "excursion operators" such that each character of $\mathcal{B}$ determines a unique global Langlands parameter. Unlike previous works, our method is independent on the Arthur–Selberg trace formula. We use the following two ingredients:

- the classifying stacks of shtukas, introduced by Drinfeld for $GL_r$ Drinfeld [1980, 1987b] and generalized to all reductive groups and arbitrary number of "legs" by Varshavsky [2004b] (shtukas with several legs were also considered in Lau [2004] and Châu [2006]),

- the geometric Satake equivalence, due to Lusztig, Drinfeld, Ginzburg and Mirkovic–Vilonen (see Beilinson and Drinfeld [1999] and Mirković and Vilonen [2007]) (it is a fundamental ingredient of the geometric Langlands program, whose idea comes from the fusion of particles in conformal field theory).

In the last sections we discuss recent works related to the Langlands program over function fields, notably on the independence on $\ell$ and on the geometric Langlands program. We cannot discuss the works about number fields because there are too many and it is not possible to quote them in this short text. Let us only mention that, in his lectures at this conference, Peter Scholze will explain local analogues of shtukas over $\mathbb{Q}_p$.

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## 1 Preliminaries

**1.1 Basic notions in algebraic geometry.** Let $k$ be a field. The ring of functions on the $n$-dimensional affine space $\mathbb{A}^n$ over $k$ is the ring $k[x_1, ..., x_n]$ of polynomials in

$n$ variables. For any ideal $I$, the quotient $A = k[x_1, ..., x_n]/I$ is the ring of functions on the closed subscheme of $\mathbb{A}^n$ defined by the equations in $I$ and we obtain in this way all affine schemes (of finite type) over $k$. An affine scheme over $k$ is denoted by $\mathrm{Spec}(A)$ when $A$ is the $k$-algebra of functions on it. It is equipped with the Zariski topology (generated by open subschemes of the form $f \neq 0$ for $f \in A$). It is called a variety when $A$ has no non zero nilpotent element. General schemes and varieties are obtained by gluing. The projective space $\mathbb{P}^n$ over $k$ is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by homotheties and can be obtained by gluing $n+1$ copies of $\mathbb{A}^n$ (which are the quotients of $\{(x_0, ..., x_n), x_i \neq 0\}$, for $i = 0, ..., n$). Closed subschemes (resp. varieties) of $\mathbb{P}^n$ are called projective schemes (resp varieties) over $k$. Schemes over $k$ have a dimension and a curve is a variety purely of dimension 1.

**1.2 Global fields.** A number field is a finite extension of $\mathbb{Q}$, i.e. a field generated over $\mathbb{Q}$ by some roots of a polynomial with coefficients in $\mathbb{Q}$.

A function field $F$ is the field of rational functions on an irreducible curve $X$ over a finite field $\mathbb{F}_q$.

We recall that if $q$ is a prime number, $\mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. In general $q$ is a power of a prime number and all finite fields of cardinal $q$ are isomorphic to each other (although non canonically), hence the notation $\mathbb{F}_q$.

The simplest example of a function field is $F = \mathbb{F}_q(t)$, namely the field of rational functions on the affine line $X = \mathbb{A}^1$. Every function field is a finite extension of such a field $\mathbb{F}_q(t)$.

Given a function field $F$ there exists a unique smooth projective and geometrically irreducible curve $X$ over a finite field $\mathbb{F}_q$ whose field of rational functions is $F$: indeed for any irreducible curve over $\mathbb{F}_q$ we obtain a smooth projective curve with the same field of rational functions by resolving the singularities and adding the points at infinity. For example $F = \mathbb{F}_q(t)$ is the field of rational functions of the projective line $X = \mathbb{P}^1$ over $\mathbb{F}_q$ (we have added to $\mathbb{A}^1$ the point at infinity).

For the rest of the text we fix a smooth projective and geometrically irreducible curve $X$ over $\mathbb{F}_q$. We denote by $F$ the field of functions of $X$ (but $F$ may also denote a general global field, as in the next subsection).

**1.3 Places of global fields and local fields.** A place $v$ of a global field $F$ is a non trivial multiplicative norm $F \to \mathbb{R}_{\geq 0}$, up to equivalence (where the equivalence relation identifies $\|.\|$ and $\|.\|^s$ for any $s > 0$). The completion $F_v$ of the global field $F$ for this norm is called a local field. It is a locally compact field and the inclusion $F \subset F_v$ determines $v$. Therefore a place is "a way to complete a global field into a local field".

For any local field there is a canonical normalization of its norm given by the action on its Haar measure. For any non zero element of a global field the product of the normalized norms at all places is equal to 1.

For example the places of $\mathbb{Q}$ are

- the archimedean place, where the completion is $\mathbb{R}$ (with normalized norm equal to the usual absolute value),

- for every prime number $p$, the place $p$ where the completion is $\mathbb{Q}_p$ (the normalized norm in $\mathbb{Q}_p$ of a number $r \in \mathbb{Q}^\times$ is equal to $p^{-n_p(r)}$, where $n_p(r) \in \mathbb{Z}$ is the exponent of $p$ in the decomposition of $r$ as the product of a sign and powers of the prime numbers).

Thus the local fields obtained by completion of $\mathbb{Q}$ are $\mathbb{Q}_p$, for all prime numbers $p$, and $\mathbb{R}$. A place $v$ is said to be archimedean if $F_v$ is equal to $\mathbb{R}$ or $\mathbb{C}$. These places are in finite number for number fields and are absent for function fields. For each non archimedean place $v$ we denote by $\mathcal{O}_v$ the ring of integers of $F_v$, consisting of elements of norm $\leq 1$. For example it is $\mathbb{Z}_p$ if $F_v = \mathbb{Q}_p$.

In the case of function fields, where we denote by $F$ the field of functions of $X$, the places are exactly the closed points of $X$ (defined as the maximal ideals). The closed points are in bijection with the orbits under $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $X(\overline{\mathbb{F}_q})$ (Galois groups are recalled below). For every closed point $v$ of $X$, we denote by $n_v : F^\times \to \mathbb{Z}$ the valuation which associates to a rational function other than 0 its vanishing order at $v$. We can see $\mathcal{O}_v$ as the $\mathbb{F}_q$-algebra of functions on the formal neighborhood around $v$ in $X$ and $F_v$ as the $\mathbb{F}_q$-algebra of functions on the punctured formal neighborhood. We denote by $\kappa(v)$ the residue field of $\mathcal{O}_v$; it is a finite extension of $\mathbb{F}_q$, whose degree is denoted by $\deg(v)$, therefore it is a finite field with $q^{\deg(v)}$ elements. The normalized norm on $F$ associated to $v$ sends $a \in F^\times$ to $q^{-\deg(v)n_v(a)}$.

In the example where $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \infty$, the unitary irreducible polynomials in $\mathbb{F}_q[t]$ (which is the ring of functions on $\mathbb{A}^1$) play a role analoguous to that of the prime numbers in $\mathbb{Z}$: the places of $\mathbb{P}^1$ are

- the place $\infty$, at which the completion is $\mathbb{F}_q((t^{-1}))$,

- the places associated to unitary irreducible polynomials in $\mathbb{F}_q[t]$ (the degree of such a place is simply the degree of the polynomial). For example the unitary irreducible polynomial $t$ corresponds to the point $0 \in \mathbb{A}^1$ and the completion at this place is $\mathbb{F}_q((t))$.

We recall that the local field $\mathbb{F}_q((t))$ consists of Laurent series, i.e. sums $\sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in \mathbb{F}_q$ and $a_n = 0$ for $n$ negative enough.

**1.4 Galois groups.** If $k$ is a field, we denote by $\overline{k}$ an algebraic closure of $k$. It is generated over $k$ by the roots of all polynomials with coefficients in $k$. The separable closure $k^{\mathrm{sep}} \subset \overline{k}$ consists of the elements whose minimal polynomial over $k$ has a non zero derivative. We denote by $\mathrm{Gal}(\overline{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ the group of automorphisms of $\overline{k}$ (or equivalently of $k^{\mathrm{sep}}$) which act by the identity on $k$. It is a profinite group, i.e. a projective limit of finite groups: an element of $\mathrm{Gal}(\overline{k}/k)$ is the same as a family, indexed by the finite Galois extensions $k' \subset \overline{k}$ of $k$, of elements $\theta_{k'} \in \mathrm{Gal}(k'/k)$, so that if $k'' \supset k'$, $\theta_{k''}\big|_{k'} = \theta_{k'}$. We recall that $k' \subset \overline{k}$ is said to be a finite Galois extension of $k$ if it is a finite dimensional $k$-vector subspace of $k^{\mathrm{sep}}$ and is stable under the action of $\mathrm{Gal}(\overline{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ (and then $\mathrm{Gal}(k'/k)$ is a finite group of cardinal equal to the dimension of $k'$ over $k$).

A simple example is given by finite fields: $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is equal to the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ in such a way that $1 \in \widehat{\mathbb{Z}}$ is the Frobenius generator $x \mapsto x^q$ (which is an automorphism of $\overline{\mathbb{F}_q}$ equal to identity on $\mathbb{F}_q$).

We recall that for any $\mathbb{F}_q$-algebra, $x \mapsto x^q$ is a morphism of $\mathbb{F}_q$-algebras, in particular $(x+y)^q = x^q + y^q$. For any scheme $S$ over $\mathbb{F}_q$ we denote by $\mathrm{Frob}_S : S \to S$ the morphism acting on functions by $\mathrm{Frob}_S^*(f) = f^q$.

We come back to the function field $F$ of $X$. Our main object of interest is the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F) = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

By the point of view of Grothendieck developed in SGA1, we have an equivalence between

- the category of finite sets $A$ endowed with a continuous action of $\Gamma$
- the category of finite separable $F$-algebras

where the functor from the first category to the second one maps $A$ to the finite separable $F$-algebra $((F^{\mathrm{sep}})^A)^\Gamma$ (here $(F^{\mathrm{sep}})^A$ is the direct sum of copies of $F^{\mathrm{sep}}$ indexed by $A$ and $\Gamma$ acts on each copy and permutes them at the same time). We write $\eta = \mathrm{Spec}(F)$ and $\overline{\eta} = \mathrm{Spec}(\overline{F})$. Then, for any dense open $U \subset X$, $\Gamma$ has a profinite quotient $\pi_1(U, \overline{\eta})$ such that a continuous action of $\Gamma$ on a finite set $A$ factors through $\pi_1(U, \overline{\eta})$ if and only if $\mathrm{Spec}(((F^{\mathrm{sep}})^A)^\Gamma)$ extends (uniquely) to an étale covering of $U$. We will not explain the notion of étale morphism in general and just say that a morphism between smooth varieties over a field is étale if and only if its differential is everywhere invertible. Thus we have an equivalence between

- the category of finite sets $A$ endowed with a continuous action of $\pi_1(U, \overline{\eta})$
- the category of finite étale coverings of $U$.

For any place $v$ the choice of an embedding $\overline{F} \subset \overline{F_v}$ provides an inclusion $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$ (well defined up to conjugation). We denote by $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{F}/F)$ the image of any element of $\mathrm{Gal}(\overline{F}_v/F_v)$ lifting the Frobenius generator Frobenius $x \mapsto x^{q^{\deg(v)}}$ in $\mathrm{Gal}(\overline{\kappa(v)}/\kappa(v)) = \widehat{\mathbb{Z}}$. When $U$ is open dense in $X$ as above and $v$ is a place in $U$, the image of $\mathrm{Frob}_v$ in $\pi_1(U, \overline{\eta})$ is well defined up to conjugation.

