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ON EXPLICIT ASPECT OF PLURICANONICAL MAPS OF PROJECTIVE VARIETIES

JUNGKAI A. CHEN (陳榮凱) AND MENG CHEN (陈猛)

Abstract

In this survey article, we introduce the development of birational geometry associated to pluricanonical maps. Especially, we explain various aspects of explicit studies of threefolds including the key idea of theory of baskets and other applications.

1 Introduction

In the realm of birational geometry, divisors are perhaps the most important objects. Classically they were used to keep track of the property of zeros and poles and naturally developed into a convenient tool to study functions with preassigned conditions. Thus one may study linear systems associated to various interesting divisors. Among all divisors, the canonical divisor K, together with m-th canonical divisor mK for any $m \in \mathbb{Z}$, plays the central role. The behavior of pluricanonical maps φ_m or pluricanonical systems $|mK|$ is intensively studied in the minimal model program (in short, MMP) as well as other "canonical" classification problems. In fact, many very important concepts in algebraic geometry such as Kodaira dimension, Iitaka fibration, canonical volume, extremal contractions in minimal model program and so on, are defined on the basis of specific properties of the canonical divisor K.

More explicitly let V be any nonsingular projective variety of dimension n. For any $m \in \mathbb{Z}$, denote by $\varphi_{m,V}$ the m-canonical map of V. The following three types of problems are in the core of birational geometry:

Question 1.1. For any integer $n \geq 3$, find a practical integer r_n so that, for all nonsingular projective n-folds of general type, φ_m is birational onto its image for all $m \geq r_n$.

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Question 1.2. For any integers $n \geq 3$ and $n > \kappa \geq 0$, find integers $M_{n,\kappa}$ and $d_{n,\kappa}$ such that, for all nonsingular projective n-folds with Kodaira dimension κ, the m-th canonical map φ_m defines an Iitaka fibration for all $m \geq M_{n,\kappa}$ and divisible by $d_{n,\kappa}$.

Question 1.3. For any integer $n \geq 3$, find an integer m_n so that, for all canonical (terminal) weak $\mathbb{Q}$-Fano n-folds (i.e. $-K$ being $\mathbb{Q}$-Cartier, nef and big), φ_{-m} is birational onto its image for all $m \geq m_n$.

First we recall some results on the quest of existence for r_n, $M_{n,\kappa}$ and m_n.

- For varieties V of general type, the remarkable theorem, proved separately by Hacon and McKernan [2006], Takayama [2006] and Tsuji [2006], asserts that there is a constant $\tilde{r}_n$ depending only on the dimension n ($n > 2$) such that $\varphi_{m,V}$ is birational for all $m \geq \tilde{r}_n$.

- For varieties of intermediate Kodaira dimension, say $0 < \kappa(V) < n$, the effective Iitaka fibration conjecture predicts that there exists a constant $\tilde{c}_n$ depending only on n such that $\varphi_{m,V}$ defines an Iitaka fibration for all $m \geq \tilde{c}_n$ and divisible. The canonical bundle formula of Fujino and Mori [2000] serves as the fundamental tool in this situation. The up-to-date result, due to Birkar and Zhang [2016] says that there exists a uniform number $M(n, b_F, \beta_{\tilde{F}})$ so that φ_m gives an Iitaka fibration for all $m \geq M(n, b_F, \beta_{\tilde{F}})$ and divisible. The numbers b_F and $\beta_{\tilde{F}}$ are defined as follows. Let F be the general fiber of Iitaka fibration. The number b_F, called the *index of fiber*, is the smallest positive integer so that $|bK_F| \neq \varnothing$. One has a covering $\tilde{F} \to F$ by $|mK_F|$. Then $\beta_{\tilde{F}}$, called the *middle Betti number*, is defined as the $(n-\kappa)$-th Betti number of the $n-\kappa$ dimensional variety $\tilde{F}$. One may refer to Viehweg and Zhang [2009], G. Todorov and Xu [2009], Pacienza [2009], X. Jiang [2013], Di Cerbo [2014] and Birkar and Zhang [2016] for more details along this direction.

- For canonical (resp. terminal) weak $\mathbb{Q}$-Fano threefolds, the boundedness was proved by Kawamata [1992] under the condition that the Picard number $\rho = 1$ and by Kollár, Miyaoka, Mori, and Takagi [2000] for the general case with $\rho > 1$. Recent breakthrough of Birkar [2016] asserts that even for $n \geq 4$ there is a constant $\tilde{m}_n$ depending only on n such that $\varphi_{-m,V}$ is birational for all $m \geq \tilde{m}_n$.

It is interesting to study the explicit aspect of pluricanonical maps of projective varieties in high dimensions. Some recent advances show that r_3, m_3, and $M_{3,\kappa}$ have realistic bounds which are very close to being optimal. The purpose of this survey article is to introduce and to sketch some of the key ideas and techniques developed from those explicit studies of 3-folds. We expect that such detailed and explicit studies of 3-folds will pave a solid path toward the understanding of higher dimensional birational geometry.

