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NONHYPERBOLIC ERGODIC MEASURES

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Abstract

We discuss some methods for constructing nonhyperbolic ergodic measures and their applications in the setting of nonhyperbolic skew-products, homoclinic classes, and robustly transitive diffeomorphisms.

to Wellington de Melo in memoriam

1 The transitive and nonhyperbolic setting

Irrational rotations of the circle $\mathbb{T}^1$ and Anosov maps of the two-torus $\mathbb{T}^2$ are emblematic examples of *transitive* systems (existence of a dense orbit). Small perturbations of Anosov systems are also transitive. This property fails however for irrational rotations. Anosov diffeomorphisms are also paradigmatic examples of hyperbolic maps and, by definition, hyperbolicity persists by small perturbations. Our focus are systems which are *robustly transitive*. In dimension three or higher, there are important examples of those systems that fail to be hyperbolic. They are one of the main foci of this paper. A second focus is on nonhyperbolic elementary pieces of dynamics. We discuss how their lack of hyperbolicity is reflected at the ergodic level by the existence of nonhyperbolic ergodic measures. We also study how this influences the structure of the space of measures. In this discussion, we see how this sort of dynamics gives rise to robust cycles and blenders.

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1.1 Paradigmatic examples. Let us start with an important class of examples, which are also the simplest ones, called *skew-products*. Consider the space $\Sigma_N = \{0, \dots, N-1\}^{\mathbb{Z}}$, $N \geq 2$, of bi-infinite sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ endowed with the usual metric $d(\xi,\eta) \stackrel{\text{def}}{=} 2^{-n(\xi,\eta)}$, where $n(\xi,\eta) \stackrel{\text{def}}{=} \inf\{|\ell| : \xi_i \neq \eta_i \text{ for } i = -\ell, \dots, \ell\}$, and the *shift map* $\sigma\colon \Sigma_N \to \Sigma_N$, $\sigma(\xi_i) \stackrel{\text{def}}{=} (\xi_i')$, $\xi_i' \stackrel{\text{def}}{=} \xi_{i+1}$. This map is transitive and has a dense subset of periodic points. Consider now a compact manifold K and a family of diffeomorphisms $f_\xi\colon K \to K$, $\xi \in \Sigma_N$, depending "nicely" on ξ. Associated to these maps we consider the *skew-product*

$$F\colon \Sigma_N \times K \to \Sigma_N \times K, \quad F(\xi,x) \stackrel{\text{def}}{=} (\sigma(\xi), f_\xi(x)). \tag{1-1}$$

The maps f_ξ are called *fiber maps*. The simplest case occurs when $f_\xi = f_{\xi_0}$ and then the system is called a *(one-)step skew-product*. There is a differentiable version of this model. Take (for instance) an Anosov diffeomorphism $A\colon \mathbb{T}^2 \to \mathbb{T}^2$ and fiber maps $f_X\colon K \to K$, $X \in \mathbb{T}^2$, depending "nicely" on X, and define

$$\Phi\colon \mathbb{T}^2 \times K \to \mathbb{T}^2 \times K, \quad \Phi(X,x) \stackrel{\text{def}}{=} (A(X), f_X(x)). \tag{1-2}$$

In many cases, these systems are an important source of nonhyperbolic and transitive dynamics. It may happen that these systems fail to be transitive, for instance, when f_ξ is the identity for all ξ. However, their appropriate perturbations are robustly transitive and nonhyperbolic examples, see Bonatti and Díaz [1996].

To continue with our discussion, we will introduce notions related to hyperbolicity and ergodicity. In what follows, M denotes a closed compact Riemannian manifold and $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M equipped with the C^1-uniform metric. Given $f \in \mathrm{Diff}^1(M)$, a closed set Λ is invariant if $f(\Lambda) = \Lambda$. A property is called *generic* if there is a residual subset of diffeomorphisms satisfying it. The phrase *"for generic diffeomorphisms it holds"* means *"there is a residual subset of diffeomorphisms such that..."*.

1.2 Weak forms of hyperbolicity. Let $f \in \mathrm{Diff}^1(M)$ and Λ be an f-invariant set. A Df-invariant splitting over Λ, $T_\Lambda M = E \oplus F$, is *dominated* if there are constants $C > 0$ and $\lambda < 1$ with[1]

$$||Df^{-n}|_{F_{f^n(x)}}||\,||Df^n|_{E_x}|| < C\lambda^n, \quad \text{for all } x \in \Lambda \text{ and } n \in \mathbb{N},$$

where $||\cdot||$ stands for the norm. The dimension of E is called the *index* of the splitting. A special type of dominated splitting is the hyperbolic one, when E is *uniformly contracting*

[1] The order of the bundles is relevant: the first one is the "most contracting" one.

($\|Df^n|_{E_x}\| < C\lambda^n$) and F is *uniformly expanding* ($\|Df^{-n}|_{F_{f^n(x)}}\| < C\lambda^n$). In such a case, we write $E = E^{\mathrm{s}}$ and $F = E^{\mathrm{u}}$ and call these bundles *stable* and *unstable* ones, respectively.

A Df-invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_r$ with several bundles is *dominated* if for all $j \in \{1, \dots, r-1\}$ the splitting $TM = E_1^j \oplus E_{j+1}^r$ is dominated, where $E_i^j \stackrel{\text{def}}{=} E_i \oplus \cdots \oplus E_j$, $i \le j$. When studying dominated splittings it is important to consider those with *undecomposable* bundles (i.e., the bundles cannot be decomposed further in a dominated way). Such a splitting is uniquely defined and called the *finest dominated splitting* of Λ, for details see Bonatti, Díaz, and Pujals [2003].

The set Λ is *hyperbolic* if there is a dominated splitting $T_\Lambda M = E^{\mathrm{s}} \oplus E^{\mathrm{u}}$ (one of these bundles may be trivial). The *index* of a transitive hyperbolic set Λ is the dimension of its stable bundle, denoted by $\mathrm{ind}(\Lambda)$. The set Λ is *partially hyperbolic* if there is a dominated splitting $T_\Lambda M = E^{\mathrm{s}} \oplus E \oplus E^{\mathrm{u}}$ (at most one of the bundles E^{s}, E^{u} may be trivial). For instance, if in (1-2) the rates of expansion and contraction of the maps f_X are "appropriate" then $\mathbb{T}^2 \times K$ is a partially hyperbolic set of Φ with a partially hyperbolic splitting with three nontrivial directions whose intermediate direction E has dimension $\dim(K)$.

1.3 Oseledecs' theorem and nonhyperbolicity. A measure μ is f-invariant if $\mu(A) = \mu(f^{-1}(A))$ for every Borel set A. We denote by $\mathcal{M}(f)$ the set of f-invariant probability measures and equip it with the weak$*$ topology. A measure $\mu \in \mathcal{M}(f)$ is *ergodic* if for every set B with $B = f^{-1}(B)$ it holds $\mu(B) \in \{0, 1\}$. We denote by $\mathcal{M}_{\mathrm{erg}}(f)$ the subset of $\mathcal{M}(f)$ of ergodic measures.