### 1.5 A lemma of Drinfeld [1980].

Let $U \subset X$ open dense as above. For any $i \in I$ we denote by $\mathrm{Frob}_i$ the "partial Frobenius" morphism $U^I \to U^I$ which sends $(x_j)_{j\in I}$ to $(x'_j)_{j\in I}$ with $x'_i = \mathrm{Frob}_U(x_i)$ and $x'_j = x_j$ for $j \neq i$. For any scheme $T$ and any morphism $T \to U^I$, we say that a morphism $a : T \to T$ is "above" $\mathrm{Frob}_i$ if the square

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d] & T \arrow[d] \\ U^I \arrow[r, "\mathrm{Frob}_i"] & U^I \end{tikzcd}$$

is commutative.

**Lemma 1.1.** *We have an equivalence of categories between*

- *the category of finite sets $A$ endowed with a continuous action of* $(\pi_1(U,\overline{\eta}))^I$*,*
- *the category of finite étale coverings $T$ of $U^I$, equipped with partial Frobenius morphisms, i.e. morphisms $F_{\{i\}}$ above* $\mathrm{Frob}_i$*, commuting with each other, and whose composition is* $\mathrm{Frob}_T$.

The functor from the first category to the second one is the following: if the action of $(\pi_1(U,\overline{\eta}))^I$ on $A$ factorizes through $\prod_{i\in I}\mathrm{Gal}(U_i/U)$ where for each $i$, $U_i$ is a finite étale Galois covering of $U$ (and $\mathrm{Gal}(U_i/U)$ is its automorphism group), then the image by the functor is $(\prod_{i\in I}U_i)\times_{\prod_{i\in I}\mathrm{Gal}(U_i/U)}A$, equipped with the partial Frobenius morphisms $F_{\{i\}}$ given by $\left(\mathrm{Frob}_{U_i}\times\prod_{j\neq i}\mathrm{Id}_{U_j}\right)\times\mathrm{Id}_A$.

**1.6 Split connected reductive groups and bundles.** We denote by $\mathbb{G}_m=GL_1$ the multiplicative group. A split torus over a field $k$ is an algebraic group $T$ which is isomorphic to $\mathbb{G}_m^r$ for some $r$.

A connected reductive group over a field $k$ is a connected, smooth, affine algebraic group whose extension to $\overline{k}$ has a trivial unipotent radical (i.e. any normal, smooth, connected, unipotent subgroup scheme of it is trivial). A connected reductive group $G$ over $k$ is said to be split if it has a split maximal torus $T$. Then (after choosing a Borel subgroup containing $T$) the lattices $\mathrm{Hom}(\mathbb{G}_m,T)$ and $\mathrm{Hom}(T,\mathbb{G}_m)$ are called the coweight and weight lattices of $G$. The split connected reductive groups over a field $k$ are exactly the quotients by central finite subgroup schemes of products of $\mathbb{G}_m$, simply-connected split groups in the four series $SL_{n+1}$, $Spin_{2n+1}$, $Sp_{2n}$, $Spin_{2n}$, and five simply-connected split exceptional groups.

Let $G$ be a split connected reductive group over a field $k$, and $X$ a scheme over $k$. Then a $G$-bundle over $X$ is a morphism $Y\to X$, together which an action of $G$ on the fibers which is simply transitive. A $GL_r$-bundle $\mathcal{E}$ gives rise to the vector bundle of rank $r$ equal to $\mathcal{E}\times_{GL_r}\mathbb{A}^r$ and the notions are equivalent.

## 2 Reminder on automorphic forms

For the moment we take $G=GL_r$. When the global field is $\mathbb{Q}$, an automorphic form (without level at finite places) is a function on the quotient $GL_r(\mathbb{Z})\backslash GL_r(\mathbb{R})$ (the best known example is the particular case of modular functions, for which $r=2$). This quotient classifies the free $\mathbb{Z}$-modules (or, equivalently, projective $\mathbb{Z}$-modules) $M$ of rank $r$ equipped with a trivialization $M\otimes_{\mathbb{Z}}\mathbb{R}=\mathbb{R}^r$ (i.e. an embedding of $M$ as a lattice in $\mathbb{R}^r$). Indeed if we choose a basis of $M$ over $\mathbb{Z}$ its embedding in $\mathbb{R}^r$ is given by a matrix in $GL_r(\mathbb{R})$ and the change of the basis of $M$ gives the quotient by $GL_r(\mathbb{Z})$.

Now we come back to our function field $F$. To explain the analogy with $\mathbb{Q}$ we choose a place $v$ of $X$ (of degree 1 to simplify) playing the role of the archimedean place of $\mathbb{Q}$ (but this choice is not natural and will be forgotten in five lines). An analogue of a projective $\mathbb{Z}$-module $M$ of rank $r$ equipped with a trivialization $M\otimes_{\mathbb{Z}}\mathbb{R}=\mathbb{R}^r$ is a vector bundle of rank $r$ over $X$ equipped with a trivialization on the formal neighborhood around $v$. Now we forget the trivialization on the formal neighborhood around $v$ (because we do not want to introduce a level at $v$) and then we forget the choice of $v$.

Thus an automorphic form (without level at any place) for $GL_r$ is a function on the set $\mathrm{Bun}_{GL_r}(\mathbb{F}_q)$ of isomorphism classes of vector bundles of rank $r$ over $X$.

Now we consider the case of a general group $G$. From now on we denote by $G$ a connected reductive group over $F$, assumed to be split for simplicity. An automorphic form (without level) for $G$ is a function on the set $\mathrm{Bun}_G(\mathbb{F}_q)$ of isomorphism classes of $G$-bundles over $X$.

**Remark 2.1.** This remark can be skipped. In fact the $G$-bundles over $X$ have finite automorphism groups. Therefore it is more natural to consider $\mathrm{Bun}_G(\mathbb{F}_q)$ as a groupoid, i.e. a category where all arrows are invertible. It is the groupoid of points over $\mathbb{F}_q$ of the Artin stack $\mathrm{Bun}_G$ over $\mathbb{F}_q$ whose groupoid of $S$-points (with $S$ a scheme over $\mathbb{F}_q$) classifies the $G$-bundles over $X \times S$. We refer to Laumon and Moret-Bailly [2000] for the notion of Artin stack and we say only that examples of Artin stacks are given by the quotients of algebraic varieties by algebraic groups. In Artin stacks the automorphism groups of the points are algebraic groups (for example in the case of a quotient they are the stabilizers of the action). The Quot construction of Grothendieck implies that $\mathrm{Bun}_G$ is an Artin stack (locally it is even the quotient of a smooth algebraic variety by a smooth algebraic group). The automorphisms groups of points in the groupoid $\mathrm{Bun}_G(\mathbb{F}_q)$ are finite, in fact they are the points over $\mathbb{F}_q$ of automorphisms groups of points in $\mathrm{Bun}_G$, which are algebraic groups of finite type.

It is convenient to impose a condition relative to the center $Z$ of $G$. From now on we fix a subgroup $\Xi$ of finite index in $\mathrm{Bun}_Z(\mathbb{F}_q)$ (for example the trivial subgroup if $Z$ is finite) and we consider functions on $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$. However, except when $G$ is a torus, $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$ is still infinite. To obtain vector spaces of finite dimension we now restrict ourselves to cuspidal automorphic forms.

For any field $E \supset \mathbb{Q}$, we denote by $C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, E)$ the $E$-vector space of finite dimension consisting of *cuspidal* functions on $\mathrm{Bun}_G(\mathbb{F}_q)/\Xi$. It is defined as the intersection of the kernel of all "constant term" morphisms $C_c(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, E) \to C(\mathrm{Bun}_M(\mathbb{F}_q)/\Xi, E)$ (which are given by the correspondence

$$\mathrm{Bun}_G(\mathbb{F}_q) \leftarrow \mathrm{Bun}_P(\mathbb{F}_q) \to \mathrm{Bun}_M(\mathbb{F}_q)$$

and involve only finite sums), for all proper parabolic subgroups $P$ of $G$ with associated Levi quotient $M$ (defined as the quotient of $P$ by its unipotent radical). For readers who do not know these notions, we recall that in the case of $GL_r$ a parabolic subgroup $P$ is conjugated to a subgroup of upper block triangular matrices and that the associated Levi quotient $M$ is isomorphic to the group of block diagonal matrices. It is legitimate in the Langlands correspondence to restrict oneself to cuspidal automorphic forms because all automorphic forms for $G$ can be understood from cuspidal automorphic forms for $G$ and for the Levi quotients of its parabolic subgroups.

Let $\ell$ be a prime number not dividing $q$. To simplify the notations we assume that $\mathbb{Q}_\ell$ contains a square root of $q$ (otherwise replace $\mathbb{Q}_\ell$ everywhere by a finite extension containing a square root of $q$). For Galois representations we have to work with coefficients in $\mathbb{Q}_\ell$ and $\overline{\mathbb{Q}_\ell}$, and not $\mathbb{Q}$, $\overline{\mathbb{Q}}$ and even $\mathbb{C}$ (to which $\overline{\mathbb{Q}_\ell}$ is isomorphic algebraically but not topologically) because the Galois representations which are continuous with coefficients in $\mathbb{C}$ always have a finite image (unlike those with coefficients in $\overline{\mathbb{Q}_\ell}$) and are not enough to match automorphic forms in the Langlands correspondence. Therefore, even if the notion of cuspidal automorphic form is (in our case of function fields) algebraic, to study the Langlands correspondence we will consider cuspidal automorphic forms with coefficients in $E = \mathbb{Q}_\ell$ or $\overline{\mathbb{Q}_\ell}$.

## 3 Class field theory for function fields

It was developped by Rosenlicht and Lang (see Serre [1975]). Here we consider only the unramified case.

Let Pic be the relative Picard scheme of $X$ over $\mathbb{F}_q$, whose definition is that, for any scheme $S$ over $\mathbb{F}_q$, $\mathrm{Pic}(S)$ (the set of morphisms $S \to \mathrm{Pic}$) classifies the isomorphism classes $[\mathcal{E}]$ of line bundles $\mathcal{E}$ on $X \times S$ (a line bundle is a vector bundle of rank 1, so it is the same as a $\mathbb{G}_m$-bundle). The relation with $\mathrm{Bun}_{GL_1}$ is that $\mathrm{Bun}_{GL_1}$ can be identified with the quotient of Pic by the trivial action of $\mathbb{G}_m$.

Let $\mathrm{Pic}^0$ be the neutral component of Pic, i.e. the kernel of the degree morphism $\mathrm{Pic} \to \mathbb{Z}$. It is an abelian variety over $\mathbb{F}_q$, also called the jacobian of $X$.

Class field theory states (in the unramified case to which we restrict ourselves in this text) that there is a canonical isomorphism

$$\big(\pi_1(X, \overline{\eta})\big)^{\mathrm{ab}} \times_{\widehat{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}(\mathbb{F}_q) \tag{3-1}$$

characterized by the fact that for any place $v$ of $X$, it sends $\mathrm{Frob}_v$ to $[\mathcal{O}(v)]$, where $\mathcal{O}(v)$ is the line bundle on $X$ whose sections are the functions on $X$ with a possible pole of order $\leq 1$ at $v$.

The isomorphism (3-1) implies that for any $a \in \mathrm{Pic}(\mathbb{F}_q)$ of non zero degree we can associate to any (multiplicative) character $\chi$ of the finite abelian group $\mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$ (with values in any field, e.g. $\overline{\mathbb{Q}_\ell}$ for $\ell$ prime to $q$) a character $\sigma(\chi)$ of $\pi_1(X, \overline{\eta})$. We now give a geometric construction of $\sigma(\chi)$, which is in fact the key step in the proof of the isomorphism (3-1).