Throughout, all varieties are considered over an algebraically closed field k of characteristic zero.

2 Theory of weighted baskets

The understanding of terminal and canonical singularities plays the essential role in the development of three dimensional geometry. The milestone work of the existence of flips in dimension three, due to Mori [1988], built on the classification of terminal singularities and extremal neighborhoods. Moreover, Reid showed that each terminal singularities can be deformed into cyclic quotient singularities. Hence, the collection of deformed quotient singularities carries ample information of singularities. This leads to the notion of *baskets of terminal orbifold points*. For simplicity, a terminal orbifold point of type $\frac{1}{r}(1,-1,b)$ will be denoted as (b,r) with $b \le r/2$. A basket, which is a collection of terminal orbifold points, is written as $\mathcal{B} = \{n_i \times (b_i, r_i)\}$ where n_i denotes the multiplicities.

The subsequent result of singular Riemann-Roch formula (see Reid [1987]) can be derived by computing the contribution of basket of singularities, say

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D-K_X)(2D-K_X) + \frac{1}{12}(D.c_2(X)) + \sum_{P\in B(X)}\left(-i_P\cdot\frac{r_P^2-1}{12r_P} + \sum_{j=1}^{i_P-1}\frac{\overline{jb_P}(r_P-\overline{jb_P})}{2r_P}\right),$$

where $c_2(X)$ is defined in the sense of intersection theory, $B(X) = \{(b_P, r_P)\}$ is the basket data of X and i_P is the local index of D such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(i_P K_X)$ near P.

By applying the singular Riemann-Roch formula to $D = K_X$, then one gets

$$(K_X.c_2(X)) = -24\chi(\mathcal{O}_X) + \sum_{P\in B_X}\left(r_P - \frac{1}{r_P}\right).$$

This leads to various results. For example, Kawamata and Morrison's result on the global index of 3-folds with $K_X \equiv 0$ was then derived (see Section 5).

By taking $D = mK_X$ and replacing $(K_X.c_2(X))$ with $\chi(\mathcal{O}_X)$ and the contribution of singularities, we get the following plurigenus formula Reid [ibid.]:

$$\chi_m = \frac{1}{12}m(m-1)(2m-1)K^3 + (1-2m)\chi + l(m), \tag{2-1}$$

where $\chi = \chi(\mathcal{O}_X)$, $K^3 = K_X^3$, $\chi_m = \chi(\mathcal{O}_X(mK_X))$ and

$$l(m) = \sum_{P\in B_X}\sum_{j=1}^{m-1}\frac{\overline{jb_P}(r_P-\overline{jb_P})}{2r_P}. \tag{2-2}$$

It is clear from the Riemann-Roch formula that the triple (B_X, χ_2, χ) determine χ_m for all $m \geq 3$. We call the triple $\mathbb{B} = \{B, \chi_2, \chi\}$ a *weighted basket*, where B is a basket of orbifold points, χ_2 is a non-negative integer and χ is an integer. For any $m \geq 3$, χ_m can be inductively and formally defined by means of (2-1). Note that the rational number K^3, which is also uniquely determined by $\mathbb{B}$, is called the volume of $\mathbb{B}$.

Given a basket

$$B = \{(b_1, r_1), (b_2, r_2), \dots, (b_k, r_k)\},$$

we call the basket

$$B' := \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \dots, (b_k, r_k)\}$$

a packing of B, written as $B \succcurlyeq B'$. If $b_1 r_2 - b_2 r_1 = 1$, then we call $B \succcurlyeq B'$ a *prime packing*.

Then we introduced the "canonical sequence of prime unpackings of a basket" in J. A. Chen and M. Chen [2010b]:

$$B^{(0)}(B) \succcurlyeq B^{(5)}(B) \succcurlyeq \dots \succcurlyeq B^{(n)}(B) \succcurlyeq \dots \succcurlyeq B,$$

so that $B^{(n)}$ consists of orbifold points (b_i, r_i) with either $r_i \leq n$ or $b_i = 1$. The basket $B^{(0)}$, called *the initial basket*, consists of orbifold points of the form $(1, r_i)$. Take an orbifold point (b, r) for example. Let $q = \lfloor \frac{r}{b} \rfloor$. Then $\frac{1}{q} \geq \frac{b}{r} \geq \frac{1}{q+1}$. Indeed, let $\beta := r - qb$, then the initial basket of (b, r) is $\{(b - \beta) \times (1, q), \beta \times (1, q+1)\}$.