Given $\mu \in \mathcal{M}_{\mathrm{erg}}(f)$, by the Oseledecs Theorem (Oseledec [1968]) there are numbers $k = k(\mu) \in \{1, \dots, \dim(M)\}$ and $\chi_1(\mu) < \chi_2(\mu) < \cdots < \chi_k(\mu)$, called the *Lyapunov exponents* of μ, and a Df-invariant splitting $F_1 \oplus F_2 \oplus \cdots \oplus F_k$, called the *Oseledets splitting of* μ, such that for μ-almost every point $x \in M$ it holds

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \chi_i(\mu), \quad \text{for every } i \in \{1, \dots, k\} \text{ and } v \in F_i \setminus \{\bar{0}\}.$$

The dimension of F_j is the *multiplicity* of the exponent $\chi_j(\mu)$. The number of negative exponents, counted with multiplicity, is the *index of* μ, denoted $\mathrm{ind}(\mu)$. The measure μ is *hyperbolic* if $\chi_j(\mu) \neq 0$ for every $j \in \{1, \dots, k\}$. Otherwise, μ is called *nonhyperbolic*. If $\chi_{j(\mu)}(\mu) = 0$ then the dimension of $F_{j(\mu)}$ is the *number of zero exponents* of μ. Let j_{s} be the largest i with $\chi_i(\mu) < 0$ and j_{u} the smallest i with $\chi_i(\mu) > 0$. Note that either $j_{\mathrm{u}} = j_{\mathrm{s}} + 1$ (if the measure is hyperbolic) or $j_{\mathrm{u}} = j_{\mathrm{s}} + 2$ (otherwise). In this latter case, we let $j_{\mathrm{c}} = j_{\mathrm{s}} + 1$. We let $E^{\mathrm{cs}} \stackrel{\text{def}}{=} F_1 \oplus \cdots \oplus F_{j_{\mathrm{s}}}$, $E^{\mathrm{c}} \stackrel{\text{def}}{=} F_{j_{\mathrm{c}}}$, and $E^{\mathrm{cu}} \stackrel{\text{def}}{=} F_{j_{\mathrm{u}}} \oplus \cdots \oplus F_k$. In general, E^{cs} is not uniformly contracting and E^{cu} is not uniformly expanding.

Consider now μ ergodic and the finest dominated splitting $T_{\mathrm{supp}(\mu)} M = E_1 \oplus \cdots \oplus E_k$ over $\mathrm{supp}(\mu)$. By definition of domination, vectors in different bundles E_i have different exponents. Thus, every bundle F_j of the Oseledets splitting is contained in some E_{i_j}. The latter inclusion may be proper and then the Oseledets splitting is not dominated. We will see in Section 2.4 that the domination of the splitting and its type have dynamical consequences.

1.4 Nonhyperbolic settings. The *nonwandering set* $\Omega(f)$ of f is the set of points x such that for every neighborhood U of x there is some $n > 0$ with $f^n(U) \cap U \neq \varnothing$. The set $\Omega(f)$ is closed and f-invariant. When $\Omega(f)$ is hyperbolic we say that f is *hyperbolic*. A hyperbolic set is *nontrivial* if it contains some non-periodic orbit.

In the late 60s, Abraham and Smale exhibited open sets of diffeomorphisms consisting of nonhyperbolic ones, thus proving the non-density of hyperbolic systems, Abraham and Smale [1970]. Recall that the Kupka-Smale genericity theorem claims that periodic points of generic diffeomorphisms are hyperbolic and their invariant manifolds meet transversely. Note that the stable and unstable sets of nontrivial hyperbolic sets, in general, are not manifolds and hence we cannot speak of general position of these sets. The construction in Abraham and Smale [ibid.] shows an open set of diffeomorphisms such that the hyperbolic structures of two hyperbolic sets do not fit together nicely: invariant stable and unstable sets of (nontrivial) hyperbolic sets may not intersect “coherently”. These non-coherent intersections are the germ of the notion of a *robust heterodimensional cycle* to be discussed in Section 2.2.

1.5 Robustly nonhyperbolic transitive diffeomorphisms. The construction in Abraham and Smale [ibid.] was followed by a series of examples of transitive diffeomorphisms which fail to be hyperbolic, see Shub [1971] and Mañé [1978] which fit into the class of systems nowadays called *robustly nonhyperbolic transitive diffeomorphisms.*

We say that $f \in \mathrm{Diff}^1(M)$ is C^1*-robustly transitive* if it has a C^1-neighborhood $\mathfrak{N}(f)$ such that every $g \in \mathfrak{N}(f)$ is transitive. We denote by $\mathbf{RTN}(M) \subset \mathrm{Diff}^1(M)$ the (open) set of robustly transitive and nonhyperbolic diffeomorphisms. A typical feature of these systems is the coexistence of saddles of different *indices* (dimension of the stable direction). These systems always exhibit a dominated splitting, Mañé [1982], Díaz, Pujals, and Ures [1999], and Bonatti, Díaz, and Pujals [2003], but they may fail to be partially hyperbolic, Bonatti and Viana [2000]. These findings showed the necessity of weaker notions of hyperbolicity such as partial hyperbolicity and dominated splitting, among others.

1.6 Hyperbolic flavors in nonhyperbolic dynamics. Although the examples described above are nonhyperbolic they do exhibit some “hyperbolic features”. To start this discussion, recall that the set of f invariant probability measures $\mathcal{M}(f)$ is a Choquet simplex whose extremal elements are the ergodic measures. Density of ergodic measures in $\mathcal{M}(f)$

implies that either $\mathcal{M}(f)$ is a singleton or a nontrivial simplex whose extreme points are dense (*the* Poulsen simplex). Sigmund addressed the natural questions of the density of the ergodic measures in $\mathcal{M}(f)$ and the properties of generic invariant measures. Assuming that f is *Axiom A* (i.e., $\Omega(f)$ is hyperbolic and the periodic points are dense in $\Omega(f)$) he proved that the periodic measures (and thus the ergodic ones) are dense in $\mathcal{M}(f)$, Sigmund [1970]. Here a measure is *periodic* if it is the invariant probability measure supported on a periodic orbit. Moreover, the sets of ergodic measures and of measures with entropy zero are both residual in $\mathcal{M}(f)$. For an updated discussion and more references, see Gelfert and Kwietniak [n.d.].

It is now pertinent to recall the foundational talk by Mañé about ergodic properties of C^1-generic diffeomorphisms at the International Congress of Mathematicians of 1983, Mañé [1984]. Mañé proved that ergodic measures of C^1-generic diffeomorphisms are approached in the weak$*$ topology by periodic (and hence hyperbolic) measures, Mañé [1982]. Mañé's view of generic measures of C^1-diffeomorphisms was completed and substantially expanded in Abdenur, Bonatti, and Crovisier [2011] (see Abdenur, Bonatti, and Crovisier [ibid., Theorem 3.8]). In this context, another important result is Abdenur, Bonatti, and Crovisier [ibid., Theorem 3.5]: for every isolated transitive invariant set Λ of a C^1-generic diffeomorphism every generic measure supported in Λ is ergodic, hyperbolic, and its support is Λ. These results support the principle that, in the ergodic level, under a C^1-perspective hyperbolicity is somewhat widespread also in nonhyperbolic settings. However, it may be not necessarily ubiquitous if the viewpoint is changed. For instance, in the conservative setting nonhyperbolic dynamics can be robust (elliptic behavior in KAM theory) or locally generic (dichotomies all Lyapunov exponents are zero versus hyperbolicity, Bochi [2002] and Bochi and Viana [2002]). This discussion leads the following question

To what extent is the behavior of a generic dynamical system hyperbolic?

posed in Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005] and reformulated with different flavors in the literature (see, for example, the program in Palis [2008] and the conjectures in Pesin [2007]). In Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005] there is taken an ergodic point of view and "hyperbolicity" means that *all* ergodic measures are hyperbolic. The results in Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [ibid.] (see Section 3.1) inaugurated a fertile line of research about the construction of nonhyperbolic ergodic measures. Note that, by the Kupka-Smale genericity theorem, nonhyperbolic ergodic measures of generic diffeomorphisms have uncountable support. Also note that the hyperbolicity of $\Omega(f)$ implies the hyperbolicity of the ergodic measures. However, the converse is false in general as there are examples of nonhyperbolic diffeomorphisms whose all ergodic measures are hyperbolic. Nevertheless, all these examples are very specific and easily breakable. Thus, one hopes that the "big majority" of nonhyperbolic systems must exhibit nonhyperbolic ergodic measures which "truly" detect the nonhyperbolic behavior: capturing the whole dynamics (large support), the entropy of the system (large or positive entropy), and the number of nonhyperbolic directions (number of zero exponents). We will discuss these points in the next sections.

The generic measures investigated in Abdenur, Bonatti, and Crovisier [2011] were not studied in terms of their entropy[2]. There are several settings of nonhyperbolic chaotic systems which show "hyperbolic-like features from the entropy point of view". To justify this assertion, let us consider three-dimensional robustly nonhyperbolic transitive diffeomorphisms (see Section 1.5) with a partially hyperbolic splitting with one-dimensional central direction. There are three types of such diffeomorphisms: (compact case) having a global foliation consisting of circles tangent to the center direction (as the ones in Shub [1971], corresponding to systems as in (1-2)), (mixed case) having at least one invariant (or periodic) circle tangent to the center direction, Bonatti and Díaz [1996], and (non-compact case) without any invariant circle (certain derived from Anosov (DA) diffeomorphisms in Mañé [1978]). By Cowieson and Young [2005], these diffeomorphisms always have (ergodic) measures of maximal entropy (equal to the topological one).