The Lang isogeny $L : \mathrm{Pic} \to \mathrm{Pic}^0$ is such that, for any scheme $S$ over $\mathbb{F}_q$ and every line bundle $\mathcal{E}$ on $X \times S$, $[\mathcal{E}] \in \mathrm{Pic}(S)$ is sent by $L$ to $[\mathcal{E}^{-1} \otimes (\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E})] \in \mathrm{Pic}^0(S)$. We note that $[(\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E})] \in \mathrm{Pic}(S)$ is the image by $\mathrm{Frob}_{\mathrm{Pic}}$ of $[\mathcal{E}] \in \mathrm{Pic}(S)$. The Lang isogeny is surjective and its kernel is $\mathrm{Pic}(\mathbb{F}_q)$. For any finite set $I$ and any family $(n_i)_{i \in I} \in \mathbb{Z}^I$ satisfying $\sum_{i \in I} n_i = 0$, we consider the Abel–Jacobi morphism $AJ : X^I \to \mathrm{Pic}^0$ sending $(x_i)_{i \in I}$ to the line bundle $\mathcal{O}(\sum_{i \in I} n_i x_i)$. We form the fiber product

$$\begin{tikzcd}
\mathrm{Cht}_{I,(n_i)_{i \in I}} \arrow[r] \arrow[d, "\mathfrak{p}"] \arrow[dr, phantom, "\square"] & \mathrm{Pic} \arrow[d, "L"] \\
X^I \arrow[r, "AJ"'] & \mathrm{Pic}^0
\end{tikzcd}$$

and see that $\mathfrak{p}$ is a Galois covering of $X^I$ with Galois group $\mathrm{Pic}(\mathbb{F}_q)$. Thus, up to an automorphism group $\mathbb{F}_q^\times$ which we neglect, for any scheme $S$ over $\mathbb{F}_q$, $\mathrm{Cht}_{I,(n_i)_{i \in I}}(S)$ classifies

- morphisms $x_i : S \to X$

- a line bundle $\mathcal{E}$ on $X \times S$
- an isomorphism $\mathcal{E}^{-1} \otimes (\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{E}) \simeq \mathcal{O}(\sum_{i\in I} n_i x_i)$.

Moreover $\mathrm{Cht}_{I,(n_i)_{i\in I}}$ is equipped with partial Frobenius morphisms $F_{\{i\}}$ sending $\mathcal{E}$ to $\mathcal{E} \otimes \mathcal{O}(n_i x_i)$. The morphism $F_{\{i\}}$ is above $\mathrm{Frob}_i : X^I \to X^I$, because

$$(\mathrm{Frob}_S \times \mathrm{Id}_X)^*(\mathcal{O}(x_i)) = \mathcal{O}(\mathrm{Frob}_S(x_i))$$

Taking the quotient by $a^{\mathbb{Z}}$ we obtain a *finite* Galois covering

$$\begin{array}{c} \mathrm{Cht}_{I,(n_i)_{i\in I}}/a^{\mathbb{Z}} \\ \big\downarrow{\scriptstyle \mathfrak{p}} \\ X^I \end{array}$$

with Galois group $\mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$ and equipped with the partial Frobenius morphisms $F_{\{i\}}$. Then Drinfeld's lemma gives rise to a morphism $\alpha_{I,(n_i)_{i\in I}} : \pi_1(X,\overline{\eta})^I \to \mathrm{Pic}(\mathbb{F}_q)/a^{\mathbb{Z}}$. The character $\sigma(\chi)$ of $\pi_1(X,\overline{\eta})$ is characterized by the fact that for any $I$ and $(n_i)_{i\in I}$ with sum 0, $\chi \circ \alpha_{I,(n_i)_{i\in I}} = \boxtimes_{i\in I}\sigma(\chi)^{n_i}$ and this gives in fact a construction of $\sigma(\chi)$.

## 4 The Langlands correspondence for split tori

Split tori are isomorphic to $\mathbb{G}_m^r$, so there is nothing more than in the case of $\mathbb{G}_m = GL_1$ explained in the previous section. Nevertheless the isomorphism of a split torus with $\mathbb{G}_m^r$ is not canonical (because the automorphism group of $\mathbb{G}_m^r$ is non trivial, equal to $GL_r(\mathbb{Z})$). Let $T$ be a split torus over $F$. To obtain a canonical correspondence we introduce the Langlands dual group $\widehat{T}$, defined as the split torus over $\mathbb{Q}_\ell$ whose weights are the coweights of $T$ and reciprocally. In other words the lattice $\Lambda = \mathrm{Hom}(\widehat{T}, \mathbb{G}_m)$ is equal to $\mathrm{Hom}(\mathbb{G}_m, T)$. Then the Langlands correspondence gives a bijection $\chi \mapsto \sigma(\chi)$ between

- characters $\mathrm{Bun}_T(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^\times$ with finite image
- continuous morphisms $\pi_1(X,\overline{\eta}) \to \widehat{T}(\overline{\mathbb{Q}_\ell})$ with finite image

characterized by the fact that for any place $v$ of $X$ and any $\lambda \in \Lambda$ the image of $\sigma(\chi)(\mathrm{Frob}_v)$ by $\widehat{T}(\overline{\mathbb{Q}_\ell}) \xrightarrow{\lambda} \overline{\mathbb{Q}_\ell}^\times$ is equal to the image of $\mathcal{O}(v)$ by

$$\mathrm{Pic}(\mathbb{F}_q) \xrightarrow{\lambda} \mathrm{Bun}_T(\mathbb{F}_q) \xrightarrow{\chi} \overline{\mathbb{Q}_\ell}^\times$$

(this condition is the particular case for tori of the condition of "compatibility with the Satake isomorphism" which we will consider later for all reductive groups).

The construction of $\sigma(\chi)$ works as in the previous section, except that $a^{\mathbb{Z}}$ has to be replaced by a subgroup $\Xi$ of $\mathrm{Bun}_T(\mathbb{F}_q)$ of finite index which is included in the kernel of $\chi$, and we now have to use schemes of $T$-shtukas, defined using $T$-bundles instead of line bundles.

## 5 Reminder on the dual group

Let $G$ be a split reductive group over $F$. We denote by $\widehat{G}$ the Langlands dual group of $G$. It is the split reductive group over $\mathbb{Q}_\ell$ characterized by the fact that its roots and weights are the coroots and coweights of $G$, and reciprocally. Here are some examples:

| $G$ | $\widehat{G}$ |
|---|---|
| $GL_n$ | $GL_n$ |
| $SL_n$ | $PGL_n$ |
| $SO_{2n+1}$ | $Sp_{2n}$ |
| $Sp_{2n}$ | $SO_{2n+1}$ |
| $SO_{2n}$ | $SO_{2n}$ |

and if $G$ is one of the five exceptional groups, $\widehat{G}$ is of the same type. Also the dual of a product of groups is the product of the dual groups.

**Definition 5.1.** A global Langlands parameter is a conjugacy class of morphisms $\sigma$ : $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(U, \overline{\eta})$ for some open dense $U \subset X$, defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.

We say that $\sigma$ is semisimple if for any parabolic subgroup containing its image there exists an associated Levi subgroup containing it. Since $\overline{\mathbb{Q}_\ell}$ has characteristic 0 this means equivalently that the Zariski closure of its image is reductive, Serre [2005].

We now define the Hecke operators (the spherical ones, also called unramified, i.e. without level). They are similar to the Laplace operators on graphs.

Let $v$ be a place of $X$. If $\mathcal{G}$ and $\mathcal{G}'$ are two $G$-bundles over $X$ we say that $(\mathcal{G}', \phi)$ is a modification of $\mathcal{G}$ at $v$ if $\phi$ is an isomorphism between the restrictions of $\mathcal{G}$ and $\mathcal{G}'$ to $X \setminus v$. Then the relative position is a dominant coweight $\lambda$ of $G$ (in the case where $G = GL_r$ it is the $r$-uple of elementary divisors). Let $\lambda$ be a dominant coweight of $G$. We get the Hecke correspondence

$$\begin{tikzcd} & \mathcal{H}_{v,\lambda} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_G(\mathbb{F}_q) & & \mathrm{Bun}_G(\mathbb{F}_q) \end{tikzcd}$$

where $\mathcal{H}_{v,\lambda}$ is the groupoid classifying modifications $(\mathcal{G}, \mathcal{G}', \phi)$ at $v$ with relative position $\lambda$ and $h^{\leftarrow}$ and $h^{\rightarrow}$ send this object to $\mathcal{G}'$ and $\mathcal{G}$. Then the Hecke operator acts on functions by pullback by $h^{\leftarrow}$ followed by pushforward (i.e. sums in the fibers) by $h^{\rightarrow}$. In other words

$$T_{\lambda,v} : C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell) \to C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$$
$$f \mapsto \Big[\mathcal{G} \mapsto \sum_{(\mathcal{G}', \phi)} f(\mathcal{G}')\Big]$$

where the finite sum is taken over all the modifications $(\mathcal{G}', \phi)$ of $\mathcal{G}$ at $v$ with relative position $\lambda$.

These operators form an abstract commutative algebra $\mathcal{H}_v$, the so-called spherical (or unramified) Hecke algebra at $v$, and this algebra acts on $C_c^{\text{cusp}}(\text{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$. This algebra $\mathcal{H}_v$ is equal to $C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{Q}_\ell)$ and it is possible to write its action with the help of adèles. The actions of these algebras $\mathcal{H}_v$ for different $v$ commute with each other.

The Satake isomorphism Satake [1963] and Cartier [1979] can be viewed Gross [1998] as a canonical isomorphism

$$[V] \mapsto T_{V,v}$$

from the Grothendieck ring of representations of $\widehat{G}$ (with coefficients in $\mathbb{Q}_\ell$) to the unramifed Hecke algebra $\mathcal{H}_v$, namely we have $T_{V\oplus V',v} = T_{V,v} + T_{V',v}$ and $T_{V\otimes V',v} = T_{V,v}T_{V',v}$. If $V$ is an irreducible representation of $\widehat{G}$, $T_{V,v}$ is a combination of the $T_{\lambda,v}$ for $\lambda$ a weight of $V$.

## 6 Presentation of the main result of V. Lafforgue [2012]

We now explain the construction in V. Lafforgue [ibid.] of a canonical decomposition of $\overline{\mathbb{Q}_\ell}$-vector spaces

$$C_c^{\text{cusp}}(\text{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma \tag{6-1}$$

where the direct sum is taken over global Langlands parameters $\sigma : \pi_1(X, \overline{\eta}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. This decomposition is respected by and compatible with the action of Hecke operators. In fact we construct a commutative $\mathbb{Q}_\ell$-algebra

$$\mathcal{B} \subset \text{End}(C_c^{\text{cusp}}(\text{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell))$$

containing the image of $\mathcal{H}_v$ for all places $v$ and such that each character $\nu$ of $\mathcal{B}$ with values in $\overline{\mathbb{Q}_\ell}$ corresponds in a unique way to a global Langlands parameter $\sigma$.

Since $\mathcal{B}$ is commutative we deduce a canonical spectral decomposition

$$C_c^{\text{cusp}}(\text{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\nu \mathfrak{H}_\nu$$

where the direct sum is taken over characters $\nu$ of $\mathcal{B}$ with values in $\overline{\mathbb{Q}_\ell}$ and $\mathfrak{H}_\nu$ is the generalized eigenspace associated to $\nu$. By associating to each $\nu$ a global Langlands parameter $\sigma$ we deduce the decomposition (6-1) we want to construct. We show in V. Lafforgue [ibid.] that any $\sigma$ obtained in this way factorizes through $\pi_1(X, \overline{\eta})$, and that the decomposition (6-1) is compatible with the Satake isomorphism at every place $v$ of $X$, in the sense that for every representation $V$ of $\widehat{G}$, $T_{V,v}$ acts on $\mathfrak{H}_\sigma$ by multiplication by $\text{Tr}_V(\sigma(\text{Frob}_v))$.