The packing of baskets naturally induces the packing of weighted baskets, namely we define

$$\{B, \chi_2, \chi\} \succcurlyeq \{B', \chi_2, \chi\}$$

if $B \succcurlyeq B'$. Furthermore, any given weighted basket $\mathbb{B}$ has the corresponding "canonical sequence":

$$\mathbb{B}^{(0)} \succcurlyeq \mathbb{B}^{(5)} \succcurlyeq \dots \succcurlyeq \mathbb{B}^{(n)} \succcurlyeq \dots \succcurlyeq \mathbb{B}.$$

As revealed in J. A. Chen and M. Chen [ibid., Section 3], the intrinsic properties of the canonical sequence provide many new inequalities among the Euler characteristic and other characteristics of a given weighted basket $\mathbb{B}$, of which the most interesting one is:

$$2\chi_5 + 3\chi_6 + \chi_8 + \chi_{10} + \chi_{12} \geq \chi + 10\chi_2 + 4\chi_3 + \chi_7 + \chi_{11} + \chi_{13} + R, \tag{2-3}$$

where R is certain non-negetative combination of all initial baskets with higher indices.

Although the above notions were introduced in a very formal way, it was proved to be quite effective for various geometric problems, which we will discuss in next sections. Also, even though the notion of packings was introduced to study the numerical behavior rather than its geometric meaning at the beginning. The relation appears in divisorial contractions to points.

Example 2.1. Let $P = \frac{1}{9}(2,7,1) \in X \cong \mathbb{C}^3/\mu_9$ be a terminal quotient singularity. Let $Y \to X$ be the weighted blowup with weights $\frac{1}{9}(2,7,1)$. Then the basket of Y is $B_Y = \{(1,2),(3,7)\}$ and the basket of X is $\{(4,9)\}$ which is a packing of B_Y. This is an example of *Kawamata blowup* (cf. Kawamata [1996]).

Example 2.2. Let $P \in X \cong (xy + z^{15} + u^2 = 0) \subset \mathbb{C}^4/\mu_5$ (of type $\frac{1}{5}(3,2,1,5)$) be a $cA/5$ singularity. Let $Y \to X$ be a weighted blowup with weights $\frac{1}{5}(3,7,1,5)$. Then the basket of Y is $\{(3,7),(1,3)\}$ and the basket of X is $\{2 \times (2,5)\}$. They have the same initial baskets $\{2 \times (1,2), 2 \times (1,3)\}$.

3 Pluricanonical maps of threefolds of general type

We consider 3-folds of general type in this section. Since minimal models exist and the problems are birational in nature, we usually work on minimal projective 3-folds of general type with $\mathbb{Q}$-factorial terminal singularities, unless otherwise stated. Denote by r_X the Cartier index of X. Define the *canonical stability index*

$$r_{\mathrm{s}}(X) = \min\{t \,|\, \varphi_{m,X} \text{ is birational for all } m \geq t\}.$$

Clearly $r_3 = \max\{r_{\mathrm{s}}(X) \,|\, X \text{ is a minimal 3-fold of general type}\}$.

3.1 The case $r_X = 1$.

When X is smooth and minimal, it is Wilson [1980] who first proved that $r_{\mathrm{s}}(X) \leq 25$. Then this was improved, chronoligically, by Benveniste [1986] ($r_{\mathrm{s}}(X) \leq 9$), Matsuki [1986] ($r_{\mathrm{s}}(X) \leq 7$) and the second author M. Chen [1998] ($r_{\mathrm{s}}(X) \leq 6$).

In fact, the method of Benveniste, Matsuki and the second author can be easily extended to the situation, when X is Gorenstein and minimal, by using a special partial resolution due to Reid [1983] and Miyaoka [1987]. Note also that S. Lee [2000] proved the optimal base point freeness of $|4K|$.

Example 3.1. Let $X = S \times C$ where S is a minimal surface of general type of $(1,2)$-type and C is a complete curve of genus ≥ 2. Then $r_{\mathrm{s}}(X) = 5$ according to Bombieri [1973]. Thus, for Gorenstein minimal 3-folds of general type, the best one can expect is that $r_{\mathrm{s}}(X) \leq 5$ holds.

Question 1.1 in the case of $n = 3$ and $r_X = 1$ was finally solved in 2007 by the authors and De-Qi Zhang:

Theorem 3.2. *(J. A. Chen, M. Chen, and Zhang [2007, Theorem 1.1]) Let X be a minimal projective 3-fold of general type with $r_X = 1$. Then $\varphi_{m,X}$ is a biratioal morphism for every integer $m \geq 5$.*

3.2 Kollár's method.

The work of Kollár on push-forward of dualizing sheaves provide various important applications in the study of higher dimensional geometry. One of the them is to reduce the birationality problem to non-vanishing of plurigenera as in the following theorem:

Theorem 3.3. *(Kollár [1986, Corollary 4.8]) Let V be a nonsingular projective 3-fold of general type with $P_k(V) \geq 2$ for some integer $k > 0$. Then φ_{11k+5} is birational.*

The key contribution of Theorem 3.3 in the process of solving Question 1.1 ($n = 3$) is that it reduces to the problem to find an effective integer k so that $P_k \geq 2$, which is the standard task of Riemann-Roch formula.