We just discuss the "compact case". For skew-products as in (1-2) where $K = \mathbb{T}^1$, the entropy of the diffeomorphism is equal to the entropy of the base map and there is a C^1-open and dense subset of such systems having finitely many (ergodic) measures of maximal entropy, all hyperbolic, F. Rodriguez Hertz, M. A. Rodriguez Hertz, Tahzibi, and Ures [2012]. In some robustly transitive cases, there are exactly two such measures, Ures, Viana, and J. Yang [n.d.]. The spirit of these results is summarized in the following rigidity result in Tahzibi and J. Yang [n.d.] for partially hyperbolic diffeomorphisms with a central foliation by circles: *if there are high-entropy invariant measures with central Lyapunov exponent arbitrarily close to zero then the dynamics is conjugate to an isometric extension of an Anosov homeomorphism.* This result, based on the invariance principle in Avila and Viana [2010], holds for C^2-diffeomorphisms and involves some natural conditions such as dynamical coherence, transitive Anosov diffeomorphism in the base, and existence of global holonomies. See Díaz, Gelfert, and Rams [2017a] for results in the same spirit, just assuming C^1-regularity, in the skew-product setting,

To conclude, we see how hyperbolic features can also be found in the topology of $\mathcal{M}_{\mathrm{erg}}(f)$. For instance, under certain conditions (isolation and homoclinic relations, see Section 2.3) there are certain elementary components of the space of invariant measures with the same index which are Poulsen simplices and in which ergodic measures are *entropy-dense* (i.e., any measure can be approximated also in entropy by ergodic ones), see

[2]Quoting Mañé, Mañé [1984], "the generic elements of $\mathfrak{M}_{\mathrm{erg}}(f)$ fail to reflect the dynamic complexity of f". More precisely, generic measures of C^1-generic diffeomorphisms supported on a transitive isolated set have zero entropy, see Gelfert and Kwietniak [n.d., Theorem 8.1] and also Abdenur, Bonatti, and Crovisier [2011, Theorem 3.1].

Gorodetski and Pesin [2017]. Such elementary components are also studied in Bochi, Bonatti, and Gelfert [n.d.]. For an example where such components can be easier described, assume that there is a partially hyperbolic splitting $E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$, where $\dim(E^{\mathrm{c}}) = 1$ (as in (1-1) when $K = \mathbb{T}^1$). Then the index of any $\mu \in \mathcal{M}_{\mathrm{erg}}(f)$ is either $\dim(E^{\mathrm{s}})$ or $\dim(E^{\mathrm{s}}) + 1$ and the exponent $\chi_{E^{\mathrm{c}}}(\mu)$ is the only exponent of μ that can be zero. Thus, the space $\mathcal{M}_{\mathrm{erg}}(f)$, splits as

$$\mathcal{M}_{\mathrm{erg}}(f) = \mathcal{M}^{\mathrm{c}}_{\mathrm{erg},<0}(f) \,\dot{\cup}\, \mathcal{M}^{\mathrm{c}}_{\mathrm{erg},0}(f) \,\dot{\cup}\, \mathcal{M}^{\mathrm{c}}_{\mathrm{erg},>0}(f),$$

corresponding to the ergodic measures whose exponent $\chi_{E^{\mathrm{c}}}(\mu)$ is negative, zero, and positive, respectively. Under additional "transitive-like" hypotheses, the components $\mathcal{M}^{\mathrm{c}}_{\mathrm{erg},<0}(f)$ and $\mathcal{M}^{\mathrm{c}}_{\mathrm{erg},>0}(f)$ both have the above mentioned properties and, moreover, any element in $\mathcal{M}^{\mathrm{c}}_{\mathrm{erg},0}(f)$ is approximated weak$*$ and in entropy by measures in either of the other two, see Díaz, Gelfert, and Rams [2017b]. Investigations in this direction can be also found in Bonatti and Zhang [n.d.(b)].

2 Robustly nonhyperbolic dynamics

In this section, we review the main ingredients and tools that appear in our nonhyperbolic setting. One general underlying theme is how, in our setting, nonhyperbolic dynamics forces the existence of robust cycles. To establish this, the main object is the blender. Recall first that, given $f \in \mathrm{Diff}^1(M)$ and a hyperbolic set Λ_f of f, there is a C^1-neighborhood $\mathcal{N}(f) \subset \mathrm{Diff}^1(M)$ such that every $g \in \mathcal{N}(f)$ has a hyperbolic set Λ_g (called the *continuation* of Λ_f) such that $g|_{\Lambda_g}$ is is conjugate to $f|_{\Lambda_f}$ and such that the map $g \mapsto \Lambda_g$ is continuous (this map is also uniquely defined). A special case occurs when the set is a periodic orbit.

2.1 Blenders. In very rough terms, a *blender* is a "semi-local plug" providing a hyperbolic set whose stable set "behaves" as a manifold of dimension greater than its index. Blenders were introduced in Bonatti and Díaz [1996], formalizing the arguments in Díaz [1995], to construct new types of robustly transitive diffeomorphisms. Blenders were also used in several contexts, for example generation of robust heterodimensional cycles and tangencies, stable ergodicity, construction of nonhyperbolic measures, among others. Each of these cases involves a specific type of blender. Here we follow the definition of a *geometrical blender* introduced in Bochi, Bonatti, and Díaz [2016].

We need some preliminary ingredients. The set $\mathfrak{D}^i(M)$ of i-dimensional (closed) discs C^1-embedded in M is endowed with a natural distance, Bochi, Bonatti, and Díaz [ibid., Section 3.1]. Given a family of disks $\mathfrak{D}$ in $\mathfrak{D}^i(M)$ we denote by $\mathcal{V}_\eta(\mathfrak{D})$ its η-neighborhood. The (non-empty) family $\mathfrak{D}$ is *strictly f-invariant* if for every $D \in \mathfrak{D}$ there is $\eta > 0$ such that the image $f(D)$ of any disk $D \in \mathcal{V}_\eta(\mathfrak{D})$ contains a disk of $\mathfrak{D}$.

We consider a transitive set Γ that is locally maximal in an open neighborhood V of it and is simultaneously hyperbolic and also partially hyperbolic,

$$\Gamma = \bigcap_{n\in\mathbb{Z}} f^n(\overline{V}), \qquad T_\Gamma M = E^{\mathrm{s}} \oplus E^{\mathrm{wu}} \oplus E^{\mathrm{uu}}, \qquad E^{\mathrm{u}} = E^{\mathrm{wu}} \oplus E^{\mathrm{uu}},$$

where each of the bundles E^{s}, E^{wu}, E^{uu} is nontrivial. The set Γ is a *dynamical* cu-*blender* if there are a strictly f-invariant family of discs $\mathfrak{D} \subset \mathfrak{D}^i(M)$, $i = \dim(E^{\mathrm{uu}})$, $\epsilon > 0$, and a strong unstable cone field $\mathcal{C}^{\mathrm{uu}}$ around E^{uu} such that every disc in $\mathcal{V}_\epsilon(\mathfrak{D})$ is contained in V and tangent to $\mathcal{C}^{\mathrm{uu}}$.

An important property of blenders is that they are C^1-robust: If Γ_f is a dynamical blender of f then for every g C^1-close to f the continuation Γ_g is also a blender. An important consequence of the definition of a blender is that its local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f)$ (i.e., the set of points whose forward orbits are contained in V) intersects every disk of the family $\mathfrak{D}$, see Bochi, Bonatti, and Díaz [2016, Section 3.4]. This property corresponds to the assertion above "a blender is a hyperbolic set whose local stable manifold behaves as a manifold of dimension $\dim(E^{\mathrm{s}}) + 1$".