The elements of $\mathcal{B}$ are constructed with the help of the $\ell$-adic cohomology of stacks of shtukas and are called *excursion operators*.

In the case of $GL_r$, since every semisimple linear representation is determined up to conjugation by its character and since the Frobenius elements $\text{Frob}_v$ are dense in

$\mathrm{Gal}(\overline{F}/F)$ by the Chebotarev theorem, the decomposition (6-1) is uniquely determined by its compatibility with the Satake isomorphism.

On the contrary, for some groups $G$ other than $GL_r$, according to Blasius [1994] and Lapid [1999] it may happen that different global Langlands parameters correspond to the same characters of $\mathcal{H}_v$ for every place $v$. This comes from the fact that it is possible to find finite groups $\Gamma$ and couples of morphisms $\sigma, \sigma' : \Gamma \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ such that $\sigma$ and $\sigma'$ are not conjugated but that for any $\gamma \in \Gamma$, $\sigma(\gamma)$ and $\sigma'(\gamma)$ are conjugated Larsen [1994, 1996].

Thus for a general group $G$, the algebra $\mathcal{B}$ of excursion operators may not be generated by the Hecke algebras $\mathcal{H}_v$ for all places $v$ and the compatibility of the decomposition (6-1) with Hecke operators may not characterize it in a unique way. Therefore we wait for the construction of the excursion operators (done in section 8) before we write the precise statement of our main result, which will be theorem 8.4.

## 7 The stacks of shtukas and their $\ell$-adic cohomology

The $\ell$-adic cohomology of a variety (over any algebraically closed field of characteristic $\neq \ell$) is very similar to the Betti cohomology of a complex variety, but it has coefficients in $\mathbb{Q}_\ell$ (instead of $\mathbb{Q}$ for the Betti cohomology). For its definition Grothendieck introduced the notions of site and topos, which provide an extraordinary generalization of the usual notions of topological space and sheaf of sets on it.

To a topological space $X$ we can associate the category whose

- objects are the open subsets $U \subset X$
- arrows $U \to V$ are the inclusions $U \subset V$

and we have the notion of a covering of an open subset by a family of open subsets. A site is an abstract category with a notion of covering of an object by a family of arrows targeting to it, with some natural axioms. A topos is the category of sheaves of sets on a site (a sheaf of sets $\mathcal{F}$ on a site is a contravariant functor of "sections of $\mathcal{F}$" from the category of the site to the category of sets, satisfying, for each covering, a gluing axiom). Different sites may give the same topos.

To define the étale cohomology of an algebraic variety $X$ we consider the étale site

- whose objects are the étale morphisms

$$\begin{tikzcd} U \arrow[d] \\ X \end{tikzcd}$$

- whose arrows are given by commutative triangles of étale morphisms,

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

- with the obvious notion of covering.

The étale cohomology is defined with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, whence $\mathbb{Z}_\ell$ by passing to the limit, and $\mathbb{Q}_\ell$ by inverting $\ell$.

The stacks of shtukas, introduced by Drinfeld, play a role analoguous to Shimura varieties over number fields. But they exist in a much greater generality. Indeed, while the Shimura varieties are defined over the spectrum of the ring of integers of a number field and are associated to *a minuscule coweight* of the dual group, the stacks of shtukas exist over *arbitrary powers* of the curve $X$, and can be associated to *arbitrary coweights*, as we will see now. One simple reason for this difference between function fields and number fields is the following: in the example of the product of two copies, the product $X \times X$ is taken over $\mathbb{F}_q$ whereas nobody knows what the product $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ should be, and over what to take it.

Let $I$ be a finite set and $W = \boxtimes_{i\in I} W_i$ be an irreducible $\mathbb{Q}_\ell$-linear representation of $\widehat{G}^I$ (in other words each $W_i$ is an irreducible representation of $\widehat{G}$).

We define $\operatorname{Cht}_{I,W}$ as the reduced Deligne–Mumford stack over $X^I$ whose points over a scheme $S$ over $\mathbb{F}_q$ classify shtukas, i.e.

- points $(x_i)_{i\in I} : S \to X^I$, called the legs of the shtuka (“les pattes du chtouca” in French),
- a $G$-bundle $\mathcal{G}$ over $X \times S$,
- an isomorphism

$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$

(where $\Gamma_{x_i}$ denotes the graph of $x_i$), such that

(7-1) the relative position at $x_i$ of the modification $\phi$ is bounded by the dominant coweight of $G$ corresponding to the dominant weight of $W_i$.

The notion of Deligne–Mumford stack is in algebraic geometry what corresponds to the topological notion of orbifold. Every quotient of an algebraic variety by a finite étale group scheme is a Deligne–Mumford stack and in fact $\operatorname{Cht}_{I,W}$ is locally of this form.

**Remark 7.1.** Compared to the notion of Artin stacks mentioned in remark 2.1, a Deligne–Mumford stack is a particular case where the automorphism groups of geometric points are finite groups (instead of algebraic groups).

**Remark 7.2.** In the case of $GL_1$, *resp.* split tori, we had defined schemes of shtukas. With the above definition, the stacks of shtukas are the quotients of these schemes by the trivial action of ${\mathbb{F}_q}^\times$, *resp.* $T(\mathbb{F}_q)$.

We denote by $H_{I,W}$ the $\mathbb{Q}_\ell$-vector space equal to the “Hecke-finite” subspace of the $\ell$-adic intersection cohomology with compact support, in middle degree, of the fiber of $\operatorname{Cht}_{I,W}/\Xi$ over a generic geometric point of $X^I$ (or, in fact equivalently, over a generic geometric point of the diagonal $X \subset X^I$). To give an idea of intersection cohomology, let us say that for a smooth variety it is the same as the $\ell$-adic cohomology and that for (possibly singular) projective varieties it is Poincaré self-dual. An element of this $\ell$-adic intersection cohomology is said to be Hecke-finite if it belongs to a sub-$\mathbb{Z}_\ell$-module of finite type stable by all Hecke operators $T_{\lambda,v}$ (or equivalently by all Hecke operators $T_{V,v}$). Hecke-finiteness is a technical condition but Cong Xue has proven Xue [2017] that $H_{I,W}$ can equivalently be defined by a cuspidality condition (defined using stacks of shtukas for parabolic subgroups of $G$ and their Levi quotients) and that it has finite dimension over $\mathbb{Q}_\ell$.

Drinfeld has constructed "partial Frobenius morphisms" between stacks of shtukas. To define them we need a small generalization of the stacks of shtukas where we require a factorization of $\phi$ as a composition of several modifications. Let $(I_1, ..., I_k)$ be an ordered partition of $I$. An example is the coarse partition $(I)$ and in fact the stack $\mathrm{Cht}_{I,W}$ previously defined is equal to $\mathrm{Cht}_{I,W}^{(I)}$ in the following definition.

**Definition 7.3.** We define $\mathrm{Cht}_{I,W}^{(I_1,...,I_k)}$ as the *reduced* Deligne–Mumford stack whose points over a scheme $S$ over $\mathbb{F}_q$ classify

$$\Big((x_i)_{i\in I}, \mathcal{G}_0 \xrightarrow{\phi_1} \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0)\Big) \tag{7-2}$$

with

- $x_i \in (X \smallsetminus N)(S)$ for $i \in I$,
- for $i \in \{0, ..., k-1\}$, $\mathcal{G}_i$ is a $G$-bundle over $X \times S$ and we write $\mathcal{G}_k = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0)$ to prepare the next item,
- for $j \in \{1, ..., k\}$
$$\phi_j : \mathcal{G}_{j-1}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I_j}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}_j\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I_j}\Gamma_{x_i})}$$
is an isomorphism such that the relative position of $\mathcal{G}_{j-1}$ with respect to $\mathcal{G}_j$ at $x_i$ (for $i \in I_j$) is bounded by the dominant coweight of $G$ corresponding to the dominant weight of $W_i$.

We can show that the obvious morphism $\mathrm{Cht}_{I,W}^{(I_1,...,I_k)} \to \mathrm{Cht}_{I,W}$ (which forgets the intermediate modifications $\mathcal{G}_1, ..., \mathcal{G}_{k-1}$) gives an isomorphism at the level of intersection cohomology. The interest of $\mathrm{Cht}_{I,W}^{(I_1,...,I_k)}$ is that we have the partial Frobenius morphism $\mathrm{Frob}_{I_1} : \mathrm{Cht}_{I,W}^{(I_1,...,I_k)} \to \mathrm{Cht}_{I,W}^{(I_2,...,I_k,I_1)}$ which sends (7-2) to

$$\Big((x_i')_{i\in I}, \mathcal{G}_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{k-1}} \mathcal{G}_{k-1} \xrightarrow{\phi_k} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_0) \xrightarrow{(\mathrm{Id}_X\times\mathrm{Frob}_S)^*(\phi_1)} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}_1)\Big)$$

where $x_i' = \mathrm{Frob}(x_i)$ if $i \in I_1$ and $x_i' = x_i$ otherwise. Taking $I_1$ to be a singleton we get the action on $H_{I,W}$ of the partial Frobenius morphisms. Thanks to an extra work (using the Hecke-finiteness condition and Eichler–Shimura relations), we are able in

V. Lafforgue [2012] to apply Drinfeld's lemma, and this endows the $\mathbb{Q}_\ell$-vector space $H_{I,W}$ with a continuous action of $\mathrm{Gal}(\overline{F}/F)^I$.

For $I = \varnothing$ and $W = \mathbf{1}$ (the trivial representation), we have

$$H_{\varnothing,\mathbf{1}} = C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell). \tag{7-3}$$

Indeed the $S$-points over $\mathrm{Cht}_{\varnothing,\mathbf{1}}$ classify the $G$-bundles $\mathcal{G}$ over $X \times S$, equipped with an isomorphism

$$\phi : \mathcal{G} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G}).$$

If we see $\mathcal{G}$ as a $S$-point of $\mathrm{Bun}_G$, $(\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})$ is its image by $\mathrm{Frob}_{\mathrm{Bun}_G}$. Therefore $\mathrm{Cht}_{\varnothing,\mathbf{1}}$ classifies the fixed points of $\mathrm{Frob}_{\mathrm{Bun}_G}$ and it is discrete (i.e. of dimension 0) and equal to $\mathrm{Bun}_G(\mathbb{F}_q)$. Therefore the $\ell$-adic cohomology of $\mathrm{Cht}_{\varnothing,\mathbf{1}}/\Xi$ is equal to $C_c(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)$ and in this particular case it is easy to see that Hecke-finiteness is equivalent to cuspidality, so that (7-3) holds true.

Up to now we defined a vector space $H_{I,W}$ for every isomorphism class of irreducible representation $W = \boxtimes_{i\in I} W_i$ of $\widehat{G}^I$. A construction based on the geometric Satake equivalence enables to

**a)** define $H_{I,W}$ functorialy in $W$

**b)** understand the fusion of legs

as explained in the next proposition.

**Proposition 7.4.** *a) For every finite set $I$,*

$$W \mapsto H_{I,W}, \quad u \mapsto \mathcal{H}(u)$$

*is a $\mathbb{Q}_\ell$-linear functor from the category of finite dimensional representations of $\widehat{G}^I$ to the category of finite dimensional and continuous representations of* $\mathrm{Gal}(\overline{F}/F)^I$.

*This means that for every morphism $u : W \to W'$ of representations of $\widehat{G}^I$, we have a morphism $\mathcal{H}(u) : H_{I,W} \to H_{I,W'}$ of representations of* $\mathrm{Gal}(\overline{F}/F)^I$.