Kollár's method of proving Theorem 3.3 is as follows. If one takes a sub-pencil $\Lambda \subset |kK_V|$, modulo a further birational modification if necessary, one gets a surjective morphism $f : V \longrightarrow \Gamma \cong \mathbb{P}^1$. One has the inclusion $\mathcal{O}(1) \hookrightarrow f_*\omega_V^k$ and then, for any $p \geq 5$,

$$f_*\omega_{V/\Gamma}^p \otimes \mathcal{O}(1) \hookrightarrow f_*\omega_V^{(2p+1)k+p}.$$

Since the 5-canonical map of the general fiber is birational and by the semi-positivity of $f_*\omega_{V/\Gamma}^p$, one sees that φ_{11k+5} is birational by simply taking $p = 5$.

Theorem 3.3 was considerably improved by the second author M. Chen [2004a, Theorem 0.1] that, under the same condition as that of Theorem 3.3, φ_{5k+6} is birational. Some further optimal results were proved in M. Chen [2003], M. Chen [2004a], M. Chen [2007], and J. A. Chen and M. Chen [2010a].

3.3 The case of $r_X \geq 2$.

Turning to the general situation that minimal model contains some singularities of index ≥ 2. Suppose $\chi(\mathcal{O}_X) < 0$. Reid's Riemann-Roch formula implies $P_2(X) \geq 4$. Hence the question is solvable by Kollár's theorem. Suppose that $P_k \geq 2$ for some $k \leq 12$, one can apply Kollár's method as well.

It remains to consider $\chi(\mathcal{O}_X) \geq 0$ and $P_k(X) \leq 1$ for all $2 \leq k \leq 12$. The key inequality (2-3) reads:

$$(3\text{-}1) \qquad 2P_5 + 3P_6 + P_8 + P_{10} + P_{12} \geq \chi(\mathcal{O}_X) + 10P_2 + 4P_3 + P_7 + P_{11} + P_{13},$$

which directly implies that $\chi(\mathcal{O}_X) \leq 8$. This also means that P_{13} is upper bounded by 7. In practice, one may obtain more precise bounds for both $\chi(\mathcal{O}_X)$ and P_{13}. It turns out that the 12-th weighted basket $\mathbb{B}^{12}(X)$ has only finite possibilities and so does $\mathbb{B}(X)$. It is then possible to answer Question 1.1, which was the main work in the authors' papers J. A. Chen and M. Chen [2010b,a, 2015a].

To improve or to reach the possible optimal bound, one needs to study the birational geometry explicitly. Known useful techniques include some effective method to estimate the lower bound of K_X^3 and better birationality criterion of $\varphi_{m,X}$ under the assumption that $P_{m_0} \geq 2$ for some number $m_0 > 0$, which is a kind of improvement to Kollár's method. We briefly describe the technique below. Indeed, if $P_{m_0} \geq 2$, then one has an induced fibration from a sub-pencil of $|m_0 K|$, say $f : X' \longrightarrow \Gamma$, where X' is a smooth model of X and Γ is a smooth complete curve. Denote by $\pi : X' \longrightarrow X$ the birational morphism. Pick a general fiber F of f and denote by $\sigma : F \longrightarrow F_0$ the contraction onto its minimal model. Naturally one has

$$m_0 \pi^*(K_X) \equiv pF + E_{m_0},$$

for some integer $p > 0$ and effective $\mathbb{Q}$-divisor E_{m_0}. The key point is to prove the so-called "canonical restriction inequality" (see M. Chen [2003], M. Chen [2004a], M. Chen [2007], M. Chen and Zhang [2008], M. Chen and Zuo [2008], and J. A. Chen and M. Chen [2010a, 2015a] for its development history) as follows:

$$\pi^*(K_X)|_F \geq \frac{p}{m_0 + p} \sigma^*(K_{F_0}) \tag{3-2}$$

modulo $\mathbb{Q}$-linear equivalence. The inequality (3-2) directly gives an effective lower bound of K_X^3 and it is crucial as well in proving the birationality of $\varphi_{m,X}$.

Here are the main theorems of the authors as the answer to Question 1.1 ($n = 3$):

Theorem 3.4. *(J. A. Chen and M. Chen [2010b,a, 2015a]) Let X be a minimal projective 3-fold of general type. Then*

(1) $K_X^3 \geq \frac{1}{1680}$;

(2) $\varphi_{m,X}$ *is birational for all* $m \geq 61$;

(3) $P_{12} \geq 1$ *and* $P_{24} \geq 2$.

(4) $K_X^3 \geq \frac{1}{420}$ *(optimal) if* $\chi(\mathcal{O}_X) \leq 1$.