2.2 Heterodimensional cycles. A pair of saddles of different indices, p_f and q_f, of a diffeomorphism f are related by a *heterodimensional cycle* if their invariant manifolds intersect cyclically. Suppose that $\mathrm{ind}(p_f) > \mathrm{ind}(q_f)$. Then, due to dimension deficiency, the intersection $W^{\mathrm{u}}(p_f, f) \cap W^{\mathrm{s}}(q_f, f)$ cannot be transverse, while, due to dimension sufficiency (the sum of the dimensions of these manifolds is bigger than $\dim(M)$), the intersection $W^{\mathrm{s}}(p_f, f) \cap W^{\mathrm{u}}(q_f, f)$ can be transverse (indeed, this is what happens "typically"). A (heterodimensional) cycle associated with transitive hyperbolic sets of different indices is defined similarly, just replacing the saddles by the corresponding transitive hyperbolic sets. By the Kupka-Smale genericity theorem, cycles associated with saddles cannot be robust, in our nonhyperbolic context they are ubiquitous and play a fundamental role (see comments below). In the spirit of Abraham and Smale [1970], we aim to get "robust cycles" and for that we need to consider cycles associated to nontrivial hyperbolic sets. Two transitive hyperbolic sets of different indices of f, Λ_f and Γ_f, have a *C^1-robust cycle* if there is a neighborhood $\mathcal{N}(f) \subset \mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{N}(f)$ the invariant sets Λ_g and Γ_g intersect cyclically.

In the case when the saddles p_f and q_f satisfy $\mathrm{ind}(p_f) = \mathrm{ind}(q_f) + 1$, there are g C^1-arbitrarily close to f having a pair of transitive hyperbolic sets Λ_g and Γ_g with a robust cycle, see Bonatti and Díaz [2008]. However, these sets may be not related to the saddles in the initial cycle. If some of these saddles have a nontrivial homoclinic class then one can take these sets such that $\Lambda_g \ni p_g$ and $\Gamma_g \ni q_g$ and say that the cycle is C^1*-stabilized*, see Bonatti, Díaz, and Kiriki [2012, Theorem 1].

Let us explain the mechanism for robust cycles when $\dim(M) = 3$, $\mathrm{ind}(p_f) = 2$, and $\mathrm{ind}(q_f) = 1$. The unfolding of the cycle generates a blender Γ_g and the saddle p_g is "related" to Γ_g. This relation has two parts: first, $W^{\mathrm{s}}(p_g, g)$ intersects transversely $W^{\mathrm{u}}(\Gamma_g, g)$ (this is possible by dimension sufficiency), second $W^{\mathrm{u}}(p_g, g)$ contains a disk of the family $\mathfrak{D}$ and hence it intersects $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_g, g)$.

2.3 Homoclinic relations and classes. A pair of hyperbolic periodic saddles p_f and q_f of a diffeomorphism f are *homoclinically related* if the invariant manifolds of the orbits $\mathcal{O}(p_f)$ and $\mathcal{O}(q_f)$ intersect transversely in a cyclic way. In particular, two saddles that are homoclinically related have the same index and their continuations (in a small neighborhood) are also homoclinically related.

The *homoclinic class* of a saddle p_f of f, denoted by $H(p_f, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of $\mathcal{O}(p_f)$. The hyperbolic periodic points of f form a dense subset of $H(p_f, f)$. A homoclinic class is also a transitive set. A homoclinic class $H(p_f, f)$ may contain saddles of indices different from the one of p_f (thus these saddles cannot be homoclinically related to p_f) and hence may fail to be hyperbolic. In many relevant cases (as in the Axiom A case) the homoclinic classes are the "elementary pieces of dynamics", see Bonatti [2011, Sections 3 and 5] for an in-depth discussion.

Let us recall some properties of homoclinic classes. The map $p_f \mapsto H(p_f, f)$ is upper semi-continuous and hence this map is generically continuous. Also, C^1-generically, two homoclinic classes are either disjoint or equal, Carballo, Morales, and Pacifico [2003]. To recall an even stronger version of this result, consider saddles p_f and q_f of f, then there is a C^1-neighborhood $\mathfrak{U}(f)$ of f such that either it holds $H(p_g, g) = H(q_g, g)$ for every for C^1-generic $g \in \mathfrak{U}(f)$ or $H(p_g, g) \cap H(q_g, g) = \varnothing$ for every C^1-generic $g \in \mathfrak{U}(f)$, see for instance Abdenur, Bonatti, Crovisier, Díaz, and Wen [2007, Lemma 2.1]. In the first case, we say that p_f and q_f *are C^1-persistently in the same homoclinic class*.

Consider an open set $\mathcal{N} \subset \mathrm{Diff}^1(M)$ so that there is a saddle p_f such that $H(p_f, f)$ contains a saddle q_f with $\mathrm{ind}(p_f) = \mathrm{ind}(q_f) \pm 1$ for every $f \in \mathcal{N}$. The *connecting lemma,* Hayashi [1997], provides a dense subset $\mathfrak{D}$ of $\mathcal{N}$ such that p_f and q_f are involved in a cycle for every $f \in \mathfrak{D}$. Since $H(p_f, f)$ is nontrivial, the stabilization of cycles above gives an open and dense subset $\mathcal{O}$ of $\mathcal{N}$ such that every $f \in \mathcal{O}$ has a robust cycle associated to hyperbolic sets containing p_f and q_f.

2.4 Weak forms of hyperbolicity and homoclinic classes. The existence of a dominated splitting and its type provide important dynamical information. For instance, for homoclinic classes of C^1-generic diffeomorphisms there is the dichotomy "existence of a dominated splitting versus accumulation of the class by sinks or sources", Bonatti, Díaz, and Pujals [2003]. We highlight two results in this spirit.

The Oseledets splitting of an ergodic measure μ may be non-dominated, but if it is dominated then it has an extension to the whole support of μ. Recall that any hyperbolic measures whose Oseledets splitting is dominated is supported on the homoclinic class of a point of $\mathrm{ind}(\mu)$ (this is a version of the Anosov closing lemma, see Crovisier [2011, Proposition 1.4] based on Abdenur, Bonatti, and Crovisier [2011, Lemma 8.1]).

Proposition 2.1 (Cheng, Crovisier, Gan, Wang, and D. Yang [n.d., Proposition 1.4]). *There are the following possibilities for a nonhyperbolic homoclinic class $H(p_f, f)$ of a C^1-generic diffeomorphism:*

(i) Index-variability: *$H(p_f, f)$ contains saddles with different indices.*

(ii) *All periodic points of $H(p_f, f)$ have the same index k. There are two cases:*

(a) *There is a dominated splitting $T_{H(p_f,f)}M = E \oplus F$ of index k. Moreover, either $E = E^{\mathrm{s}}$ is uniformly contracting or E has a dominated splitting $E = E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ where E^{s} is uniformly contracting and E^{c} is one-dimensional.*

(b) *There is no dominated splitting of $T_{H(p_f,f)}M$ of index k.*

Let us observe that all known examples of C^1-generic nonhyperbolic homoclinic classes fall into item i). In case ii.b) there are different sub-cases according to the different types of dominated splittings, for details see Cheng, Crovisier, Gan, Wang, and D. Yang [ibid., Proposition 1.4].

3 Tools to build nonhyperbolicity

In this section, we present two methods for constructing nonhyperbolic ergodic measures with uncountable support. In the first one, the measure is obtained as a limit of periodic measures. The second one provides a set of positive entropy supporting only nonhyperbolic measures. We also discuss "mixed" methods.

3.1 GIKN method. We present the GIKN-method introduced in Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005] guaranteeing the ergodicity (and non-triviality) of accumulation points of a sequence of periodic measures.