*b) For each map $\zeta : I \to J$ between finite sets, we have an isomorphism*

$$\chi_\zeta : H_{I,W} \xrightarrow{\sim} H_{J,W^\zeta}$$

*which is*

- *functorial in $W$, where $W$ is a representation of $\widehat{G}^I$ and $W^\zeta$ denotes the representation of $\widehat{G}^J$ on $W$ obtained by composition with the diagonal morphism*

$$\widehat{G}^J \to \widehat{G}^I, (g_j)_{j\in J} \mapsto (g_{\zeta(i)})_{i\in I}$$

- $\mathrm{Gal}(\overline{F}/F)^J$*-equivariant, where* $\mathrm{Gal}(\overline{F}/F)^J$ *acts on the LHS by the diagonal morphism*

$$\mathrm{Gal}(\overline{F}/F)^J \to \mathrm{Gal}(\overline{F}/F)^I, \ (\gamma_j)_{j\in J} \mapsto (\gamma_{\zeta(i)})_{i\in I},$$

- *and compatible with composition, i.e. for every* $I \xrightarrow{\zeta} J \xrightarrow{\eta} K$ *we have* $\chi_{\eta\circ\zeta} = \chi_\eta \circ \chi_\zeta$.

The statement b) is a bit complicated, here is a basic example of it. For every finite set $I$ we write $\zeta_I : I \to \{0\}$ the tautological map (where $\{0\}$ is an arbitrary choice of notation for a singleton). If $W_1$ and $W_2$ are two representations of $\widehat{G}$, the statement of b) provides a canonical isomorphism

$$\chi_{\zeta_{\{1,2\}}} : H_{\{1,2\},W_1\boxtimes W_2} \xrightarrow{\sim} H_{\{0\},W_1\otimes W_2} \tag{7-4}$$

associated to $\zeta_{\{1,2\}} : \{1,2\} \to \{0\}$. We stress the difference between $W_1 \boxtimes W_2$ which is a representation of $(\widehat{G})^2$ and $W_1 \otimes W_2$ which is a representation of $\widehat{G}$.

Another example of b) is the isomorphism on the left in

$$H_{\{0\},\mathbf{1}} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} H_{\varnothing,\mathbf{1}} \stackrel{(7\text{-}3)}{=} C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell) \tag{7-5}$$

which is associated to $\zeta_\varnothing : \varnothing \to \{0\}$ (the idea of the isomorphism $\chi_{\zeta_\varnothing}$ is that $H_{\varnothing,\mathbf{1}}$ *resp.* $H_{\{0\},\mathbf{1}}$ is the cohomology of the stack of shtukas without legs, *resp.* with a inactive leg, and that they are equal). Thanks to (7-5) we are reduced to construct a decomposition

$$H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} = \bigoplus_\sigma \mathfrak{H}_\sigma. \tag{7-6}$$

**Idea of the proof of proposition 7.4.** We denote by $\mathrm{Cht}_I$ the inductive limit of Deligne–Mumford stacks over $X^I$, defined as $\mathrm{Cht}_{I,W}$ above, but without the condition (7-1) on the relative position. In other words, and with an extra letter $\mathcal{G}'$ to prepare the next definition, the points of $\mathrm{Cht}_I$ over a scheme $S$ over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- two $G$-bundles $\mathcal{G}$ and $\mathcal{G}'$ over $X \times S$,
- a modification $\phi$ at the $x_i$, i.e. an isomorphism

$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$

- an isomorphism $\theta : \mathcal{G}' \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})$.

We introduce the "prestack" $\mathfrak{M}_I$ of "modifications on the formal neighborhood of the $x_i$", whose points over a scheme $S$ over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- two $G$-bundles $\mathcal{G}$ and $\mathcal{G}'$ on the formal completion $\widehat{X \times S}$ of $X \times S$ in the neighborhood of the union of the graphs $\Gamma_{x_i}$,
- a modification $\phi$ at the $x_i$, i.e. an isomorphism

$$\phi : \mathcal{G}\big|_{(\widehat{X\times S})\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'\big|_{(\widehat{X\times S})\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}.$$

The expert reader will notice that for any morphism $S \to X^I$, $\mathfrak{M}_I \times_{X^I} S$ is the quotient of the affine grassmannian of Beilinson–Drinfeld over $S$ by $\Gamma(\widehat{X \times S}, G)$. We have a formally smooth morphism $\epsilon_I : \mathrm{Cht}_I \to \mathfrak{M}_I$ given by restricting $\mathcal{G}$ and $\mathcal{G}'$ to the formal neighborhood of the graphs of the $x_i$ and forgetting $\theta$.

The geometric Satake equivalence, due to Lusztig, Drinfeld, Ginzburg and Mirković–Vilonen Beilinson and Drinfeld [1999] and Mirković and Vilonen [2007], is a fundamental statement which *constructs* $\widehat{G}$ from $G$ and is the cornerstone of the geometric Langlands program. It is a canonical equivalence of tensor categories between

- the category of perverse sheaves on the fiber of $\mathfrak{M}_{\{0\}}$ above any point of $X$ (where $\{0\}$ is an arbitrary notation for a singleton)
- the tensor category of representations of $\widehat{G}$.

For the non expert reader we recall that perverse sheaves, introduced in Beĭlinson, Bernstein, and Deligne [1982], behave like ordinary sheaves and have, in spite of their name, very good properties. An example is given by intersection cohomology sheaves of closed (possibly singular) subvarieties, whose total cohomology is the intersection cohomology of this subvarieties.

The tensor structure on the first category above is obtained by "fusion of legs", thanks to the fact that $\mathfrak{M}_{\{1,2\}}$ is equal to $\mathfrak{M}_{\{0\}} \times \mathfrak{M}_{\{0\}}$ outside the diagonal of $X^2$ and to $\mathfrak{M}_{\{0\}}$ on the diagonal. The first category is tannakian and $\widehat{G}$ is defined as the group of automorphisms of a natural fiber functor.

This equivalence gives, for every representation $W$ of $\widehat{G}^I$, a perverse sheave $\mathcal{S}_{I,W}$ on $\mathfrak{M}_I$, with the following properties:

- $\mathcal{S}_{I,W}$ is functorial in $W$,
- for every surjective map $I \to J$, $\mathcal{S}_{J,W^\zeta}$ is canonically isomorphic to the restriction of $\mathcal{S}_{I,W}$ to $\mathfrak{M}_I \times_{X^I} X^J \simeq \mathfrak{M}_J$, where $X^J \to X^I$ is the diagonal morphism,
- for every irreducible representation $W$, $\mathcal{S}_{I,W}$ is the intersection cohomology sheaf of the closed substack of $\mathfrak{M}_I$ defined by the condition (7-1) on the relative position of the modification $\phi$ at the $x_i$.

Then we define $H_{I,W}$ as the "Hecke-finite" subspace of the cohomology with compact support of $\epsilon_I^*(\mathcal{S}_{I,W})$ on the fiber of $\mathrm{Cht}_I/\Xi$ over a geometric generic point of $X^I$ (or, in fact equivalently, over a geometric generic point of the diagonal $X \subset X^I$). The first two properties above imply a) and b) of the proposition. The third one and the smoothness of $\epsilon_I$ ensure that, for $W$ irreducible, $\epsilon_I^*(\mathcal{S}_{I,W})$ is the intersection cohomology sheaf of $\mathrm{Cht}_{I,W}$ and therefore the new definition of $H_{I,W}$ generalizes the first one using the intersection cohomology of $\mathrm{Cht}_{I,W}$.

## 8 Excursion operators and the main theorem of V. Lafforgue [2012]

Let $I$ be a finite set. Let $(\gamma_i)_{i\in I} \in \mathrm{Gal}(\overline{F}/F)^I$. Let $W$ be a representation of $\widehat{G}^I$ and $x \in W$ and $\xi \in W^*$ be invariant by the diagonal action of $\widehat{G}$. We define the endomorphism $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ of (7-5) as the composition

$$H_{\{0\},\mathbf{1}} \xrightarrow{\mathcal{H}(x)} H_{\{0\},W^{\zeta_I}} \xrightarrow[\sim]{\chi_{\zeta_I}^{-1}} H_{I,W} \xrightarrow{(\gamma_i)_{i\in I}} H_{I,W} \xrightarrow[\sim]{\chi_{\zeta_I}} H_{\{0\},W^{\zeta_I}} \xrightarrow{\mathcal{H}(\xi)} H_{\{0\},\mathbf{1}} \tag{8-1}$$

where $\mathbf{1}$ denotes the trivial representation of $\widehat{G}$, and $x : \mathbf{1} \to W^{\zeta_I}$ and $\xi : W^{\zeta_I} \to \mathbf{1}$ are considered as morphisms of representations of $\widehat{G}$ (we recall that $\zeta_I : I \to \{0\}$ is the obvious map and that $W^{\zeta_I}$ is simply the vector space $W$ equipped with the diagonal action of $\widehat{G}$).

Paraphrasing (8-1) this operator is the composition

- of a creation operator associated to $x$, whose effect is to create legs at the same (generic) point of the curve,
- of a Galois action, which moves the legs on the curve independently from each other, then brings them back to the same (generic) point of the curve,
- of an annihilation operator associated to $\xi$.

It is called an "excursion operator" because it moves the legs on the curve (this is what makes it non trivial).

To $W, x, \xi$ we associate the matrix coefficient $f$ defined by

$$f((g_i)_{i\in I}) = \langle \xi, (g_i)_{i\in I} \cdot x\rangle. \tag{8-2}$$

We see that $f$ is a function on $\widehat{G}^I$ invariant by left and right translations by the diagonal $\widehat{G}$. In other words $f \in \mathcal{O}(\widehat{G}\backslash\widehat{G}^I/\widehat{G})$, where $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ denotes the coarse quotient, defined as the spectrum of the algebra of functions $f$ as above. Unlike the stacky quotients considered before, the coarse quotients are schemes and therefore forget the automorphism groups of points.

For every function $f \in \mathcal{O}(\widehat{G}\backslash\widehat{G}^I/\widehat{G})$ we can find $W, x, \xi$ such that (8-2) holds. We show easily that $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ does not depend on the choice of $W, x, \xi$ satisfying (8-2), and therefore we denote it by $S_{I,f,(\gamma_i)_{i\in I}}$.

The conjectures of Arthur and Kottwitz on multiplicities in vector spaces of automorphic forms and in the cohomologies of Shimura varieties Arthur [1989] and Kottwitz [1990] give, by extrapolation to stacks of shtukas, the following heuristics.

**Remark 8.1.** Heuristically we conjecture that for every global Langlands parameter $\sigma$ there exists a $\overline{\mathbb{Q}_\ell}$-linear representation $\mathfrak{A}_\sigma$ of its centralizer $S_\sigma \subset \widehat{G}$ (factorizing through $S_\sigma/Z_{\widehat{G}}$), so that we have a $\mathrm{Gal}(\overline{F}/F)^I$-equivariant isomorphism

$$H_{I,W} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \overset{?}{=} \bigoplus_\sigma \left(\mathfrak{A}_\sigma \otimes_{\overline{\mathbb{Q}_\ell}} W_{\sigma^I}\right)^{S_\sigma} \tag{8-3}$$

where $W_{\sigma^I}$ is the $\overline{\mathbb{Q}_\ell}$-linear representation of $\mathrm{Gal}(\overline{F}/F)^I$ obtained by composition of the representation $W$ of $\widehat{G}^I$ with the morphism $\sigma^I : \mathrm{Gal}(\overline{F}/F)^I \to \widehat{G}(\overline{\mathbb{Q}_\ell})^I$, and $S_\sigma$ acts diagonally. We conjecture that (8-3) is functorial in $W$, compatible to $\chi_\zeta$ and that it is equal to the decomposition (7-6) when $W = \mathbf{1}$ (so that $\mathfrak{H}_\sigma = (\mathfrak{A}_\sigma)^{S_\sigma}$).