Remark 3.5. The statements are birational in nature. More precisely, φ_m are birationally equivalent on different birational models and the canonical volume $\mathrm{Vol}(V)$ is a birational invariant, equals to K_X^3 of a minimal model X. Therefore, the above statements can be also read as: Let V be a nonsingular projective 3-fold of general type. Then $\mathrm{Vol}(V) \geq \frac{1}{1680}$ and $\varphi_{m,V}$ is birational for all $m \geq 61$, etc.

Define the *pluricanonical section index* $\delta(X)$ to be the minimal integer so that $P_\delta \geq 2$. The author proved the following results:

Theorem 3.6. *(J. A. Chen and M. Chen [2010b,a, 2015a]) Let X be a minimal projective 3-fold of general type. Then*

(1) $\delta(X) \le 18$*;*

(2) $\delta(X) = 18$ *if and only if* $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$*;*

(3) $\delta(X) \neq 16,\ 17$*;*

(4) $\delta(X) = 15$ *if and only if* $\mathbb{B}(X)$ *belongs to one of the types in J. A. Chen and M. Chen [2015a,* Table F–1*];*

(5) $\delta(X) = 14$ *if and only if* $\mathbb{B}(X)$ *belongs to one of the types in J. A. Chen and M. Chen [ibid.,* Table F–2*];*

(6) $\delta(X) = 13$ *if and only if* $\mathbb{B}(X) = \{B_{41}, 0, 2\}$

where

$$B_{2a} = \{4 \times (1,2), (4,9), (2,5), (5,13), 3 \times (1,3), 2 \times (1,4)\} \textit{ and}$$
$$B_{41} = \{5 \times (1,2), (4,9), 2 \times (3,8), (1,3), 2 \times (2,7)\}.$$

Example 3.7. Consider Fletcher's example, which is a general weighted hypersurface X_{46} of weighted degree 46 in weighted projective space $\mathbb{P}(4,5,6,7,23)$ (cf. Iano-Fletcher [2000]). Note that φ_{26} is not birational, $\chi(\mathcal{O}_X) = 1$ and $K_X^3 = \frac{1}{420}$. This provides an example of 3-fold for Theorem 3.4(4).

Recently the second author M. Chen [2016] showed $r_3 \le 57$ on the basis of above classifications. Therefore, $27 \le r_3 \le 57$.

For 3-folds with $\delta = 1$, the second author proved the following optimal results:

Theorem 3.8. *(M. Chen [2003, 2007]) Let X be a minimal projective 3-fold of general type with $p_g(X) \ge 2$. Then*

(1) $K_X^3 \ge \frac{1}{3}$*;*

(2) $\varphi_{8,X}$ *is birational onto its image.*

3.4 On irregular 3-folds of general type. Let X be a minimal projective 3-fold of general type with $q(X) > 0$. One may consider the Albanese map of X. A pioneer work on this topic was due to J. A. Chen and Hacon [2002], who developed the Fourier-Mukai theory to study irregular varieties and proved the following theorem:

Theorem 3.9. *(J. A. Chen and Hacon [2002, 2007]) Let X be a minimal irregular 3-fold of general type. Then*

(1) $|mK_X + P|$ *gives a birational map for all* $m \geq 7$ *(resp.* $m \geq 5$*) and for all (resp. general)* $P \in \mathrm{Pic}^0(X)$.

(2) when $\chi(\omega_X) > 0$, $|mK_X + P|$ *gives a birational map for all* $m \geq 5$ *and for all* $P \in \mathrm{Pic}^0(X)$.

Theorem 3.9(ii) is clearly optimal. For Theorem 3.9(i), the authors and Jiang proved the following theorem:

Theorem 3.10. *(J. A. Chen, M. Chen, and Zhang [2007]) Let* X *be a minimal irregular 3-fold of general type. Then* $\varphi_{6,X}$ *is birational.*

Besides, the authors gave the following effective lower bound for K_X^3:

Theorem 3.11. *(J. A. Chen and M. Chen [2008b, Corollary 1.2]) Let* X *be a minimal irregular 3-fold of general type. Then* $K_X^3 \geq \frac{1}{22}$.

Question 3.12. Are the results in Theorem 3.10 and Theorem 3.11 optimal?

4 The anti-canonical geometry of $\mathbb{Q}$-Fano 3-folds

A normal projective 3-fold X is called a *weak* $\mathbb{Q}$*-Fano 3-fold* (resp. $\mathbb{Q}$*-Fano 3-fold*) if the anti-canonical divisor $-K_X$ is nef and big (resp. ample). A *canonical* (resp. *terminal)* weak $\mathbb{Q}$-Fano 3-fold is a weak $\mathbb{Q}$-Fano 3-fold with at worst canonical (resp. terminal) singularities.