Given a periodic point p of period $\pi(p)$ of f, consider the periodic measure $\mu_{\mathcal{O}(p)}$ supported on the orbit $\mathcal{O}(p)$ of p. Consider a sequence of periodic points $(p_n)_n$ of f with increasing periods $\pi(p_n)$ and the measures $\mu_n = \mu_{\mathcal{O}(p_n)}$. Assume that in this process for every n there is a "large proportion" of points of $\mathcal{O}(p_{n+1})$ which are close to the previous orbit $\mathcal{O}(p_n)$ ("shadowing part"). A careful selection of these proportion times assures the convergence of μ_n, but without extra assumptions, there is a risk of obtaining a periodic measure as a limit. Thus, in the construction, it is also assumed that at each step there is also proportion of the orbit $\mathcal{O}(p_{n+1})$ that is far from the previous one ("tail part"). A careful choice of the proportion of the "shadowing" and "tail" parts forces that the limit measure is non-periodic. This involves some quantitative estimates. Given $\gamma, \tau > 0$, we say that the periodic orbit $\mathcal{O}(p)$ is a (γ, τ)-*good approximation* of $\mathcal{O}(q)$ is there are a subset $\Gamma \subset \mathcal{O}(p)$ and a (surjective) projection $\varrho\colon \Gamma \to \mathcal{O}(q)$ such that: (i) $\operatorname{dist}(f^i(x), f^i(\varrho(x)) < \gamma$ for every $x \in \Gamma$ and every $0 \le i \le \pi(q)$, (ii) $\#(\Gamma) \ge \tau \cdot \pi(p)$, and (iii) $\#(\varrho^{-1}(x))$ is independent of $x \in \mathcal{O}(q)$. Here $\#(A)$ denotes the cardinality of A. The next result corresponds to Bonatti, Díaz, and Gorodetski [2010, Lemmas 2.3 and 2.5], which reformulate Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005, Lemma 2 and Section 8].

Lemma 3.1 (Nontrivial ergodic limit of periodic measures)**.** *Consider a sequence of periodic orbits $(p_n)_n$ of f with increasing periods. Suppose that there are sequences of strictly positive numbers $(\gamma_n)_n$ and $(\tau_n)_n$ such that for each n the orbit $\mathcal{O}(p_{n+1})$ is (γ_n, τ_n)-good approximation of $\mathcal{O}(p_n)$, where*

$$\sum_{n=1}^{\infty} \gamma_n < \infty \quad \textit{and} \quad \prod_{n=1}^{\infty} \tau_n > 0.$$

Then $\mu_{\mathcal{O}(p_n)} \to \mu$ in the weak$*$ topology, where μ is ergodic, non-periodic, and

$$\operatorname{supp}(\mu) = \bigcap_{k=1}^{\infty} \left(\overline{\bigcup_{\ell=k}^{\infty} \mathcal{O}(p_\ell)} \right).$$

Assume now that there is a one-dimensional, Df-invariant, and continuous direction field E defined on the whole space such that $|\chi_E(\mu_{n+1})| < \alpha |\chi_E(\mu_n)|$, for some $\alpha \in (0, 1)$. Then the limit measure satisfies $\chi_E(\mu) = 0$ and thus is nonhyperbolic, see Bonatti, Díaz, and Gorodetski [2010, Proposition 2.7]. Note that if the periodic orbits are constructed scattered throughout the whole manifold then the limit measure has full support. Note that the constructed measures have a somewhat repetitive pattern, as a result the limit measure has zero entropy, see Kwietniak and Łącka [n.d.] where this type of limit measures are studied and general results are obtained.

3.2 The flip-flop method.

Given a compact subset K of M and continuous map $\varphi\colon K \to \mathbb{R}$ we consider its *Birkhoff averages* defined by

$$\varphi_n(x) \overset{\text{def}}{=} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)), \quad n \in \mathbb{N}, \quad \text{and} \quad \varphi_\infty(x) \overset{\text{def}}{=} \lim_{n\to\infty} \varphi_n(x),$$

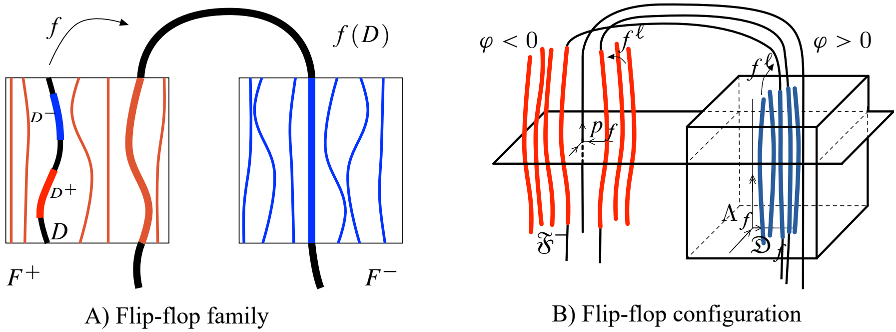

Figure 1: Flip-flops

provided this limit exists. In our setting, $\varphi = \log ||Df|_E||$, where E is a continuous Df-invariant line-field defined on K. Hence, $\varphi_\infty(x) = 0$ corresponds to $\chi_E(x) = 0$. We strive for a criterion implying that φ_∞ is zero uniformly on some compact set of positive entropy. For that we consider *controlled averages* and *flip-flop families*.

Let $\beta > 0$, $t \in \mathbb{N}$, and $T \in \mathbb{N} \cup \{\infty\}$. A point $x \in K$ is (β, t, T)-*controlled* if $\bigcup_{i=1}^{T} f^i(x) \subset K$ and if there is a subset $\mathcal{C} \subset \mathbb{N}$ of *control times* such that

- $0 \in \mathcal{C}$, $T \in \mathcal{C}$ if $T < \infty$, and $\mathcal{C}$ is infinite if $T = \infty$, and
- given $k < \ell$ two consecutive times in $\mathcal{C}$, then $\ell - k \leq t$ and $|\varphi_{\ell-k}(f^k(x))| \leq \beta$. The point x is *controlled at all scales* if there are sequences (t_i) of natural numbers and (β_i) of positive numbers, with $t_i \nearrow \infty$ and $\beta_i \searrow 0$, such that x is (β_i, t_i, T)-controlled for every i. The following holds, Bochi, Bonatti, and Díaz [2016, Lemma 2.2]: Assume that $x \in K$ is controlled at all scales and denote by $\omega(x)$ the ω-limit set of x. Then $\varphi_\infty(y) = 0$ for every $y \in \omega(x)$ and this limit is uniform on $\omega(x)$.

We now introduce the main ingredient to get "controlled" orbits. A family $\mathfrak{F} = \mathfrak{F}^+ \dot\cup \mathfrak{F}^-$ of compact subsets of K is called *flip-flop* if it satisfies the following properties (see Figure 1 A)):

- Let $F^+ \overset{\text{def}}{=} \bigcup_{D \in \mathfrak{F}^+} D$ and $F^- \overset{\text{def}}{=} \bigcup_{D \in \mathfrak{F}^-} D$. There is $\alpha > 0$ such that

$$\varphi(x^-) < -\alpha < 0 < \alpha < \varphi(x^+), \quad \text{for every } x^+ \in F^+ \text{ and } x^- \in F^-.$$

- For every $D \in \mathfrak{F}$ there are compact sets $D^+, D^- \subset D$ such that $f(D^+) \in \mathfrak{F}^+$ and $f(D^-) \in \mathfrak{F}^-$. Moreover, there is a constant $\lambda > 1$ such that

$$d(f(x), f(y)) \geq \lambda\, d(x, y) \quad \text{for every } x, y \in D^\pm.$$

The following result relates flip-flop families to zero Birkhoff averages: Given any D in a flip-flop family, there is $x \in D$ that is controlled at all scales and such that the restriction of f to $\omega(x)$ has positive topological entropy, Bochi, Bonatti, and Díaz [ibid., Theorem 2.1]. By the variational principle for entropy, the set $\omega(x)$ supports an ergodic measure of positive entropy. Let us explain some ingredients of this result.

An $\mathfrak{F}$-*segment of length* T is a sequence $\mathbf{D} = \{D_i\}_{i=0}^{T}$ such that $f(D_i) = D_{i+1}$, each D_i is contained in an element of $\mathfrak{F}$, and $D_T \in \mathfrak{F}$. Given $\beta > 0$ and $t \le T$, we say that $\mathbf{D}$ is (β, t)-*controlled* if there exists a set of control times $\mathcal{C} \subset \{0, \dots, T\}$ containing 0 and T such that $|\varphi_{\ell-k}(f^k(x))| \le \beta$ for every $x \in D_0$ and every pair $k < \ell$ of consecutive control times in $\mathcal{C}$.