In the heuristics (8-3) the endomorphism $S_{I,f,(\gamma_i)_{i\in I}} = S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ of

$$H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} H_{\varnothing,\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \overset{?}{=} \bigoplus_{\sigma} (\mathfrak{A}_\sigma)^{S_\sigma}$$

acts on $(\mathfrak{A}_\sigma)^{S_\sigma}$ by the composition

$$(\mathfrak{A}_\sigma)^{S_\sigma} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}} (\mathfrak{A}_\sigma \otimes \mathbf{1})^{S_\sigma} \xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes x} (\mathfrak{A}_\sigma \otimes W_{\sigma^I})^{S_\sigma} \xrightarrow{(\sigma(\gamma_i))_{i\in I}} (\mathfrak{A}_\sigma \otimes W_{\sigma^I})^{S_\sigma}$$
$$\xrightarrow{\mathrm{Id}_{\mathfrak{A}_\sigma} \otimes \xi} (\mathfrak{A}_\sigma \otimes \mathbf{1})^{S_\sigma} \xrightarrow[\sim]{\chi_{\zeta_\varnothing}^{-1}} (\mathfrak{A}_\sigma)^{S_\sigma}$$

i.e. by the scalar

$$\langle \xi, (\sigma(\gamma_i))_{i\in I} \cdot x\rangle = f\big((\sigma(\gamma_i))_{i\in I}\big).$$

In other words we should have

(8-4) $\qquad \mathfrak{H}_\sigma \overset{?}{=}$ eigenspace of the $S_{I,f,(\gamma_i)_{i\in I}}$ with the eigenvalues $f\big((\sigma(\gamma_i))_{i\in I}\big)$.

The heuristics (8-4) clearly indicates the path to follow. We show in V. Lafforgue [2012] that the $S_{I,f,(\gamma_i)_{i\in I}}$ generate a commutative $\mathbb{Q}_\ell$-algebra $\mathfrak{B}$ and satisfy some properties implying the following proposition.

**Proposition 8.2.** *For each character $\nu$ of $\mathfrak{B}$ with values in $\overline{\mathbb{Q}_\ell}$ there exists a unique global Langlands parameter $\sigma$ such that for all $I$, $f$ and $(\gamma_i)_{i\in I}$, we have*

(8-5)
$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}).$$

The unicity of $\sigma$ in the previous proposition comes from the fact that, for any integer $n$, taking $I = \{0, ..., n\}$, the coarse quotient $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ identifies with the coarse quotient of $(\widehat{G})^n$ by diagonal conjugation by $\widehat{G}$, and therefore, for any $(\gamma_1, ..., \gamma_n) \in (\mathrm{Gal}(\overline{F}/F))^n$, (8-5) applied to $(\gamma_i)_{i\in I} = (1, \gamma_1, ..., \gamma_n)$ determines $(\sigma(\gamma_1), ..., \sigma(\gamma_n))$ up to conjugation and semisimplification (thanks to Richardson [1988]). The existence and continuity of $\sigma$ are justified thanks to relations and topological properties satisfied by the excursion operators.

Since $\mathfrak{B}$ is commutative we obtain a canonical spectral decomposition $H_{\{0\},\mathbf{1}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} = \bigoplus_\nu \mathfrak{H}_\nu$ where the direct sum is taken over characters $\nu$ of $\mathfrak{B}$ with values in $\overline{\mathbb{Q}_\ell}$. Associating to each $\nu$ a unique global Langlands parameter $\sigma$ as in the previous proposition, we deduce the decomposition (7-6) we wanted to construct. We do not know if $\mathfrak{B}$ is reduced.

Moreover the unramified Hecke operators are particular cases of excursion operators: for every place $v$ and for every irreducible representation $V$ of $\widehat{G}$ with character $\chi_V$, the unramified Hecke operator $T_{V,v}$ is equal to the excursion operator $S_{\{1,2\},f,(\mathrm{Frob}_v,1)}$ where $f \in \mathcal{O}(\widehat{G}\backslash(\widehat{G})^2/\widehat{G})$ is given by $f(g_1, g_2) = \chi_V(g_1 g_2^{-1})$, and $\mathrm{Frob}_v$ is a Frobenius element at $v$. This is proven in V. Lafforgue [2012] by a geometric argument (essentially a computation of the intersection of algebraic cycles in a stack of shtukas). It implies the compatibility of the decomposition (7-6) with the Satake isomorphism at all places.

**Remark 8.3.** By the Chebotarev density theorem, the subalgebra of $\mathcal{B}$ generated by all the Hecke algebras $\mathcal{H}_v$ is equal to the subalgebra generated by the excursion operators with $\sharp I = 2$. The remarks at the end of section 6 show that in general it is necessary to consider excursion operators with $\sharp I > 2$ to generate the whole algebra $\mathcal{B}$.

Finally we can state the main theorem.

**Theorem 8.4.** *We have a canonical decomposition of $\overline{\mathbb{Q}_\ell}$-vector spaces*

$$C_c^{\text{cusp}}(\text{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma, \tag{8-6}$$

*where the direct sum in the RHS is indexed by global Langlands parameters , i.e. $\widehat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy classes of morphisms $\sigma : \text{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(X, \overline{\eta})$, defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.*

*This decomposition is uniquely determined by the following property : $\mathfrak{H}_\sigma$ is equal to the generalized eigenspace associated to the character $\nu$ of $\mathcal{B}$ defined by*

$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}. \tag{8-7}$$

*This decomposition is respected by the Hecke operators and is compatible with the Satake isomorphism at all places $v$ of $X$.*

Everything is still true with a level (a finite subscheme $N$ of $X$). We denote by $\text{Bun}_{G,N}(\mathbb{F}_q)$ the set of isomorphism classes of $G$-bundles over $X$ trivialized on $N$. Then we have a canonical decomposition

$$C_c^{\text{cusp}}(\text{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma, \tag{8-8}$$

where the direct sum is taken over global Langlands parameters $\sigma : \pi_1(X \smallsetminus N, \overline{\eta}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. This decomposition is respected by all Hecke operators and compatible with the Satake isomorphism at all places of $X \smallsetminus N$. If $G$ is split we have, by Thắng [2011],

$$\text{Bun}_{G,N}(\mathbb{F}_q) = G(F)\backslash G(\mathbb{A})/K_N \tag{8-9}$$

(where $\mathbb{A}$ is the ring of adèles, $\mathbb{O}$ is the ring of integral adèles, $\mathcal{O}_N$ the ring of functions on $N$ and $K_N = \text{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N)))$. When $G$ is non necessarily split the RHS of (8-9) must be replaced by a direct sum, indexed by the finite group $\ker^1(F, G)$, of adelic quotients for inner forms of $G$ and in the definition of global Langlands parameters we must replace $\widehat{G}$ by the $L$-group (see Borel [1979] for $L$-groups).

We have a statement similar to theorem 8.4 with coefficients in $\overline{\mathbb{F}_\ell}$ instead of $\overline{\mathbb{Q}_\ell}$.

We can also consider the case of metaplectic groups thanks to the metaplectic variant of the geometric Satake equivalence due to Finkelberg and Lysenko [2010], Lysenko [2014], and Gaitsgory and Lysenko [2016].

**Remark 8.5.** Drinfeld gave an idea to prove something like the heuristics (8-3) but it is a bit difficult to formulate the result. Let Reg be the left regular representation of $\widehat{G}$ with coefficients in $\mathbb{Q}_\ell$ (considered as an inductive limit of finite dimensional representations). We can endow the $\mathbb{Q}_\ell$-vector space $H_{\{0\},\text{Reg}}$ (of infinite dimension in general) with

a) a structure of module over the algebra of functions on the "affine space $\mathcal{S}$ of morphisms $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}$ with coefficients in $\mathbb{Q}_\ell$-algebras",

b) an algebraic action of $\widehat{G}$ (coming from the right action of $\widehat{G}$ on Reg) which is compatible with conjugation by $\widehat{G}$ on $\mathcal{S}$.

The space $\mathcal{S}$ is not rigorously defined and the rigorous definition of structure a) is the following. For any finite dimensional $\mathbb{Q}_\ell$-linear representation $V$ of $\widehat{G}$, with underlying vector space $\underline{V}$, $H_{\{0\},\mathrm{Reg}} \otimes \underline{V}$ is equipped with an action of $\mathrm{Gal}(\overline{F}/F)$, making it an inductive limit of finite dimensional continuous representations of $\mathrm{Gal}(\overline{F}/F)$, as follows. We have a $\widehat{G}$-equivariant isomorphism

$$\begin{aligned} \theta : \mathrm{Reg} \otimes \underline{V} &\simeq \mathrm{Reg} \otimes V \\ f \otimes x &\mapsto [g \mapsto f(g) g.x] \end{aligned}$$

where $\widehat{G}$ acts diagonally on the RHS, and to give a meaning to the formula the RHS is identified with the vector space of algebraic functions $\widehat{G} \to V$. Therefore we have an isomorphism

$$H_{\{0\},\mathrm{Reg}} \otimes \underline{V} = H_{\{0\},\mathrm{Reg} \otimes \underline{V}} \xrightarrow[\sim]{\theta} H_{\{0\},\mathrm{Reg} \otimes V} \simeq H_{\{0,1\},\mathrm{Reg} \boxtimes V}$$

where the first equality is tautological (since $\underline{V}$ is just a vector space) and the last isomorphism is the inverse of the fusion isomorphism $\chi_{\zeta_{\{0,1\}}}$ of (7-4). Then the action of $\mathrm{Gal}(\overline{F}/F)$ on the LHS is defined as the action of $\mathrm{Gal}(\overline{F}/F)$ on the RHS corresponding to the leg 1. If $V_1$ and $V_2$ are two representations of $\widehat{G}$, the two actions of $\mathrm{Gal}(\overline{F}/F)$ on $H_{\{0\},\mathrm{Reg}} \otimes \underline{V_1} \otimes \underline{V_2}$ associated to the actions of $\widehat{G}$ on $V_1$ and $V_2$ commute with each other and the diagonal action of $\mathrm{Gal}(\overline{F}/F)$ is the action associated to the diagonal action of $\widehat{G}$ on $V_1 \otimes V_2$. This gives a structure as we want in a) because if $V$ is as above, $x \in V$, $\xi \in V^*$, $f$ is the function on $\widehat{G}$ defined as the matrix coefficient $f(g) = \langle \xi, g.x \rangle$, and $\gamma \in \mathrm{Gal}(\overline{F}/F)$ then we say that $F_{f,\gamma} : \sigma \mapsto f(\sigma(\gamma))$, considered as a "function on $\mathcal{S}$", acts on $H_{\{0\},\mathrm{Reg}}$ by the composition

$$H_{\{0\},\mathrm{Reg}} \xrightarrow{\mathrm{Id} \otimes x} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\gamma} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\mathrm{Id} \otimes \xi} H_{\{0\},\mathrm{Reg}}.$$

Any function $f$ on $\widehat{G}$ can be written as such a matrix coefficient, and the functions $F_{f,\gamma}$ when $f$ and $\gamma$ vary are supposed to "generate topologically all functions on $\mathcal{S}$". The property above with $V_1$ and $V_2$ implies relations among the $F_{f,\gamma}$, namely that $F_{f,\gamma_1\gamma_2} = \sum_\alpha F_{f_1^\alpha,\gamma_1} F_{f_2^\alpha,\gamma_2}$ if the image of $f$ by comultiplication is $\sum_\alpha f_1^\alpha \otimes f_2^\alpha$. In Xinwen [2017] Xinwen Zhu gives an equivalent construction of the structure a). Structures a) and b) are compatible in the following sense: the conjugation $g F_{f,\gamma} g^{-1}$ of the action of $F_{f,\gamma}$ on $H_{\{0\},\mathrm{Reg}}$ by the algebraic action of $g \in \widehat{G}$ is equal to the action of $F_{f^g,\gamma}$ where $f^g(h) = f(g^{-1}hg)$.