Weak $\mathbb{Q}$-Fano varieties form a fundamental class in minimal model program and various aspects of birational geometry. Given a canonical weak $\mathbb{Q}$-Fano 3-fold X, the *m-th anti-canonical map* $\varphi_{-m,X}$ (or simply φ_{-m}) is the rational map defined by the linear system $|-mK_X|$. It is worthwhile to mention that the behavior of $\varphi_{-m,X}$ in not necessarily birationally invariant, which makes Question 1.3 much harder to work on.

We may always study on a terminal weak $\mathbb{Q}$-Fano 3-fold. Take the weighted basket

$$\mathbb{B}(X) = \{B_X, P_{-1}, \chi(\mathcal{O}_X)\}.$$

By the duality and the vanishing of higher cohomology, we have $\chi_m = -P_{-(m-1)}$ for all $m \geq 2$. Hence the basket theory introduced in Section 2 has a parallel version in Fano case. In fact, the basket theory works very well in classifying weak $\mathbb{Q}$-Fano 3-folds with small invariants.

4.1 Lower bound of the anti-canonical volume.

In 2008, the authors applied the basket theory to prove the following theorem:

Theorem 4.1. *(J. A. Chen and M. Chen [2008a, Theorem 1.1]) Let X be a terminal (or canonical) weak $\mathbb{Q}$-Fano 3-fold. Then*

(1) $P_{-4} > 0$ with possibly one exception of a basket of singularities;

(2) $P_{-6} > 0$ and $P_{-8} > 1$;

(3) $-K_X^3 \geq \frac{1}{330}$. Furthermore $-K_X^3 = -\frac{1}{330}$ if and only if the basket of singularities is $\{(1,2),(2,5),(1,3),(2,11)\}$.

Theorem 4.1.(3) is optimal according to the example of general hypersurface $X_{66} \subset \mathbb{P}(1,5,6,22,33)$ (cf. Iano-Fletcher [2000]).

4.2 The anti-pluricanonical birationality.

The second author started to study the constant for anti-pluricanonical birationality in M. Chen [2011] in 2011. A practical upper bound for m_3 for $\mathbb{Q}$-Fano 3-folds with Picard number one was proved by the second author and C. Jiang in 2016:

Theorem 4.2 (M. Chen and C. Jiang [2016, Theorem 1.6])**.** *Let X be a terminal $\mathbb{Q}$-Fano 3-fold of Picard number one. Then $\varphi_{-m,X}$ is birational for all $m \geq 39$.*

Theorem 4.3 (M. Chen and C. Jiang [ibid., Theorem 1.8, Remark 1.9])**.** *Let X be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then $\varphi_{-m,X}$ is birational for all $m \geq 97$.*

The result in Theorem 4.2 is very close to be optimal according to Fletcher's example Iano-Fletcher [2000]. The numerical bound in Theorem 4.3, however, might be far from optimal. Recently the second author and Jiang had an improvement on this problem:

Theorem 4.4 (M. Chen and C. Jiang [2017, Theorem 1.9])**.** *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then there exists a terminal weak $\mathbb{Q}$-Fano 3-fold X birational to V such that*

1. $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geq 37$;

2. $\varphi_{-m,X}$ is birational for all $m \geq 52$.

Theorem 4.5 (M. Chen and C. Jiang [ibid., Theorem 1.10])**.** *Let V be a canonical weak $\mathbb{Q}$-Fano 3-fold. Then, for any K-Mori fiber space Y of V,*

1. $\dim \overline{\varphi_{-m}(Y)} > 1$ for all $m \geq 37$;

2. $\varphi_{-m,Y}$ is birational for all $m \geq 52$.

One notes that an intensive classification using the basket theory developed by the authors was done in proving Theorem 4.4 and Theorem 4.5.

It is a very interesting question to ask what the optimal value of m_3 is, which is crucial in studying the anti-canonical geometry of weak $\mathbb{Q}$-Fano 3-folds. One only knows $33 \leq m_3 \leq 97$ so far.

5 Threefolds with Kodaira dimension $0 \leq \kappa < 3$

For varieties of Kodaira dimension 0, the question is then to find a uniform bound $M_{n,0}$ such that $|mK| \neq \varnothing$ for all m divisible by $M_{n,0}$ and for all n-dimensional varieties with $\kappa = 0$. It is well-known that $M_{2,0} = 12$.

For threefolds with Kodaira dimension 0, Kawamata proved that $0 \leq \chi(\mathcal{O}_X) \leq 4$. Comparing $\chi(\mathcal{O}_X)$ with $(K_X \cdot c_2)$, one knows the indices of singularities in its minimal model and hence it follows that a uniform bound $M_{3,0}$ exists (cf. Kawamata [1986]). By careful classification of possible singularities, Morrison shows that $|mK_X| \neq \varnothing$ if m is divisible by the Beauville's number (cf. Morrison [1986])

$$\mathrm{lcm}\{m|\phi(m) \leq 20\} = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

Note that 20 is chosen as $b_3(A)$ for any abelian threefold A. Indeed, by Oguiso's examples (cf. Oguiso [1993]), the minimal universal number $M_{3,0}$ is the Beauville's number.