We now relate flip-flop families, Birkhoff averages, and entropy. First, we encode orbits using their "itineraries". Let $\tau \in \mathbb{N}$. Given x in $F^+ \cup F^-$, $\mathbf{s} = (s_n) \in \{+,-\}^{\mathbb{N}}$, and $T \in \mathbb{N} \cup \{\infty\}$, the point x *follows the* τ-*pattern* $\mathbf{s}$ *up to time* T if $f^{n+1}(x) \in F^{s_n}$ for every $0 \le n < T$ with $n = 0 \pmod{\tau}$. Fix $D \in \mathfrak{F}$, Bochi, Bonatti, and Díaz [ibid., Lemma 2.12], gives a sequence of $\mathfrak{F}$-segments $(\mathbf{D}_k^+)$, $\mathbf{D}_k^+ = \{D_i^k\}_{i=0}^{T_k^+}$, such that $D_0^k \subset D$ and every point in D_0^k is (β_i, t_i, T_k^+)-controlled for every i with $1 \le i \le k$, where $\beta_i \to 0$ and $t_i \to \infty$. For each k pick x_k in D_0^k and let $x_\infty \in D$ be any accumulation point of (x_k). By Bochi, Bonatti, and Díaz [ibid., Section 2.5.1], the point x_∞ is (β_i, t_i, ∞)-controlled for every i.

Fix any $\mathbf{s} \in \{+,-\}^{\mathbb{N}}$ with dense σ-orbit. For each k we select an $\mathfrak{F}$-segment $\mathbf{D}_k^+$ whose τ-pattern is $\mathbf{s}$, consider $x_k \in D_0^k$, and a limit point $x = x_\infty$. As $\omega(x) \subset \bigcap_{i \ge 0} f^{-\tau i}(F^+ \cup F^-)$, we can define the projection $\pi : \omega(x) \to \{-,+\}^{\mathbb{N}}$, $\pi(y) = \mathbf{s}(y)$, where $\mathbf{s}(y)$ is the itinerary of y (i.e., $f^{\tau i}(y) \in F^{s_i(y)}$). The map π is continuous and satisfies $\pi \circ f^\tau = \sigma \circ \pi$. Using that $\mathbf{s}$ has a dense orbit one gets that π is onto. Hence, the restriction of f^τ to $\omega(x)$ is semi-conjugate to the one-sided full-shift on 2 symbols and thus has positive entropy.

Finally, we observe that Bonatti, Díaz, and Bochi [n.d.] introduces a criterion relaxing the control at all scales-one, inspired by the GIKN-method and called *control at any scale with a long sparse tail.* It guarantees that any weak$*$ limit measure μ of a sequence of measures of the form $\nu_n(x_0) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x_0)}$, where δ_z is the Dirac measure at z, is such that μ-almost every point y has a dense orbit in the ambient space and $\varphi_\infty(y) = 0$. Again, the "role of the tail" is to spread the support of the measure. The construction implies that almost every ergodic measure η of the ergodic decomposition of μ satisfies $\int \varphi \, d\eta = 0$, see Bonatti, Díaz, and Bochi [ibid., Theorem 1].

4 Building nonhyperbolic measures and sets

We discuss some applications of the tools provided in Section 3.

4.1 Applications of the GIKN-method. We will see that there exist large (locally residual) sets in the space of C^1-diffeomorphisms of any compact manifold of dimension greater than 2 such that any diffeomorphisms in it has a nonhyperbolic ergodic measure (with zero entropy) that is the limit of periodic ones. In some cases, these measures have a zero exponent with multiplicity greater than 1. We start by discussing skew-products and see how the ingredients there pass on to the differentiable setting.

4.1.1 Step skew-products. Consider a skew-product as in (1-1) with $N = 2$, $F : \Sigma_2 \times \mathbb{T}^1 \to \Sigma_2 \times \mathbb{T}^1$, whose fiber maps are of the form $f_\xi = f_{\xi_0}$. The *iterated function system* associated to f_0 and f_1, $\mathrm{IFS}(f_0, f_1)$, is the set of maps g of the form $g = f_{\xi_{k-1}} \circ \cdots \circ f_{\xi_0}$ for some $(\xi_0, \ldots, \xi_{k-1}) \in \{0,1\}^k$ and some $k \geq 0$. The (forward) orbit of a point $x \in \mathbb{T}^1$ for $\mathrm{IFS}(f_0, f_1)$ is $\mathcal{O}^+(x) \stackrel{\text{def}}{=} \{g(x),\ g \in \mathrm{IFS}(f_0, f_1)\}$.

Theorem 4.1 (Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005, Theorem 2])**.** *Let $F : \Sigma_2 \times \mathbb{T}^1 \to \Sigma_2 \times \mathbb{T}^1$ be a step skew-product with fiber maps f_0 and f_1. Suppose that the following hypotheses hold:*

(i) (minimality) *for every $x \in \mathbb{T}^1$ the orbit $\mathcal{O}^+(x)$ is dense in $\mathbb{T}^1$,*

(ii) (expansion) *for every $x \in \mathbb{T}^1$ there is $g \in \mathrm{IFS}(f_0, f_1)$ with $|g'(x)| > 1$,*

(iii) (attracting fixed point) *there are $g \in \mathrm{IFS}(f_0, f_1)$ and $p \in \mathbb{T}^1$ with $g(p) = p$ and $|g'(p)| < 1$.*

Then F has a nonhyperbolic ergodic measure with full support.

Note that conditions (ii) and (iii) are open conditions and hence persistent by small perturbations. Although condition (i) is, *a priori*, non-open, in Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [ibid.] the authors provide an open set of pairs satisfying the conditions in the theorem.

To prove this theorem it is constructed a sequence of periodic orbits $\mathcal{O}(p_n)$ satisfying Lemma 3.1 and whose fiber Lyapunov exponents goes to zero. Hence the limit measure μ, $\mu_{\mathcal{O}(p_n)} \to \mu$, is ergodic and $\chi^c(\mu) = \lim_n \chi^c(\mu_{\mathcal{O}(p_n)}) = 0$ (as the Lyapunov exponents are given by integrals $\chi^c(\mu_{\mathcal{O}(p_n)}) = \int \log |f'_{\xi_i}|\, d\mu_{\mathcal{O}(p_n)}$).

The systems in Theorem 4.1 admit smooth realizations and thus provide diffeomorphisms with nonhyperbolic measures with uncountable support. In Kleptsyn and Nalskiĭ [2007] the methods above are used to get an open set of such systems.

To get some extra dynamical information behind the hypotheses of Theorem 4.1, let us replace condition (iii) by the slightly more restrictive condition (iii') f_0 is Morse-Smale with exactly two fixed points (say s attracting and n repelling). By minimality, the unstable set of the fixed point $S = (0^{\mathbb{Z}}, s)$ of F accumulates to the stable set of $N = (0^{\mathbb{Z}}, n)$. We also have that the stable set of $S = (0^{\mathbb{Z}}, s)$ meets the unstable set of $N = (0^{\mathbb{Z}}, n)$. Thinking of S and N as "hyperbolic saddles" of different indices, we have that S and N

are involved in an "heterodimensional quasi-cycle". This is the key observation to adapt the constructions in Gorodetskiĭ, Ilyashenko, Kleptsyn, and Nalskiĭ [2005] to the setting of homoclinic classes, see Section 4.1.3.

4.1.2 Multiple zero exponents in step skew-products. Theorem 4.1 was generalized in Bochi, Bonatti, and Díaz [2014] replacing the circle fiber by any compact manifold M, obtaining skew-products with nonhyperbolic ergodic measures with full support in the ambient space and whose fiber Lyapunov exponents are all equal to zero. The number of considered fiber maps is large to guarantee a condition called *maneuverability* (assuming $N \geq 2$ large). The property of full support is obtained by spreading the sequence of periodic points in the ambient space and using Lemma 3.1. Obtaining Lyapunov exponents which are all equal to zero is much more delicate. First, in higher dimensions, fiber exponents are not given by integrals (as in the one-dimensional case). Thus, the simultaneous convergence of all Lyapunov exponents along the orbits to zero does not imply the same property for the limit measure. A second problem is the loss of commutativity of products of matrices in higher dimensions.