The structures a) and b) give rise to a "$\mathcal{O}$-module on the stack $\mathcal{S}/\widehat{G}$ of global Langlands parameters" (such that the vector space of its "global sections on $\mathcal{S}$" is $H_{\{0\},\mathrm{Reg}}$). For any morphism $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, we want to define $\mathfrak{A}_\sigma$ as the fiber of this

$\mathcal{O}$-module at $\sigma$ (considered as a "$\overline{\mathbb{Q}_\ell}$-point of $\mathcal{S}$ whose automorphism group in the stack $\mathcal{S}/\widehat{G}$ is $S_\sigma$"). Rigorously we define $\mathfrak{A}_\sigma$ as the biggest quotient of $H_{\{0\},\mathrm{Reg}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ on which any function $F_{f,\gamma}$ as above acts by multiplication by the scalar $f(\sigma(\gamma))$, and $S_\sigma$ acts on $\mathfrak{A}_\sigma$. If the heuristics (8-3) is true it is the same as $\mathfrak{A}_\sigma$ from the heuristics. When $\sigma$ is elliptic (i.e. when $S_\sigma/Z_{\widehat{G}}$ is finite), $\sigma$ is "isolated in $\mathcal{S}$" in the sense that it cannot be deformed (among continuous morphisms whose composition with the abelianization of $\widehat{G}$ is of fixed finite order) and, as noticed by Xinwen Zhu, heuristics (8-3) is true when we restrict on both sides to the parts lying over $\sigma$. In general due to deformation of some non elliptic $\sigma$ there could a priori be nilpotents, and for example we don't know how to prove that $\mathcal{B}$ is reduced so we don't know how to prove the heuristics (8-3).

We can see the heuristics (8-3), and the structures a) and b) above, as an illustration of the general idea that, in a spectral decomposition, when the points of the spectrum naturally have automorphism groups, the multiplicities should be associated to representations of these groups. By contrast the algebra $\mathcal{B}$ of excursion operators gives the spectral decomposition with respect to the coarse quotient associated to $\mathcal{S}/\widehat{G}$, where we forget the automorphism groups $S_\sigma$.

**Remark 8.6.** The previous remark makes sense although $\mathcal{S}$ was not defined. To define a space like $\mathcal{S}$ rigorously it may be necessary to consider continuous morphisms $\sigma$ : $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}$ with coefficients in $\mathbb{Z}_\ell$-algebras where $\ell$ is nilpotent (such $\sigma$ have finite image), and $\mathcal{S}$ would be an ind-scheme over $\mathrm{Spf}\,\mathbb{Z}_\ell$. Then to define structure a) we would need to consider Reg with coefficients in $\mathbb{Z}_\ell$, and, for any representation $W$ of $\widehat{G}^I$ with coefficients in $\mathbb{Z}_\ell$, to construct $H_{I,W}$ as a $\mathbb{Z}_\ell$-module.

## 9 Local aspects: joint work with Alain Genestier

In Genestier and V. Lafforgue [2017], Alain Genestier and I construct the local parameterization up to semisimplification and the local-global compatibility.

Let $G$ be a reductive group over a local field $K$ of equal characteristics. We recall that the Bernstein center of $G(K)$ is defined, in two equivalent ways, as

- the center of the category of smooth representations of $G(K)$,
- the algebra of central distributions on $G(K)$ acting as multipliers on the algebra of locally constant functions with compact support.

On every $\overline{\mathbb{Q}_\ell}$-linear irreducible smooth representation of $G(K)$, the Bernstein center acts by a character.

The main result of Genestier and V. Lafforgue [ibid.] associates to any character $\nu$ of the Bernstein center of $G(K)$ with values in $\overline{\mathbb{Q}_\ell}$ a local Langlands parameter $\sigma_K(\nu)$ up to semisimplification , i.e. (assuming $G$ split to simplify) a conjugacy class of morphisms $\mathrm{Weil}(\overline{K}/K) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ defined over a finite extension of $\mathbb{Q}_\ell$, continuous and semisimple.

We show in Genestier and V. Lafforgue [ibid.] the local-global compatibility up to semisimplication, whose statement is the following. Let $X$ be a smooth projective and geometrically irreducible curve over $\mathbb{F}_q$ and let $N$ be a level. Then if $\sigma$ is a global

Langlands parameter and if $\pi = \bigotimes \pi_v$ is an irreducible representation of $G(\mathbb{A})$ such that $\pi^{K_N}$ is non zero and appears in $\mathfrak{H}_\sigma$ in the decomposition (8-8), then for every place $v$ de $X$ we have equality between

- the semisimplification of the restriction of $\sigma$ to $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$,
- the semisimple local parameter $\sigma_K(\nu)$ where $\nu$ is the character of the Bernstein center by which it acts on the irreducible smooth representation $\pi_v$ of $G(K)$.

We use nearby cycles on arbitrary bases (Deligne, Laumon, Gabber, Illusie, Orgogozo), which are defined on oriented products of toposes and whose properties are proven in Orgogozo [2006] (see also Illusie [2006] for an excellent survey). Technically we show that if all the $\gamma_i$ are in $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$ then the global excursion operator $S_{I,f,(\gamma_i)_{i\in I}} \in \mathrm{End}\big(C_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q)/\Xi, \mathbb{Q}_\ell)\big)$ acts by multiplication by an element $\mathfrak{z}_{I,f,(\gamma_i)_{i\in I}}$ of the $\ell$-adic completion of the Bernstein center of $G(F_v)$ which depends only on the local data at $v$. We construct $\mathfrak{z}_{I,f,(\gamma_i)_{i\in I}}$ using stacks of "restricted shtukas", which are analogues of truncated Barsotti–Tate groups.

**Remark 9.1.** In the case of $GL_r$ the local correspondence was known by Laumon, Rapoport, and Stuhler [1993] and the local-global compatibility (without semisimplification) was proven in L. Lafforgue [2002]. Badulescu and Henniart explained us that in general we cannot hope more that the local-global compatibility up to semisimplication.

## 10 Independence on $\ell$

Grothendieck motives (over a given field) form a $\overline{\mathbb{Q}}$-linear category and unify the $\ell$-adic cohomologies (of varieties over this field) for different $\ell$: a motive is "a factor in a universal cohomology of a variety". We consider here motives over $F$. We conjecture that the decomposition

$$C_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi, \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma$$

we have constructed is defined over $\overline{\mathbb{Q}}$ (instead of $\overline{\mathbb{Q}_\ell}$), indexed by motivic Langlands parameters $\sigma$, and independent on $\ell$. This conjecture seems out of reach for the moment.

The notion of motivic Langlands parameter is clear if we admit the standard conjectures. A motivic Langlands parameter defined over $\overline{\mathbb{Q}}$ would give rise to a "compatible" family of morphisms $\sigma_{\ell,\iota} : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ for any $\ell$ not dividing $q$ and any embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. When $G = GL_r$, the condition of compatibility is straightforward (the traces of the Frobenius elements should belong to $\overline{\mathbb{Q}}$ and be the same for all $\ell$ and $\iota$) and the fact that any irreducible representation (with determinant of finite order) for some $\ell$ belongs to such a family (and has therefore "compagnons" for other $\ell$ and $\iota$) was proven as a consequence of the Langlands correspondence in L. Lafforgue [ibid.]. It was generalized in the two following independent directions

- Abe [2013] used the crystalline cohomology of stacks of shtukas to construct crystalline compagnons,

- when $F$ is replaced by the field of rational functions of any smooth variety over $\mathbb{F}_q$, Deligne proved that the traces of Frobenius elements belong to a finite extension of $\mathbb{Q}$ and Drinfeld constructed these compatible families Deligne [2012], Drinfeld [2012], and Esnault and Kerz [2012].

For a general reductive group $G$ the notion of compatible family is subtle (because the obvious condition on the conjugacy classes of the Frobenius elements is not sufficient). In Drinfeld [2018] Drinfeld gave the right conditions to define compatible families and proved that any continuous semisimple morphism $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ factorizing through $\pi_1(U, \overline{\eta})$ for some open dense $U \subset X$ (and such that the Zariski closure of its image is semisimple) belongs to a unique compatible family.

## 11 Conjectures on Arthur parameters

We hope that all global Langlands parameters $\sigma$ which appear in this decomposition come from elliptic Arthur parameters, i.e. conjugacy classes of continuous semisimple morphisms $\mathrm{Gal}(\overline{F}/F) \times SL_2(\overline{\mathbb{Q}_\ell}) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ whose centralizer is finite modulo the center of $\widehat{G}$. This $SL_2$ should be related to the Lefschetz $SL_2$ acting on the intersection cohomology of compactifications of stacks of shtukas. We even hope a parameterization of the vector space of discrete automorphic forms (and not only cuspidal ones) indexed by elliptic Arthur parameters.

Moreover we expect that generic cuspidal automorphic forms appear exactly in $\mathfrak{H}_\sigma$ such that $\sigma$ is elliptic as a Langlands parameter (i.e. that it comes from an elliptic Arthur parameter with trivial $SL_2$ action). This would imply the Ramanujan–Petersson conjecture (an archimedean estimate on Hecke eigenvalues).

By Drinfeld and Kedlaya [2016] the conjectures above would also imply $p$-adic estimates on Hecke eigenvalues which would sharper than those in V. Lafforgue [2011].

## 12 Recent works on the Langlands program for function fields in relation with shtukas

In Böckle, Harris, Khare, and Thorne [2016] G. Böckle, M. Harris, C. Khare, and J. Thorne apply the results explained in this text together with Taylor–Wiles methods to prove (in the split and everywhere unramified situation) the potential automorphy of all Langlands parameters with Zariski-dense image. Thus they prove a weak form of the "Galois to automorphic" direction.

In Yun and Zhang [2017] Zhiwei Yun and Wei Zhang proved analogues of the Gross–Zagier formula, namely equality between the intersection numbers of some algebraic cycles in stacks of shtukas and special values of derivatives of L-functions (of arbitrary order equal to the number of legs).

In Xiao and Zhu [2017] Liang Xiao and Xinwen Zhu construct algebraic cycles in special fibers of Shimura varieties. Their construction was inspired by the case of the stacks of shtukas and is already new in this case (it gives a conceptual setting for the Eichler–Shimura relations used in V. Lafforgue [2012]).

## 13 Geometric Langlands program

The results explained above are based on the geometric Satake equivalence Mirković and Vilonen [2007], and are inspired by the factorization structures studied by Beilinson and Drinfeld [2004]. The geometric Langlands program was pioneered by Drinfeld [1983] and Laumon [1987a], and then developed itself in two variants, which we will discuss in turn.

**13.1 Geometric Langlands program for $\ell$-adic sheaves.** Let $X$ be a smooth projective curve over an algebraically closed field of characteristic different from $\ell$.