For threefolds with $\kappa = 2$, Ringer shows that φ_m is birational to the Iitaka fibration as long as $m \geq 48$ and divisible by 12 (cf. Ringler [2007]). The explicit effective result for threefolds with $\kappa = 1$ was obtained by Hsin-Ku Chen very recently. He show that φ_m is birational to the Iitaka fibration as long as $m \geq 96$ and divisible by 12 (cf. H.-K. Chen [2017]) and hence $M_{3,1} \leq 96$ and $d_{3,1} = 12$. There exists an example which shows that $M_{3,1} \geq 42$.

6 Further applications of the theory of baskets and the singular Riemann-Roch

6.1 Weighted complete intersections.

A weighted projective space is a natural generalization of projective spaces. Together with complete intersection inside them, weighted projective spaces provide ample examples. In Iano-Fletcher [2000], Fletcher gave a very detailed account of getting well-formed weighted complete intersection threefolds, including the classification of:

- canonically embedded codimension 1 and 2 weighted 3-folds with total weights < 100;

- weighted complete intersection $\mathbb{Q}$-Fano 3-folds of codimensions 1, 2 with total weights < 100.

Let $X = X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0, \dots, a_n)$ be a weighted complete intersection. Let $\delta_j := d_j - a_{j+\dim X}$. By modifying Fletcher's proof, it's not difficult to verify that quasi-smoothness implies that $\delta_j \geq a_{j-1}$. Then the following Theorem follows.

Theorem 6.1. *(J.-J. Chen, J. A. Chen, and M. Chen [2011, Theorem 1.3]) There is no quasi-smooth (not an intersection of a linear cone with another subvariety) weighted complete intersection $X_{d_1,\dots,d_c} \subset \mathbb{P}(a_0, \dots, a_n)$ of codimension c greater than* $\dim X + 1 + \alpha$*, where $\alpha = \sum_j d_j - \sum_i a_i$.*

In order to classify 3-fold weighted complete intersections with at worst terminal singularities, let $\mu_i := \#\{a_j | a_j = i\}$, and $\nu_i := \#\{d_j | d_j = i\}$. By Theorem 6.1, we have $\sum \mu_i \leq \alpha + A, \sum \nu_i \leq \alpha + B$ for some small integers A, B. One can classify tuples $(\mu_1, \dots, \mu_6; \nu_2, \dots, \nu_6)$ (resp. $(\mu_1, \dots, \mu_5; \nu_2, \dots, \nu_5)$) when $\alpha = 1$ (resp. $\alpha = -1$) under given conditions. It is then possible to classify initial baskets of weighted baskets with given μ_i, ν_i's. For a given weighted basket, one can compute its plurigenera by Reid's Riemann-Roch formula. By Reid's "table method", which mainly works on Poincaré series, one can determine all possible weighted complete intersections with given formal baskets.

There might be infinitely many initial baskets with given μ_i, ν_i's since the index of each individual basket might be arbitrarily large. In the Fano case with $\alpha = -1$, one can use the property $(-K_X \cdot c_2) \geq 0$ to obtain the maximal index of basket. In the case of general type, one can use $K_X^3 > 0$ to exclude most of the baskets with large indices. The details can be found in J.-J. Chen, J. A. Chen, and M. Chen [ibid.].

To summarize, the following statement holds:

Theorem 6.2. *(see J.-J. Chen, J. A. Chen, and M. Chen [ibid., Part II]) The lists of threefold weighted complete intersections in Fletcher Iano-Fletcher [2000, pp. 15.1, 15.4, 16.6, 16.7, 18.16] are complete.*

For higher dimensional weighted complete intersections, one may refer to the interesting paper of Brown and Kasprzyk [2016].

6.2 On quasi-polarized threefolds.

One can also apply the technique of baskets and singular Riemann-Roch to study some quasi-polarized threefolds (X, L). For example, C. Jiang [2016] proved the following interesting results:

- Let X be a minimal 3-fold with $K_X \equiv 0$ and L a nef and big Weil divisor. Then $|mL|$ and $|K_X + mL|$ give birational maps for all $m \geq 17$.

- Let X be a minimal Gorenstein 3-fold with $K_X \equiv 0$ and L a nef and big Weil divisor. Then $|K_X + mL|$ gives a birational map for all $m \geq 5$.

7 A brief review to explicit birational geometry of higher dimensional varieties

In dimension 4 or higher, it seems very difficult or hopeless to have desired description of terminal singularities. Therefore it is hard to study the singular Riemann-Roch formula on minimal $\mathbb{Q}$-factorial terminal n-folds. In other words, many techniques described above for 3-folds do not work in higher dimensions.

Even though little is known for dimension 4 or higher. There are some interesting results that we would like to recall here. Interested readers may find possible paths to move on.