These difficulties are bypassed in Bochi, Bonatti, and Díaz [ibid.] considering the induced skew-product on the *flag bundle* of M, $\mathfrak{F}_M$, defined as the set of the form $(x, F_1, \dots, F_{\dim(M)})$ where $x \in M$, F_i is a subspace of $T_x M$ of dimension i, and $F_1 \subset \cdots \subset F_{\dim(M)}$. To each $f \in \mathrm{Diff}^r(M)$, $r \geq 2$, there is associated the flag C^{r-1}-diffeomorphism

$$\mathfrak{F}_f : (x, F_1, \dots, F_{\dim(M)}) \mapsto \big(f(x), Df(x)(F_1), \dots, Df(x)(F_{\dim(M)})\big)$$

and to the skew-product F with the fiber maps $f_0, \dots, f_{N-1}$ the skew-product $\mathfrak{F}_F : \Sigma_N \times \mathfrak{F}_M \to : \Sigma_N \times \mathfrak{F}_M$ with the fiber maps $\mathfrak{F}_{f_0}, \dots, \mathfrak{F}_{f_{N-1}}$. The maneuverability condition implies the minimality of the flag iterated function system.

The natural projection from $\mathfrak{F}_M$ to M defines a fiber bundle. The projection of an $\mathfrak{F}_F$-invariant and ergodic probability measure ν on $\Sigma_N \times \mathfrak{F}_M$ is an F-invariant and ergodic measure μ. The fibered Lyapunov exponents of ν are given by linear functions of integrals of continuous maps (determinants). Hence these exponents vary continuously with respect to ν. Moreover, the exponents of ν and μ are related: all the Lyapunov exponents of ν are zero if and only if all the Lyapunov exponents of μ are zero. This allows to recover the continuity of the exponents, thus proving that all exponents of μ are zero.

4.1.3 Nonhyperbolic ergodic measures in homoclinic classes. We now consider nonhyperbolic homoclinic classes of C^1-generic diffeomorphisms and see that they support ergodic nonhyperbolic measures. Some of the ingredients of Section 4.1.1 will reappear:

Theorem 4.2 (Cheng, Crovisier, Gan, Wang, and D. Yang [n.d., Main Theorem]). *There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that every nonhyperbolic homoclinic class of $f \in \mathcal{R}$ supports ergodic nonhyperbolic measures.*

To see how the above result arises let us start by considering the simplest case where the class $H(p_f, f)$ contains a saddle q_f with $\mathrm{ind}(p_f) \neq \mathrm{ind}(q_f)$, a property which is called *index-variability (of the class)*. In view of Section 2.3, we can assume that there are an open set $\mathcal{U}$ and a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that $H(p_f, f) = H(q_f, f)$ for all $f \in \mathcal{U} \cap \mathcal{R}$. To $H(p_f, f)$ we associate its set $\mathrm{ind}(H(p_f, f))$ of indices (i.e., $k \in \mathbb{N}$ such that here is a saddle $q_f \in H(p_f, f)$ with $\mathrm{ind}(q_f) = k$). By Abdenur, Bonatti, Crovisier, Díaz, and Wen [2007], this set is an interval in $\mathbb{N}$. Thus, we can assume that p_f and q_f have consecutive indices, say $s+1$ and s, respectively. Since the homoclinic class $H(p_f, f)$ is nontrivial one can replace these points by points homoclinically related to them such that the eigenvalues of $Df^{\pi(p_f)}(p_f)$ and $Df^{\pi(q_f)}(q_f)$ are all real and all have multiplicity one, see Abdenur, Bonatti, Crovisier, Díaz, and Wen [ibid., Proposition 2.3]. Thus, we will assume that p_f and q_f satisfy such properties.

We now review the arguments in Díaz and Gorodetski [2009] proving that every $f \in \mathcal{R} \cap \mathcal{U}$ has a non-periodic nonhyperbolic ergodic measure supported on $H(p_f, f)$. As observed in Section 2.3, there is a C^1-dense subset $\mathfrak{D}$ of $\mathcal{U}$ such that every $g \in \mathfrak{D}$ has a heterodimensional cycle associated to p_g and q_g. The fact that the saddles have real and simple eigenvalues implies that the dynamics "associated" to the cycle is partially hyperbolic with a one-dimensional central direction E. This direction is contracting in a neighborhood of $\mathcal{O}(p_f)$ and expanding in a neighborhood of $\mathcal{O}(q_f)$. The unfolding of this cycle generates new saddles r_h (also with real and simple eigenvalues) which are homoclinically related to p_h. The orbit of r_h stays most of the time close to $\mathcal{O}(p_h)$ (i.e., $\mathcal{O}(r_h)$ shadows $\mathcal{O}(p_h)$ most of the time), but it stays also some prescribed time nearby $\mathcal{O}(q_h)$ (this is the tail part of $\mathcal{O}(r_h)$). A consequence of the "tail part" is that $0 < |\chi_E(r_h)| < |\chi_E(p_h)|$. As r_h and p_h are homoclinically related, the classes of q_h and r_h coincide and after a new perturbation one can produce a cycle related to q_φ and r_φ. This gives an inductive pattern for the creation of saddles in $H(p_h, h)$ and with decreasing Lyapunov exponents. This provides a sequence of periodic orbits satisfying Lemma 3.1. In this construction, we are in a setting similar to the one in Section 4.1.1, the points p_f and q_f playing the roles of S and N, respectively. The difference between these constructions is that now the generation of periodic points involves perturbations, while in Section 4.1.1 it does not.

In the construction above, we first identified a part of the ambient space where the dynamics is partially hyperbolic and has a nonhyperbolic one-dimensional direction. This implies that if the homoclinic class has a undecomposable central bundle of dimension two then obtained measure cannot have full support in the class. On the other hand, if the homoclinic class has a dominated splitting $T_{H(p,f)}M = F_1 \oplus E \oplus F_2$ with $\dim(E) = 1$

and $H(p_f, f)$ has saddles of indices $\dim(F_1)$ and $\dim(F_1)+1$, then one can obtain nonhyperbolic measures μ with $\mathrm{supp}(\mu) = H(p_f, f)$. To get limit measures μ with $\mathrm{supp}(\mu) = H(p_f, f)$ one chooses the saddles r_h whose tails have iterates close to q_h but also iterates scattered throughout the whole class, Bonatti, Díaz, and Gorodetski [2010]. By Lemma 3.1 one gets $\mathrm{supp}(\mu) = H(p_f, f)$. This concludes the case when the class has index-variability.

We now consider the general case without *a priori* assuming index-variability. This will then complete the discussion of the proof of Theorem 4.2. The main idea is to recover the index-variability by "changing" the indices of saddles in the class by perturbation and then to fall into the previous case. Let us discuss the case (ii.a) in Proposition 2.1 where all the saddles of $H(p_f, f)$ have index k and $H(p_f, f)$ has a dominated splitting $E \oplus F$ adapted to $\mathrm{ind}(p_f) = k$.

Recall that if Λ is a compact f-invariant set and G is a one-dimensional invariant bundle over Λ that is not uniformly contracting, then there is an ergodic measure μ such that $\chi_G(\mu) \geq 0$, see Crovisier [2011, Claim 1.7]. By Proposition 2.1, there are two possibilities for the splitting $E \oplus F$ over $H(p_f, f)$, either $E = E^{\mathrm{s}} \oplus E^{\mathrm{c}}$ (with $\dim(E^{\mathrm{c}}) = 1$) or $E = E^{\mathrm{s}}$. If the first possibility holds then, by the previous comment, we get an ergodic measure μ with $\chi_E(\mu) \geq 0$. If $\chi_E(\mu) = 0$ we are done. Otherwise, the measure μ is hyperbolic with a dominated splitting $E^{\mathrm{s}} \oplus E^{\mathrm{cu}}$, $E^{\mathrm{cu}} = E^{\mathrm{c}} \oplus F$, and hence $\mathrm{supp}(\mu)$ is contained in a homoclinic class of index $k-1$, contradicting that all saddles of $H(p_f, f)$ have index k. If the second possibility $E = E^{\mathrm{s}}$ holds, we repeat the arguments above for f^{-1} (the case $F = E^{\mathrm{u}}$ does not occur as $H(p_f, f)$ is not hyperbolic).