For any representation $W$ of $\widehat{G}^I$ the Hecke functor

$$\phi_{I,W} : D_c^b(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) \to D_c^b(\mathrm{Bun}_G \times X^I, \overline{\mathbb{Q}_\ell})$$

is given by

$$\phi_{I,W}(\mathfrak{F}) = q_{1,!}\big(q_0^*(\mathfrak{F}) \otimes \mathfrak{F}_{I,W}\big)$$

where $\mathrm{Bun}_G \xleftarrow{q_0} \mathrm{Hecke}_I \xrightarrow{q_1} \mathrm{Bun}_G \times X^I$ is the Hecke correspondence classifying modifications of a $G$-bundle at the $x_i$, and $\mathfrak{F}_{I,W}$ is defined as the inverse image of $\mathcal{S}_{I,W}$ by the natural formally smooth morphism $\mathrm{Hecke}_I \to \mathfrak{M}_I$.

Let $\sigma$ be a $\widehat{G}$-local system on $X$. Then $\mathfrak{F} \in D_c^b(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$ is said to be an eigensheaf for $\sigma$ if we have, for any finite set $I$ and any representation $W$ of $(\widehat{G})^I$, an isomorphism $\phi_{I,W}(\mathfrak{F}) \xrightarrow{\sim} \mathfrak{F} \boxtimes W_\sigma$, functorial in $W$ and compatible to exterior tensor products and fusion. The conjecture of the geometric Langlands program claims the existence of an $\sigma$-eigensheaf $\mathfrak{F}$ (it should also satisfy a Whittaker normalization condition which in particular prevents it to be 0). For $G = GL_r$ this conjecture was proven by Frenkel, Gaitsgory, and Vilonen [2002] and Gaitsgory [2004]

When $X$, $\mathrm{Bun}_G$, $\sigma$ and $\mathfrak{F}$ are defined over $\mathbb{F}_q$ (instead of $\overline{\mathbb{F}_q}$), a construction of Varshavsky [2004a] produces subspaces of cohomology classes in the stacks of shtukas and this allows to show that the function given by the trace of Frobenius on $\mathfrak{F}$ belongs to the factor $\mathfrak{H}_\sigma$ of decomposition (8-6), as explained in section 15 of V. Lafforgue [2012].

The $\ell$-adic setting is truly a geometrization of automorphic forms over function fields, and many constructions were geometrized: Braverman and Gaitsgory [2002] geometrized Eisenstein series, and Lysenko geometrized in particular Rankin–Selberg integrals Lysenko [2002], theta correspondences Lysenko [2006, 2011] and V. Lafforgue and Lysenko [2013], and several constructions for metaplectic groups Lysenko [2015a, 2017].

**13.2 Geometric Langlands program for $D$-modules.** Now let $X$ be a smooth projective curve over an algebraically closed field of characteristic 0. A feature of the setting of $D$-modules is that one can upgrade the statement of Langlands correspondence to a conjecture about an equivalence between categories on the geometric and spectral sides, respectively. See Gaitsgory [2015a] for a precise statement of the conjecture and Gaitsgory [2017] for a survey of recent progress. Such an equivalence can in principle make sense due to the fact that Galois representations into $\widehat{G}$, instead of being taken individually, now form an algebraic stack $\mathrm{LocSys}_{\widehat{G}}$ classifying $\widehat{G}$-local systems, i.e. $\widehat{G}$-bundles with connection (by contrast one does not have such an algebraic stack in the $\ell$-adic setting).

On the geometric side, one considers the derived category of D-modules on $\mathrm{Bun}_G$, or rather a stable $\infty$-category enhancing it. It is denoted $\text{D-mod}(\mathrm{Bun}_G)$ and is defined and studied in Drinfeld and Gaitsgory [2015]. The category on the spectral side is a certain modification of $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$, the (derived or rather $\infty$-) category of quasi-coherent sheaves on the stack $\mathrm{LocSys}_{\widehat{G}}$. The modification in question is denoted $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$, and it has to do with the fact that $\mathrm{LocSys}_{\widehat{G}}$ is not smooth, but rather *quasi-smooth* (a.k.a. derived locally complete intersection). The difference between $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ and $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ is measured by *singular support* of coherent sheaves, a theory developed in Arinkin and Gaitsgory [2015]. The introduction of Nilp in Arinkin and Gaitsgory [ibid.] was motivated by the case of $\mathbb{P}^1$ V. Lafforgue [n.d.] and the study of the singular support of the geometric Eisenstein series. In terms of Langlands correspondence, this singular support may also be seen as accounting for *Arthur parameters*. More precisely the singularities of $\mathrm{LocSys}_{\widehat{G}}$ are controlled by a stack $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ over $\mathrm{LocSys}_{\widehat{G}}$ whose fiber over a point $\sigma$ is the $H^{-1}$ of the cotangent complex at $\sigma$, equal to $H^2_{dR}(X, \widehat{\mathfrak{g}}_\sigma)^* \simeq H^0_{dR}(X, \widehat{\mathfrak{g}}^*_\sigma) \simeq H^0_{dR}(X, \widehat{\mathfrak{g}}_\sigma)$ where the first isomorphism is Poincaré duality and the second depends on the choice of a non-degenerate $ad$-invariant symmetric bilinear form on $\widehat{\mathfrak{g}}$. Therefore $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ is identified to the stack classifying $(\sigma, A)$, with $\sigma \in \mathrm{LocSys}_{\widehat{G}}$ and $A$ an horizontal section of the local system $\widehat{\mathfrak{g}}_\sigma$ associated to $\sigma$ with the adjoint representation of $\widehat{G}$. Then Nilp is the cone of $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$ defined by the condition that $A$ is nilpotent. By the Jacobson–Morozov theorem, any such $A$ is the nilpotent element associated to a morphism of $SL_2$ to the centralizer of $\sigma$ in $\widehat{G}$, i.e. it comes from an Arthur parameter. The singular support of a coherent sheaf on $\mathrm{LocSys}_{\widehat{G}}$ is a closed substack in $\mathrm{Sing}(\mathrm{LocSys}_{\widehat{G}})$. The category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ (compared to $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$) corresponds to the condition that the singular support of coherent sheaves has to be included in Nilp (compared to the zero section where $A = 0$). The main conjecture is that there is an equivalence of categories

$$\text{D-mod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}}). \tag{13-1}$$

Something weaker is known: by Gaitsgory [2015a], $\text{D-mod}(\mathrm{Bun}_G)$ "lives" over $\mathrm{LocSys}_{\widehat{G}}$ in the sense that $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$, viewed as a monoidal category, acts naturally on $\text{D-mod}(\mathrm{Bun}_G)$. Note that $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ acts on $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{G}})$ (one can tensor a coherent complex by a perfect one and obtain a new coherent complex) and the conjectured equivalence (13-1) should be compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{G}})$ on both sides.

Theorem 8.4 (refined in remark 8.5) can be considered as an arithmetic analogue of the fact that $\text{D-mod}(\mathrm{Bun}_G)$ "lives" over $\mathrm{LocSys}_{\widehat{G}}$ (curiously, due to the lack of an $\ell$-adic analogue of $\mathrm{LocSys}_{\widehat{G}}$, that result does not have an analogue in the $\ell$-adic geometric Langlands program, even if the vanishing conjecture proven by Gaitsgory [2004] goes in this direction). And the fact that Arthur multiplicities formula is still unproven in general is parallel to the fact that the equivalence (13-1) is still unproven.

When $G = T$ is a torus, there is no difference between $\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{T}})$ and $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\widehat{T}})$. In this case, the desired equivalence

$$\mathrm{QCoh}(\mathrm{LocSys}_{\widehat{T}}) \simeq \text{D-mod}(\mathrm{Bun}_T)$$

is a theorem, due to Laumon [1996].

The formulation of the geometric Langlands correspondence as an equivalence of categories (13-1), and even more the proofs of the results, rely on substantial developments in the technology, most of which had to do with the incorporation of the tools of higher category theory and higher algebra, developed by Lurie [2009, n.d.]. Some of the key constructions use categories of D-modules and quasi-coherent sheaves on algebro-geometric objects more general than algebraic stacks (a typical example is the moduli space of $G$-bundles on $X$ equipped with a reduction to a subgroup *at the generic point of* $X$).

**13.3 Work of Gaitsgory and Lurie on Weil's conjecture on Tamagawa numbers over function fields.** In Gaitsgory and Lurie [n.d.(a),(b)] (see also Gaitsgory [2015b]) Gaitsgory and Lurie compute the cohomology with coefficients in $\mathbb{Z}_\ell$ of the stack $\mathrm{Bun}_G$ when $X$ is any smooth projective curve over an algebraically closed field of characteristic other than $\ell$, and $G$ is a smooth affine group scheme over X with connected fibers, whose generic fiber is semisimple and simply connected. They use in particular a remarkable geometric ingredient, belonging to the same framework of factorization structures Beilinson and Drinfeld [2004] (which comes from conformal field theory) as the geometric Satake equivalence. The Ran space of $X$ is, loosely speaking, the prestack classifying non-empty finite subsets $Z$ of $X$. The affine grassmannian $\mathrm{Gr}_{Ran}$ is the prestack over the Ran space classifying such a $Z$, a $G$-bundle $\mathcal{G}$ on $X$, and a trivialization $\alpha$ of $\mathcal{G}$ on $X \setminus Z$. Then the remarkable geometric ingredient is that the obvious morphism $\mathrm{Gr}_{Ran} \to \mathrm{Bun}_G, (Z, \mathcal{G}, \alpha) \mapsto \mathcal{G}$ has contractible fibers in some sense and gives an isomorphism on homology. Note that when $k = \mathbb{C}$ and $G$ is constant on the curve, their formula implies the well-known Atiyah–Bott formula for the cohomology of $\mathrm{Bun}_G$, whose usual proof is by analytic means.

Now assume that the curve $X$ is over $\mathbb{F}_q$. By the Grothendieck–Lefschetz trace formula their computation of the cohomology of $\mathrm{Bun}_G$ over $\overline{\mathbb{F}_q}$ gives a formula for $|\mathrm{Bun}_G(\mathbb{F}_q)|$, the number of $\mathbb{F}_q$-points on the stack $\mathrm{Bun}_G$. Note that since $\mathrm{Bun}_G$ is a stack, each isomorphism class $y$ of points is weighted by $\frac{1}{\mathrm{Aut}_y(\mathbb{F}_q)}$, where $\mathrm{Aut}_y$ is the algebraic group of automorphisms of $y$, and $\mathrm{Aut}_y(\mathbb{F}_q)$ is the finite group of its $\mathbb{F}_q$-points. Although the set of isomorphism classes $y$ of points is infinite, the weighted sum converges. Gaitsgory and Lurie easily reinterpret $|\mathrm{Bun}_G(\mathbb{F}_q)|$ as the volume (with respect to some measure) of the quotient $G(\mathbb{A})/G(F)$ (where $F$ is the function field of $X$ and $\mathbb{A}$ is its ring of adèles) and prove in this way, in the case of function fields, a formula for the volume of $G(\mathbb{A})/G(F)$ as a product of local factors at all places. This formula, called the Tamagawa number formula, had been conjectured by Weil for any global field $F$.

Over number fields $\mathrm{Bun}_G$ does not make sense, only the conjecture of Weil on the Tamagawa number formula remains and it had been proven by Kottwitz after earlier works of Langlands and Lai by completely different methods (residues of Eisenstein series and trace formulas).

## 14 Homage to Alexandre Grothendieck (1928–2014)

Modern algebraic geometry was built by Grothendieck, together with his students, in the realm of categories: functorial definition of schemes and stacks, Quot construction for $\mathrm{Bun}_G$, tannakian formalism, topos, étale cohomology, motives. His vision of topos and motives already had tremendous consequences and others are certainly yet to come. He also had a strong influence outside of his school, as testified by the rise of higher categories and the work of Beilinson, Drinfeld, Gaitsgory, Kontsevich, Lurie, Voevodsky (who, sadly, passed away recently) and many others. He changed not only mathematics, but also the way we think about it.

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