7.1 Projective varieties with very large canonical volumes.

Apart from considering the number r_n, it is also interesting to consider another optimal constant r_n^+ so that, for all nonsingular projective n-folds X of general type with $p_g(X) > 0$, $\varphi_{m,X}$ is birational for all $m \geq r_n^+$. By definition $r_n^+ \leq r_n$ for any $n > 0$. One has $r_1^+ = r_1 = 3$ and $r_2^+ = r_2 = 5$ according to Bombieri. We start with the review of Bombieri's result:

Theorem 7.1. *(see Bombieri [1973]) Let S be a minimal surface of general type. Then*

(1) when $p_g(S) \geq 4$, $r_s(S) \leq r_1^+ = 3$;

(2) when $K_S^2 \geq 3$, $r_s(S) \leq r_1 = 3$.

The 3-dimensional analogy was realized by the second author and Todorov respectively:

Theorem 7.2. *Let X be a minimal projective 3-fold of general type. Then*

(1) when $p_g(X) \geq 4$, $r_s(X) \leq r_2^+ = 5$ (see M. Chen [2003, Theorem 1.2 (2)]);

(2) when $K_X^3 > 12^3$, $r_s(X) \leq r_2 = 5$ (see G. T. Todorov [2007] and M. Chen [2012]).

Recently the second author and Jiang proved the following analogy in dimensions 4 and 5:

Theorem 7.3. *(M. Chen and Z. Jiang [2017a, Theorem 1.4]) There exists a constant $K(4) > 0$ such that for any minimal 4-fold X with $K_X^3 > K(4)$, $r_s(X) \leq r_3$.*

Theorem 7.4. *(M. Chen and Z. Jiang [2017a, Theorem 1.5]) There exist two constants $L(4) > 0$ and $L(5) > 0$. For any minimal n-fold X of general type with $p_g(X) \geq L(n)$ ($n = 4, 5$), $r_{\mathrm{s}}(X) \leq r_{n-1}^{+}$.*

It is natural to ask whether such analogy holds in any dimension.

7.2 Pluricanonical maps on varieties of higher Albanese dimensions.

We recall the following interesting theorem on varieties of maximal Albanese dimensions:

Theorem 7.5. *(see J. A. Chen and Hacon [2002] and Z. Jiang and Lahoz [2013]) Let X be a smooth projective variety of maximal Albanese dimension and of general type. Then $\varphi_{m,X}$ is birational for $m \geq 3$.*

The result in the above theorem is clearly optimal. In fact there is the following generalization:

Theorem 7.6. *(see Z. Jiang and Sun [2015]) Let X be a smooth projective variety of general type and of Albanese fiber dimension one. Then $\varphi_{m,X}$ is a birational map for $m \geq 4$.*

Question 7.7. Under what condition $\varphi_{3,X}$ is birational for varieties of Albanese fiber dimension one?

Can one find optimal statement for varieties of Albanese fiber dimensions 2 and 3?

7.3 Geography of varieties of general type.

It is interesting to know how those birational invariants, such as the canonical volume, p_g, $\chi(\mathcal{O})$ and so on, reflect geometric properties of a given variety. Inequalities among birational invariants play important roles in the classification theory and many other geometrical problems.

In high dimensions, the famous inequality of Yau [1977, 1978] discloses the optimal relations between c_1 and c_2 on canonically polarized varieties, namely:

$$\frac{2(n+1)}{n}|c_1^{n-2} \cdot c_2| - |c_1^n| \geq 0$$

holds for any n-dimensional canonically polarized variety. Miyaoka [1987] proved that $3c_2 - c_1^2$ is pseudo-effective for canonically quasi-polarized (i.e. K being nef and big) varieties. One notices that Greb, Kebekus, Peternell, and Taji [2015] had an interesting generalization of the inequality of Yau and Miyaoka.

For Gorenstein minimal 3-folds of general type, the following inequality was proved by the authors J. A. Chen and M. Chen [2015b] (see M. Chen [2004b] and Catanese, M. Chen, and Zhang [2006] for historical context):

$$K^3 \geq \frac{4}{3}p_g - \frac{10}{3},$$

which is sharp thanks to examples of Kobayashi [1992]. Applying a key lemma in J. A. Chen and M. Chen [2015b], Hu [2013] proved the following optimal inequality:

$$K^3 \geq \frac{4}{3}\chi(\omega) - 2.$$

In a recent work of J. A. Chen and Lai [2017], two series of examples with small "slope" in arbitrary higher dimensions were constructed.

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Jungkai A. Chen (陳榮凱)

National Center for Theoretical Sciences
Taipei, 106, Taiwan

and

Department of Mathematics
National Taiwan University
Taipei, 106, Taiwan
jkchen@ntu.edu.tw

Meng Chen (陈猛)
School of Mathematical Sciences & Shanghai Centre for Mathematical Sciences
Fudan University
Shanghai 200433, China
mchen@fudan.edu.cn