Finally, we mention that there is the following more general version of Theorem 4.2 (see Cheng, Crovisier, Gan, Wang, and D. Yang [n.d., Theorem A]). *For a generic $f \in \mathrm{Diff}^1(M)$, if p_f is a saddle of index k and $T_{H(p_f,f)}M = E \oplus F$ is dominated, E not uniformly contracting and* $\dim(E) = k$, *and $H(p_f, f)$ does not contain saddles of index $k-1$ then there is an ergodic measure μ supported in $H(p_f, f)$ such that $\chi_k(\mu) = 0$.*

Under these conditions, Wang [n.d., Theorem A] claims that for every $\epsilon > 0$ there is a saddle q_ϵ homoclinically related to p_f and such that $|\chi_k(q_\epsilon)| < \epsilon$. Then, after a small C^1-perturbation, one can change its index and generate a cycle related the saddle q_h of index $k-1$ and p_h whose unfolding generates saddles of index $k-1$ inside $H(p_g, g)$. Thus, we recover the index-variability scenario above.

4.1.4 Ergodic measures with multiple zero exponents.

We now study generic homoclinic classes supporting nonhyperbolic ergodic measures with several zero exponents, which corresponds to the results in Section 4.1.2 in the differentiable setting.

As explained in Section 1.2, if the nonhyperbolic direction splits in a dominated way into one-dimensional sub-bundles then nonhyperbolic measures have exactly one zero exponent. Thus, in what follows, we consider homoclinic classes having a higher-dimensional and undecomposable central direction. More precisely, consider generic diffeomorphisms f having a saddle p_f whose homoclinic class have a dominated splitting $T_{H(p,f)}M = E \oplus E^{\mathrm{c}} \oplus F$, where $\mathrm{ind}(p_f) = \dim(E) = k$ and such that the class contains a saddle q_f of index $\dim(E \oplus E^{\mathrm{c}})$ and E^{c} is undecomposable. By the results in Section 4.1.3, generically, for each j with $\dim(E) < j \leq \dim(E \oplus E^{\mathrm{c}})$ there is a nonhyperbolic ergodic measure μ with $\chi_j(\mu) = 0$. Furthermore, Wang and Zhang [n.d., Corollary 1.1] claims that under these conditions there is a nonhyperbolic ergodic measure such that $\chi_{E^{\mathrm{c}}}(\mu) = 0$ (i.e., with $\dim(E^{\mathrm{c}})$ zero exponents) and also points out that the index-variability conditions are necessary (see Wang and Zhang [ibid., Section 5]). The proof of this theorem is based on the GIKN-method and all the comments in Section 4.1.2 about the difficulties to pass from dimension one to higher dimensions apply here. To discuss this result, recall first the extremely useful classical Franks' lemma for C^1-dynamics: for every (small) perturbation of the derivative of a diffeomorphism along a periodic orbit there is a small local C^1-perturbation of the diffeomorphism with such a derivative along the orbit. This result shows the importance of understanding the "perturbations of the linear part" of the dynamics.

A first ingredient in Wang and Zhang [ibid.] is Bochi and Bonatti [2012, Theorem 4.7] about perturbations of linear cocycles. Consider matrices $A_1, \dots, A_n \in \mathrm{GL}(n, \mathbb{R})$ such that $B = A_n \circ \cdots \circ A_1$ has no dominated splitting of index i. Then there is an arbitrarily small perturbations A_i' of these matrices such that the Lyapunov exponents of $B' = A_n' \circ \cdots \circ A_1'$ satisfy $\chi_i(B') = \chi_{i+1}(B')$ and $\chi_j(B) = \chi_j(B')$ if $j \notin \{i, i+1\}$. Thus, one gets a dynamics whose linear part has exponents of multiplicity two. A second argument is the improvement of Franks' lemma presented in Gourmelon [2016]: the perturbation can be done preserving parts of stable and unstable manifolds. This allows to preserve intersections of invariant manifolds throughout the perturbations.

We emphasize two ingredients in Wang and Zhang [n.d., Corollary 1.1]: i) a version of Lemma 3.1, including a comparison of the "central" Lyapunov exponents and the periodic points "heteroclinic-like" related, and ii) similar to arguments in Section 4.1.2, the exponents of the measures are linear functions of integrals of continuous maps, providing continuity of the exponents with respect to the measures.

4.2 Applications of the flip-flop method: robust zeros. In comparison with the results presented in Section 4.1, in what follows we replace locally residual sets of C^1-diffeomorphism by open ones and also obtain measures with positive entropy. Recall the definition of the set $\mathbf{RTN}(M)$ of C^1-robustly transitive and nonhyperbolic diffeomorphisms in Section 1.5.

Theorem 4.3 (Bochi, Bonatti, and Díaz [2016, Corollary 1]). *There is a C^1-open and dense subset of $\mathbf{RTN}(M)$ of diffeomorphisms with a nonhyperbolic ergodic measure with positive entropy.*

The proof of this result uses the method of controlling Birkhoff averages on sets of positive entropy and relies on finding an appropriate flip-flop family. The main ingredients for that are heterodimensional cycles and blenders.

First, there is an open and dense subset of $\mathbf{RTN}(M)$ consisting of diffeomorphisms f having a saddle p_f (assume that $\pi(p_f) = 1$) of index $s+1$ and a blender Λ_f of index s which are "heteroclinically related": $W^{\mathrm{u}}(p_f, f)$ contains a disk of the strictly f-invariant family of disks $\mathfrak{D}_f$ of the blender and $W^{\mathrm{s}}(\Lambda_f, f)$ and $W^{\mathrm{s}}(p_f, f)$ have transverse intersections. Indeed, this means that p_f and Λ_f are involved in a robust cycle. We can also assume that the eigenvalues of $Df(p_f)$ are all real and all have multiplicity one (recall Section 4.1.3). We can now identify heteroclinic orbits $\mathcal{O}_1$ (going from Λ_f to p_f) and $\mathcal{O}_2$ (going from p_f to Λ_f) in such a way in a neighborhood of $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \Lambda_f \cup \mathcal{O}(p_f)$ the dynamics is partially hyperbolic with a one-dimensional center direction E (that is contracting nearby p_f and expanding nearby Λ_f). See Figure 1 B).

The above *flip-flop* configuration yields flip-flop families associated to $\varphi = \log ||Df^m|_E||$ and some power of f, Bochi, Bonatti, and Díaz [ibid., Section 4]. The flip-flop family $\mathfrak{F} = \mathfrak{F}^- \dot\cup \mathfrak{F}^+$ is (roughly) defined as follows: $\mathfrak{F}^+$ is the family of f-invariant disks $\mathfrak{D}_f$ of the blender and $\mathfrak{F}^-$ is a small neighborhood of of $W^{\mathrm{s}}_{\mathrm{loc}}(p_f, f)$. The heteroclinic connection between the blender and the saddle implies that $\mathfrak{F}$ is a flip-flop family for some f^ℓ. Theorem 4.3 follows applying Section 3.2.

Denote by $\mathbf{RTN}^{\mathrm{c}=1}(M)$ the open subset of $\mathbf{RTN}(M)$ with a partially hyperbolic $TM = E^{\mathrm{s}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{u}}$ with three nontrivial bundles and $\dim(E^{\mathrm{c}}) = 1$. In Bonatti, Díaz, and Bochi [n.d.] and Bonatti and Zhang [n.d.(a)] it is proved that C^1-open and densely in $\mathbf{RTN}^{\mathrm{c}=1}(M)$ the diffeomorphisms have nonhyperbolic ergodic measures with full support. The method in Bonatti, Díaz, and Bochi [n.d.] uses the control at all scales with a long sparse tail criterion, while Bonatti and Zhang [n.d.(a)] uses a cocktail combining the methods in Sections 3.1 and 3.2 and a version of the shadowing lemma in Gan [2002]. Finally, in a work in progress with Bonatti and Kwietniak (Łącka announced a similar result by different methods) we prove that these measures can be obtained with full support and positive entropy.

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